Capturing the Landauer bound through the application of a detailed Jarzynski equality for entropic memory erasure

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(Received 3 September 2014; published 12 December 2014)

The states of an overdamped Brownian particle confined in a two-dimensional bilobal enclosure are considered to correspond to two binary values: 0 (left lobe) and 1 (right lobe). An ensemble of such particles represents bits of entropic information. An external bias is applied on the particles, equally distributed in two lobes, to drive them to a particular lobe erasing one kind of bit of information. It has been shown that the average work done for the entropic memory erasure process approaches the Landauer bound for a very slow erasure cycle. Furthermore, the detailed Jarzynski equality holds to a very good extent for the erasure protocol, so that the Landauer bound may be calculated irrespective of the time period of the erasure cycle in terms of the effective free-energy change for the process. The detailed Jarzynski equality applied to two subprocesses, namely the transition from entropic memory state 0 to state 1 and the transition from entropic memory state 1 to state 1, connects the work done on the system to the probability to occupy the two states under a time-reversed process. In the entire treatment, the work appears as a boundary effect of the physical confinement of the system not having a conventional potential energy barrier. Finally, an analytical derivation of the detailed and classical Jarzynski equality for Brownian movement in confined space with varying width has been proposed. Our analytical scheme supports the numerical simulations presented in this paper.

DOI: 10.1103/PhysRevE.90.062120 PACS number(s): 05.40.Jc, 05.10.Gg

I. INTRODUCTION

The computational procedure consists of a number of logic operations [1]. In the case of irreversible logic operations, one cannot properly estimate the inputs by observing the output [2]. Information is actually lost in the course of operation. This erasure of information is accompanied by dissipation of a minimum quantity of heat. Landauer first argued that erasure of a classical bit of information is associated with dissipation of at least $k_B T \ln 2$ of heat [3]. To realize this limit through an experiment or a numerical process, one has to represent the information by some physical quantity, and the act of erasing is done in an appropriate physical setup [4]. The two wells of a bistable potential may represent the two binary values of the memory, say, the left potential well represents 0 state and the right potential well represents 1 state. Then the dynamics of an overdamped Brownian particle moving in a bistable potential and subjected to an external bias may be viewed as an erasure process [5,6]. Here, the Landauer limit is recovered for a very long erasure cycle [5]. The idea of correlating information erasure with thermodynamics [2–15] has been an important area of research for quite a long time and is still a very active domain in view of both experimental and theoretical investigations. The immediate consequence of realizing the Landauer limit is to investigate whether this lower bound of work done can be found from the free-energy difference of the initial and the final states of the system in the erasure cycle. It is expected that the Jarzynski equality [16–18] would be of much help in this context as it connects the nonequilibrium work measurement with the free-energy change of a process. This has been studied recently [13] both experimentally and theoretically considering the two potential minima of the above-mentioned bistable system as two memory states, 0 and 1. It has been observed that the detailed Jarzynski equality [19], rather than the classical Jarzynski equality [16–18], serves the purpose satisfactorily.

In the present paper, we have explored a different representation of the memory state. We consider a Brownian particle confined in a two-dimensional bilobal enclosure. The particle is essentially free as there is no intrinsic potential present, the only constraint being that the particle is made to move in a geometrical confinement [20–24]. It is now well known that when a Brownian particle is made to move in a channel or a tube of varying cross section, the confinement in higher dimension gives rise to an entropic potential [21–49] in the reduced dimension. The state of the Brownian particle in two different lobes may be designated by two binary values 0 and 1 (for example, the state of the particle in the left lobe is assigned to 0 and that in the right lobe to 1). As we are interested in the statistics of work done associated with the erasure process, we consider an ensemble of such particles. Each particle corresponds to an entropic bit of information or an entropic memory state. In a recent study, the existence of the Landauer bound was investigated numerically for an entropic memory erasure procedure [15]. Here we have explored the statistics of work done, and we focus on the proper application of the Jarzynski equality to extract the actual free-energy change of the entropic memory erasure process through numerical computation. We also seek the connection of the average work done and the effective free-energy change with the Landauer bound for the erasure protocol. The interesting aspect related to the present study is that here the calculated work appears as a nontrivial boundary effect. Essentially, we study the Brownian dynamics of a free particle on which a geometrical boundary condition has been imposed. The external bias applied on the particle in the erasure cycle exploits the nonlinearity of the confinement to produce the work value associated with the process. Our object is to explore the applicability of the Jarzynski equality and the detailed Jarzynski equality for

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the memory erasure procedure where the two memory states are separated by an entropic barrier, and to obtain the value of the free-energy change for such processes from the nonequilibrium work distribution obtained as a result of irregular geometrical confinement. Another focus of the study is the reverse protocol of the memory erasure process, which is necessary to investigate the existence of the detailed Jarzynski equality [14] through which the work done for two subprocesses is connected to the percent occupancy of the two lobes under the time-reversed process. In addition to the detailed numerical analysis done on the entropic memory erasure process, we carry out an analytical investigation on the applicability of the Jarzynski equality for this protocol.

The paper is organized as follows. In Sec. II, the model of the system and the dynamics of the overdamped Brownian particle are described for the entropic memory erasure protocol. In Sec. III, the numerical results are discussed. We analytically study the validity of the detailed and the classical Jarzynski equality for the entropic memory erasure process in Sec. IV. The paper concludes in Sec. V.

II. DESCRIPTION OF THE MODEL AND THE STOCHASTIC DYNAMICS

We have considered a two-dimensional overdamped dynamics of a Brownian particle which is allowed to move in a bilobal enclosure as shown in Fig. 1(a). The following equation corresponds to the Langevin dynamics of the Brownian particle:

$$\gamma \frac{d\vec{r}}{dt} = -G\hat{e_y} + B(t)\hat{e_x} + \sqrt{\gamma k_B T}\,\vec{\xi(t)}. \tag{2.1}$$

In the above equation, $\vec{r}$ represents the position vector of the particle, and $\hat{e_x}$ and $\hat{e_y}$ stand for the unit vectors along the x and y directions, respectively. γ denotes the frictional coefficient of the system, and k_B and T are the Boltzmann constant and temperature of the bath, respectively. G corresponds to a very weak constant bias that acts along the negative y direction of the system. $\vec{\xi(t)} = (\xi_x(t),\xi_y(t))$ is a zero mean, Gaussian white noise and obeys the fluctuation-dissipation relationship. The characteristics of the noise are described by the following equations:

$$\langle \vec{\xi(t)} \rangle = 0, \quad \langle \xi_i(t)\xi_j(t')\rangle = 2\delta_{ij}\delta(t-t') \tag{2.2}$$

for $i,j = x,y$.

Apart from the usual forces described above, the particle is also subjected to an additional external bias, $B(t)\hat{e_x}$, which has a sawtooth form with respect to time. The physical confinement of the particle has been introduced by imposing the following boundary conditions on the two-dimensional dynamics of the Brownian particle. The lower and the upper boundary functions of the bilobal enclosure [28], as shown in Fig. 1(a), can be described by the following equation:

$$\begin{aligned} y_l(x) &= -y_u(x) = \omega_l(x) = -\omega_u(x) \\ &= L_y(x/L_x)^4 - 2L_y(x/L_x)^2 - c/2, \end{aligned} \tag{2.3}$$

where $\omega_l(x)$ and $\omega_u(x)$ denote the lower and the upper walls of the confinement, and $y_l(x)$ and $y_u(x)$ correspond to the lower bound and the upper bound of the y value at position x, respectively. L_x stands for the distance between the midpoint of the bottleneck and the position of the maximal width, L_y represents the narrowing of the boundary functions, and c corresponds to the remaining width at the position of the bottleneck. The local half-width of the bilobal confinement is described by the following equation:

$$\omega(x) = [\omega_u(x) - \omega_l(x)]/2. \tag{2.4}$$

These wall functions are responsible for the confined movement of the overdamped Brownian particle in the bilobal enclosure.

We now use a dimensionless description [24–34] of the system and the dynamics for the sake of simplicity for further analysis. The lengths are scaled with the characteristic length scale, L_x, i.e., $\tilde{x} = x/L_x$ and $\tilde{y} = y/L_x$, suggesting $\tilde{c} = c/L_x$. This scaling leads to the scaled boundary functions and the local half-width described as $\tilde{\omega}_l(\tilde{x}) = \omega_l(x)/L_x = -\tilde{\omega}_u(\tilde{x})$ and $\tilde{\omega}(\tilde{x}) = \omega(x)/L_x$. The time t is scaled by a distinctive time scale $t_{\rm ref}$ as $\tilde{t} = t/t_{\rm ref}$ with $t_{\rm ref} = \gamma L_x^2/k_B T_R$, where T_R corresponds to a reference temperature. $t_{\rm ref}$ actually denotes twice the time necessary for a particle to diffuse a distance L_x at temperature T_R. The forces are scaled by the characteristic force term $F_R = \gamma L_x/t_{\rm ref}$ leading to $\tilde{G} = G t_{\rm ref}/\gamma L_x$ and $\tilde{B}(\tilde{t}) = B(t)t_{\rm ref}/\gamma L_x$. To maintain brevity and notational convenience, tildes will be omitted from now on. In dimensionless form, the Langevin equation can be described

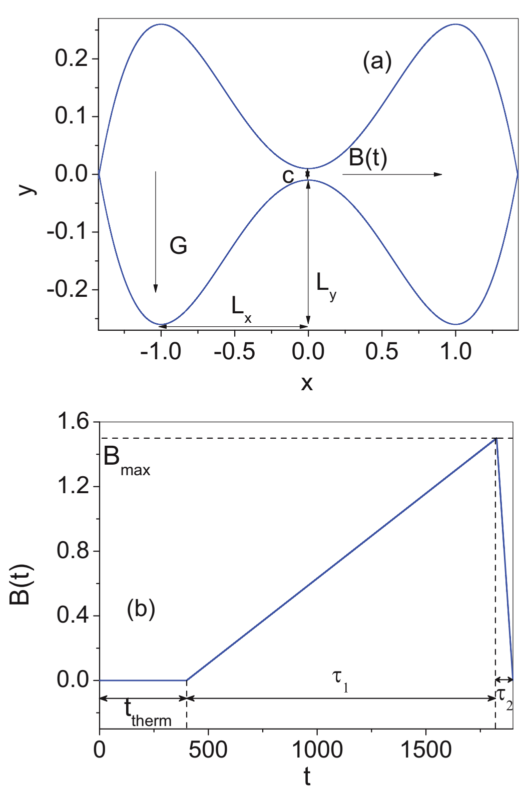

FIG. 1. (Color online) (a) The bilobal enclosure with its geometric parameters, (b) time series plot of the external bias $B(t)$ with the parameter set $B_{\max} = 1.5$, $t_{\rm therm} = 400$, $\tau_1 = 1425$, and $\tau_2 = 75$.

as follows:

$$\frac{d\overrightarrow{r}}{dt} = -G\widehat{e_y} + B(t)\widehat{e_x} + \sqrt{D}\overrightarrow{\xi(t)}, \tag{2.5}$$

where D is represented as T/T_R. D signifies the strength of noise and it is dependent both on the thermal energy and the mobility of the system. Here, we have scaled all the force terms including the Langevin force with the factor $\gamma L_x/\tau$ to make them dimensionless. This leads us to the above expression of D. The Langevin dynamics described above can be decomposed into two mutually perpendicular Langevin equations along the x and y directions as follows:

$$\begin{aligned} \frac{dx}{dt} &= B(t) + \sqrt{D}\xi_x(t), \\ \frac{dy}{dt} &= -G + \sqrt{D}\xi_y(t), \end{aligned} \tag{2.6}$$

where $\xi_x(t)$ and $\xi_y(t)$ denote the components of the Langevin force $\xi(t)$ along the x and y directions, respectively. The wall function is written as follows:

$$\omega(x) = [\omega_u(x) - \omega_l(x)]/2 = -ax^4 + bx^2 + c/2. \tag{2.7}$$

In the above equation, the aspect ratio has been defined as $a = L_y/L_x$ and $b = 2a$, implying that a and b are appropriately scaled constants. The external driving force $B(t)$ actually has a sawtooth form described as follows:

$$B(t) = 0$$

for $0 < t \leqslant t_{\text{therm}}$,

$$B(t) = B_{\max}(t - t_{\text{therm}})/\tau_1$$

for $t_{\text{therm}} < t \leqslant t_{\text{therm}} + \tau_1$,

$$B(t) = B_{\max}[1 - (t - t_{\text{therm}} - \tau_1)/\tau_2]$$

for $t_{\text{therm}} + \tau_1 < t \leqslant t_{\text{therm}} + \tau_1 + \tau_2$,

$$B(t) = 0 \tag{2.8}$$

for $t > t_{\text{therm}} + \tau_1 + \tau_2$.

t_{therm} corresponds to the thermalization time in Eqs. (2.8). The thermalization time refers to the time interval during which the system is allowed to reach a thermal equilibrium initially with the bath in the absence of any bias; i.e., after time t_{therm}, the external force $B(t)$ is turned on. Then during the time interval between t_{therm} and $t_{\text{therm}} + \tau_1$, the external bias reaches its maximum value $B_{\max}$, which corresponds to the amplitude of the driving force following the return of the value of $B(t)$ back to its initial value, i.e., $B(t) = 0$ within the time duration τ_2. The time series plot of the external bias force has been represented in Fig. 1(b).

In general, noise is inherently present in computational devices as the systems are small in size and the constituents are small in number. As a consequence, intrinsic noise has been considered to model such processes. The presence of noise turns the thermodynamic quantities such as work or heat corresponding to the erasure procedure into stochastic variables. Therefore, we essentially calculate the average values of the stochastic thermodynamic variables and their functions. For this purpose, we consider an ensemble of particles. Each particle is placed at the position of the bottleneck (0,0) initially. As expected, during the thermalization time t_{therm}, the particles get equally distributed in the two lobes, i.e., the system contains both types of binary information in equal proportion. As stated earlier, the left lobe of the confinement is assigned to logical value 0 and the right lobe to logical value 1. After the initial thermalization period t_{therm}, the external bias $B(t)$ is switched on and the particles are directed selectively to the desired lobe, leading to erasure of one kind of bits of information.

The Fokker-Planck equation [50] corresponding to the Langevin dynamics [Eq. (2.6)] in the absence of any external bias can be written as

$$\begin{aligned} \frac{\partial P(x,y,t)}{\partial t} = {} & D\frac{\partial}{\partial x}\exp\left[\frac{-u(x,y)}{D}\right] \\ & \times \frac{\partial}{\partial x}\exp\left[\frac{u(x,y)}{D}\right]P(x,y,t) \\ & + D\frac{\partial}{\partial y}\exp\left[\frac{-u(x,y)}{D}\right] \\ & \times \frac{\partial}{\partial y}\exp\left[\frac{u(x,y)}{D}\right]P(x,y,t), \end{aligned} \tag{2.9}$$

where the potential function is represented as $u(x,y) = Gy$. To consider the effect of confinement, we use the reflecting boundary condition at the wall of the enclosure. The dimensional reduction (i.e., the study of the dynamics only along the direction of interest) can be done by introducing a marginal probability distribution $C(x,t)$ along the x direction [i.e., $C(x,t) = \int dy P(x,y,t)$] and a local equilibrium probability density of y conditioned at a given x, $\rho(y;x)$, and assuming that $P(x,y,t) \cong C(x,t)\rho(y;x)$. Therefore, after reducing the transverse direction, the kinetic equation for the marginal probability distribution takes the following form:

$$\frac{\partial C(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[D\frac{\partial}{\partial x}C(x,t) + A'(x,D,G)C(x,t)\right]. \tag{2.10}$$

The effective potential $A(x,D,G)$ is obtained from the exact potential $u(x,y)$ making use of the relation described as $\int dy\, e^{-u(x,y)/D} = e^{-A(x,D,G)/D}$. For a constant force acting along the transverse direction, the potential function $A(x)$ has the following form:

$$A(x,D,G) = -D\ln\left[\frac{2D}{G}\sinh\left(\frac{G\omega(x)}{D}\right)\right]. \tag{2.11}$$

$A(x,D,G)$ corresponds to the potential that is associated with the varying cross-sectional width of the system. Therefore, the barrier created by the potential function $A(x,D,G)$ is entropic rather than energetic in origin. Equations (2.9)–(2.11) help us to realize the emergence of an entropic potential in reduced dimension when a Brownian particle is allowed to move in a higher-dimensional confinement having varying cross-sectional [22–24] width. This implies that the diffusive motion of the particle gets retarded due to the irregularity of the wall functions of the confining system, even in the absence of any conventional potential energy barrier. This dimension reduction formulation holds to a good extent in the presence of an external bias when the forcing amplitude is not too high [51].

III. NUMERICAL RESULTS AND DISCUSSION

For numerical simulation, we consider the overdamped two-dimensional Langevin dynamics of the Brownian particle described in Eq. (2.6) along with the boundary condition represented in Eq. (2.7). Equation (2.6) is solved using an improved Euler algorithm. The time step has been taken to be equal to 10^{-3}. The Langevin force has been generated using the Box-Muller algorithm [52]. We use the basic algorithm [53] proposed by Box and Muller to get normally distributed random variables representing thermal noise from uniformly distributed random numbers generated in the interval (0,1). One additional check has been incorporated in the algorithm to ensure a physically realistic value of noise. The width of the distribution is determined by the strength of the noise, i.e., D. The values of a, b, and c are set as 0.25, 0.5, and 0.02, respectively, for the entire study. The value of the very weak transverse force has been kept fixed as 0.0001. This value of G (tending to zero) ensures achieving the entropic limit of the potential. This implies that the work distribution and free-energy estimate appear purely as a result of physical confinement.

As discussed earlier, the particles get equally distributed in two lobes of the bilobal enclosure during the thermalization time, and after application of the external bias they accumulate to a particular lobe. The form of the external bias [Eq. (2.8)] is considered in such a way that it drives all the particles from the left and right lobe into the right lobe, i.e., the memory states 0 and 1 are erased to the memory state 1. We estimate the free-energy change for the entropic memory erasure process from the work value, or more specifically, the mean of the exponential function of the work done [16–18] obtained by direct numerical simulation of the exact dynamics along with appropriate boundary conditions, and we verify whether it agrees with the Landauer bound of minimum work done for the process. The statistical work done along each trajectory within the erasure cycle is numerically calculated using the following expression:

$$W = \int_{t_{\text{therm}}}^{t_{\text{therm}}+\tau} dt\, B(t)\dot{x}. \tag{3.1}$$

This form arises from the simple definition of work done by an applied field. As the effect of application of the external bias over the entire time period is intended to be captured, the force is multiplied by the velocity of the particle and is integrated over the forcing time period, t_{therm} to $t_{\text{therm}} + \tau$, where $\tau = \tau_1 + \tau_2$. $\dot{x}$ for each time step is calculated taking into account the x values of two consecutive time steps (obtained from simulation of the two-dimensional Langevin dynamics). The value of the position variable, x, and hence the velocity, $\dot{x}$, are guided by the external bias and the wall functions. This is reflected in the value of W for all trajectories, leading to a work distribution emerging purely as an effect of the physical boundary.

A. Estimation of the effective free-energy change for the entropic memory erasure process using the Jarzynski relation

We denote the free-energy change for the entropic memory erasure process, i.e., transfer of information content of the system to a unique state of memory, as $\Delta F_{\text{effective}}$. This effective free-energy change for the process is related to the nonequilibrium work done for the memory erasure protocol as

$$\langle e^{-\beta W}\rangle_{\longrightarrow 1} = e^{-\beta \Delta F_{\text{effective}}}, \tag{3.2}$$

where $\beta = 1/k_B T$, and $\langle . \rangle_{\longrightarrow 1}$ denotes the mean value of the quantities averaged over trajectories which ultimately end up in the right lobe (state 1) at the end of the erasure protocol. In the present study, we are dealing with dimensionless system parameters and dynamics. All the quantities have been made dimensionless with proper scaling factors. This suggests that the corresponding dimensionless parameter of $\gamma k_B T$ [in Eq. (2.1)] is D. Had we simulated the actual dynamics with quantities having their usual dimensions, we would have considered γ to be equal to unity, as we are not actually interested in the friction coefficient appearing in the dynamics. We focus instead on the temperature of the bath, and we analyze its effect on the work value, keeping γ fixed. Therefore, for future reference, whenever we use $k_B T$, it must be remembered that the quantity actually has the same magnitude as D. We denote the percent success rate as P_{suc}, which is the ratio of the trajectories that stays at the desired lobe at the end of the forcing time period to the total number of trajectories. The calculation of the work value for these trajectories is only considered as they correspond to the successful erasure and contribute to the effective free-energy change. Inclusion of other trajectories into the above-mentioned calculation will introduce error in the value of $\Delta F_{\text{effective}}$. For the entire numerical study, we have ensured that $P_{\text{suc}} > 95\%$. To calculate the mean of the exponential function of work done as described by the left-hand side of Eq. (3.2), we subdivide the process of jumping of particles from both the lobes to the right lobe into two subprocesses depending upon the initial position of the particles in the erasure cycle. Let us define the value of the average exponential work functions for these two subprocesses as follows:

$$A_{01} = \langle e^{-\beta W}\rangle_{0\longrightarrow 1}, \quad A_{11} = \langle e^{-\beta W}\rangle_{1\longrightarrow 1}, \tag{3.3}$$

where $\langle . \rangle_{i\longrightarrow j}$ corresponds to the mean value of the function for the trajectories that initially start at the ith lobe and end at the jth lobe after the completion of the erasure cycle, with $(i,j) \equiv (0,1)$. Consequently, the average quantity for the successful memory erasure process is given by

$$\langle e^{-\beta W}\rangle_{\longrightarrow 1} = \frac{A_{01} + A_{11}}{2}. \tag{3.4}$$

The factor 1/2 appears because of the equally distributed memory states attained after the thermalization period. We have plotted A_{01}, A_{11}, and the sum $A_{01} + A_{11}$ against the forcing time period $\tau = \tau_1 + \tau_2$ in Figs. 2(a) and 2(b) at three different temperatures for optimized $B_{\max}$ values corresponding to the temperatures and three different amplitudes of the external bias for a fixed temperature D, respectively. The optimized value of $B_{\max}$ indicates the minimum value of the forcing amplitude for which P_{suc} is at least 95% for all values of τ concerned when all other parameters are held fixed. t_{therm} has been taken to be equal to 400, and τ_1 and τ_2 have been varied over a wide range keeping the ratio

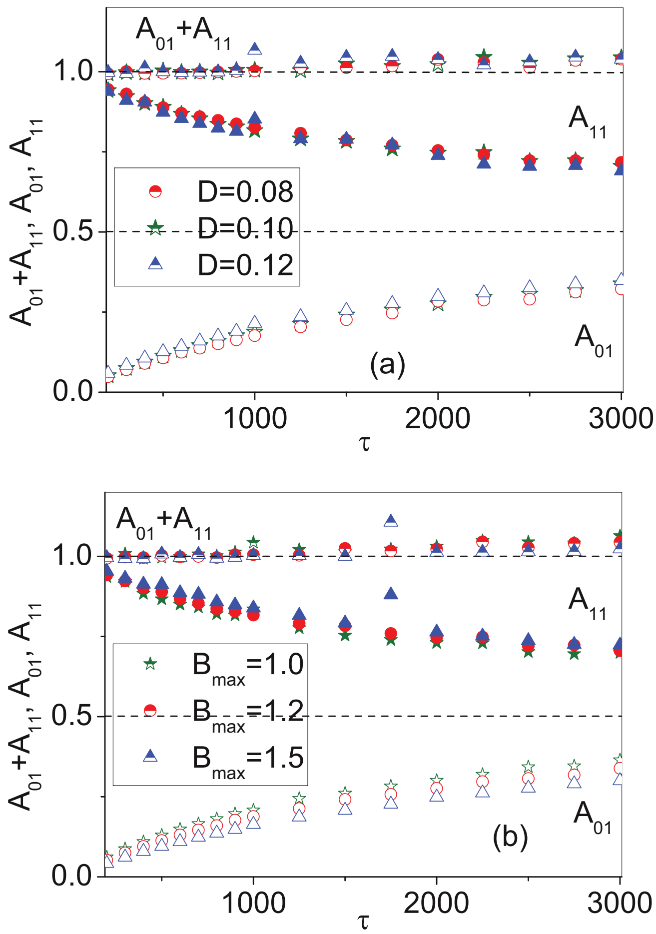

FIG. 2. (Color online) Plot of A_{01}, A_{11}, and $A_{01}+A_{11}$ against τ (a) for three different temperatures: $D=0.08$ ($B_{\max}^{\rm opt}=0.9$), $D=0.1$ ($B_{\max}^{\rm opt}=1.2$), and $D=0.12$ ($B_{\max}^{\rm opt}=1.4$); and (b) for three different amplitudes of the external bias: $B_{\max}=1.0$, 1.2, and 1.4 at $D=0.1$. The parameter set used is $a=0.25$, $b=0.5$, $c=0.02$ and $t_{\rm therm}=400$, $\tau_1:\tau_2=19:1$, and $G=0.0001$. The half-filled symbols correspond to $A_{01}+A_{11}$, open symbols to A_{01}, and completely filled symbols to A_{11}.

$\tau_1:\tau_2$ fixed at $19:1$ for the entire process. The averaging has been performed over 10^6 trajectories for all cases. It has been observed that the sum $A_{01}+A_{11}$ is constant and close to 1 irrespective of the value of noise strength or the amplitude of the external bias, and more specifically on the forcing time period. This value of $A_{01}+A_{11}$ corresponds to $\langle e^{-\beta W}\rangle_{\rightarrow 1}=1/2$, leading to $\Delta F_{\rm effective}=k_BT\ln 2$ [Eq. (3.2)]. This signifies that the Landauer bound can be recaptured in terms of the free-energy change corresponding to the entropic memory erasure procedure. The intriguing feature of this result is that the limit can be understood for any time period of the external forcing, as here we concentrate on the free-energy change of the process. In the studies [5,6,15], which address the average work done for such processes, the attainment of the Landauer limit occurs for a very slow erasure process. It is also observed that although $A_{01}+A_{11}$ remains more or less constant, A_{01} increases and A_{11} decreases with an increasing value of τ. This observation will be explained later.

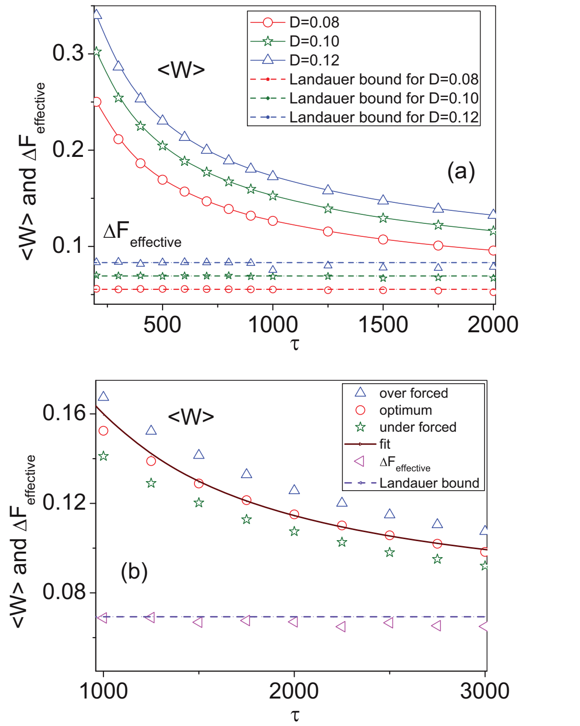

FIG. 3. (Color online) Plot of $\langle W\rangle$ and $\Delta F_{\rm effective}$ against τ (a) for three different temperatures: $D=0.08$ ($B_{\max}^{\rm opt}=0.9$), $D=0.1$ ($B_{\max}^{\rm opt}=1.2$), and $D=0.12$ ($B_{\max}^{\rm opt}=1.4$) (the large symbols denote $\langle W\rangle$, the similar small symbols represent corresponding $\Delta F_{\rm effective}$, and the dotted lines present the respective Landauer bounds) and (b) for three different forcing amplitudes: $B_{\max}=1.5$ (over forced), $B_{\max}=1.2$ (optimized), and $B_{\max}=1.0$ (under forced) at temperature $D=0.1$. The fitting with the function $k_BT\ln 2+\frac{C_1}{\tau}$ has been shown, where $C_1=90$. The parameter set used is $a=0.25$, $b=0.5$, $c=0.02$ and $t_{\rm therm}=400$, $\tau_1:\tau_2=19:1$, and $G=0.0001$.

B. Connection of $\langle W\rangle$ and $\Delta F_{\rm effective}$ with the Landauer bound

To discern the variation of $\langle W\rangle$ and $\Delta F_{\rm effective}$ with τ and their connection with the Landauer bound more clearly, we have plotted $\langle W\rangle$ and $\Delta F_{\rm effective}$ obtained from numerical simulation against τ for three different temperatures for optimized $B_{\max}$ values corresponding to the mentioned temperatures, and we compare the data with the Landauer bound accordingly in Fig. 3(a) for the same set of parameters as in the previous study. Averaging has been done over 10^6 trajectories. The value of $\langle W\rangle$ decreases with τ and approaches the Landauer limit for very high values of τ. However, for any value of τ, $\Delta F_{\rm effective}$ has a value close to the Landauer bound. It has also been observed that the variation of $\langle W\rangle$ against τ can be fitted with an expression $k_BT\ln 2+\frac{C_1}{\tau}$, where C_1 is a constant. This fit has been shown in Fig. 3(b) for an optimum value of $B_{\max}$ at a particular temperature. The variation of $\langle W\rangle$ against τ has also been presented for overforced and underforced situations in this plot [Fig. 3(b)]. The values of $\langle W\rangle$ or ΔF presented in Fig. 3 are dimensionless quantities, as we are dealing with a properly scaled system and dynamics.

C. Analysis of the results in light of the detailed Jarzynski equality

We now analyze the above results from the point of view of the overall process and subprocesses. The Landauer bound sets a lower limit for the work done associated with erasure of a classical bit of information. For memory erasure in systems in which fluctuation is present, this principle is modified [6] and the bound is obtained for the average value of the work done. As a consequence, it is expected that the mean value of work done for the erasure cycle would be at least $k_B T \ln 2$ (or $D \ln 2$ in dimensionless form), corresponding to the Landauer bound. We observe that this limit may be approached for very long cycles for erasure of an entropic bit of information. This suggests that the free-energy change for the entropic memory erasure process would be equal to $k_B T \ln 2$ as it essentially corresponds to the lower bound of the average work done for the process. Our numerical study reveals that the effective free-energy change obtained using the nonequilibrium work values for the entropic memory erasure protocol satisfies the limit. To evaluate the free-energy change of the process concerned, we have taken into consideration only the trajectories for which successful erasure takes place, and we do not focus on the process as a whole. Now, to resolve the work effect and the free-energy change for the overall process, we turn to the Jarzynski equality, which relates the equilibrium free-energy change for the entire process, ΔF, with the nonequilibrium work done, W, over an ensemble of similar paths associated with the procedure as follows:

$$\langle e^{-\beta W} \rangle = e^{-\beta \Delta F}. \tag{3.5}$$

The free-energy change ΔF in the Jarzynski equality actually corresponds to the difference in free-energy values of the equilibrium states corresponding to the final and the initial value of the switching field parameter. For the erasure cycle, the initial and the final value of the driving field parameter are $B(t=0)=0$ and $B(t = t_{\text{therm}} + \tau_1 + \tau_2 = t_{\text{therm}} + \tau) = 0$, i.e., there is no free-energy difference between the initial and the final equilibrium states of the system, which leads to $\Delta F = 0$ or $\langle e^{-\beta W} \rangle = 1$. Therefore, it is quite evident that the Landauer bound could not be retrieved had we used the classical Jarzynski equality. The correct calculation of the work done and the mean of the exponential function of work done as described in the Jarzynski equality regarding erasure of a bit of information concerns exact knowledge about the work parameter $B(t)$ and the stochastic variable of the Langevin dynamics $x(t)$ during the forcing time period only ($t = t_{\text{therm}}$ to $t = t_{\text{therm}} + \tau$). The important point to note here is that the final state in this process differs from the equilibrium state for the same value of $B(t)$. The distinct difference between the final state of the erasure process and the corresponding equilibrium state has been presented graphically in Figs. 4 and 5. Figure 4 presents the time evolution of $x(t)$ and attainment of the final state (not the equilibrium state) at the end of the erasure cycle along with the time series plot of the external bias. In Fig. 5, the same time series of $x(t)$ has been replotted up to the time when the system tends to reach equilibrium and finally will reach equilibrium after a certain time. The effective free-energy change, $\Delta F_{\text{effective}}$, denotes the change for the first process [Fig. 4(b)], not for the second one [Fig. 5(b)]. The overall

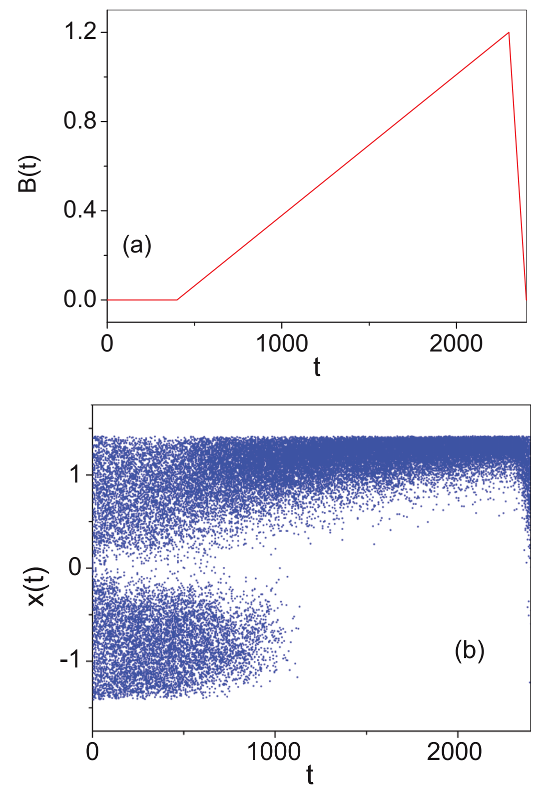

FIG. 4. (Color online) Time series plot of (a) $B(t)$ with $B_{\max} = 1.2$ and (b) $x(t)$ for 100 trajectories up to the completion of the erasure process. The parameter set used is $a = 0.25$, $b = 0.5$, $c = 0.02$ and $t_{\text{therm}} = 400$, $\tau_1 = 1900$, $\tau_2 = 100$, and $G = 0.0001$.

free-energy change, ΔF, is equal to zero, which is evident from Fig. 5(b).

To capture the work effect and free-energy difference for the memory erasure process, one has to take into consideration the detailed Jarzynski equality [13,14,19], which connects the free-energy change for a process and nonequilibrium work measurement for the same when the final state is quite distinct from the equilibrium state, through the actual and equilibrium probability density of the state,

$$\langle e^{-\beta W(t)} \rangle_{(x,t)} = \frac{\rho_{\text{eq}}(x)}{\rho(x,t)} e^{-\beta \Delta F}, \tag{3.6}$$

where the subscript (x,t) corresponds to the trajectories that pass through x at time t. $\rho(x,t)$ denotes the actual probability density at position x at time t, and $\rho_{\text{eq}}(x)$ stands for the equilibrium density at x when the external work parameter has been held fixed at the same value as that at time t. If we wish to apply the detailed Jarzynski equality to the erasure of information from both 0 and 1 memory state to memory state 1, we have to concentrate on the probability density of the state 1 (i.e., $x > 0$ corresponding to the right lobe) at time $t = t_{\text{therm}} + \tau$. It is evident from Figs. 4 and 5 that $\rho(x > 0, B(t_{\text{therm}} + \tau))$ is twice as big as $\rho_{\text{eq}}(x > 0)$. Therefore, the detailed Jarzynski equality for the successful memory erasure process [13] reads

$$\langle e^{-\beta W} \rangle_{(x>0,t_{\text{therm}}+\tau)} = \langle e^{-\beta W} \rangle_{\longrightarrow 1} = \frac{1/2}{P_{\text{suc}}} \tag{3.7}$$

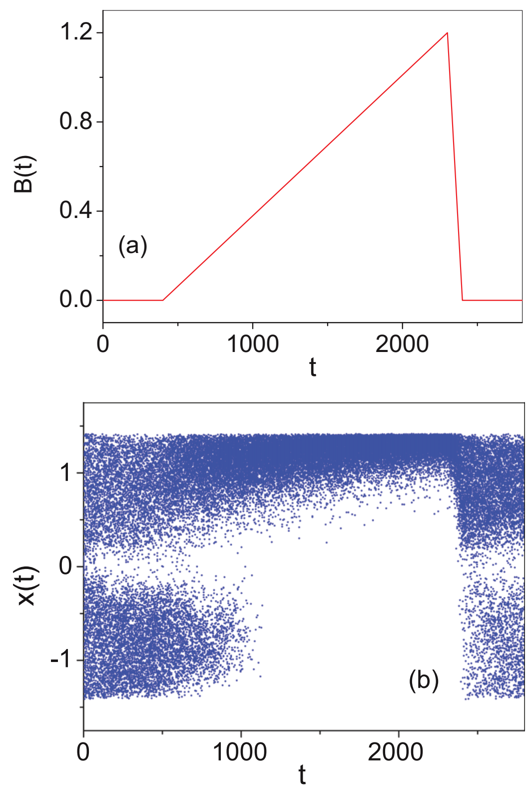

FIG. 5. (Color online) Time-series plot of (a) $B(t)$ with $B_{\max} = 1.2$ and (b) $x(t)$ for 100 trajectories tending to reach equilibrium. The parameter set used is $a = 0.25$, $b = 0.5$, $c = 0.02$ and $t_{\text{therm}} = 400$, $\tau_1 = 1900$, $\tau_2 = 100$, and $G = 0.0001$.

as the actual probability density at the state 1 is equivalent to the percent success rate of the process. Consequently, for P_{suc} close to 1 (which is the case in our study), $\langle e^{-\beta W}\rangle_{\longrightarrow 1}$ remains close to 1/2, leading to $\Delta F_{\text{effective}} \approx k_B T \ln 2$. This is what we have obtained in our numerical result. For the trajectories that ultimately end up in the wrong well (i.e., $x < 0$) at the end of the erasure cycle, the detailed Jarzynski equality can be written as

$$\langle e^{-\beta W}\rangle_{(x<0, t_{\text{therm}}+\tau)} = \langle e^{-\beta W}\rangle_{\longrightarrow 0} = \frac{1/2}{1 - P_{\text{suc}}}. \tag{3.8}$$

The Landauer limit of minimum work done for erasure of a bit of information can also be derived [13] using the above two equations [Eqs. (3.7) and (3.8)] obtained as a consequence of the detailed Jarzynski equality and Jensen's inequality ($\langle e^{-\beta W}\rangle \geqslant e^{-\beta\langle W\rangle}$), as

$$\langle W\rangle \geqslant k_B T \ln 2 + P_{\text{suc}} \ln P_{\text{suc}} + (1 - P_{\text{suc}})\ln(1 - P_{\text{suc}}). \tag{3.9}$$

This expression suggests a generalized version of the Landauer limit. Now, if we concentrate on the entire process as a whole and do not exclude the trajectories that end up in the wrong well at the completion of the erasure cycle, the average value of the exponential work function may be represented in terms of the average quantity for two individual process (for the trajectories ending up in the memory states 0 and 1) and the percent success rate of the entropic memory erasure process as follows:

$$\langle e^{-\beta W}\rangle = P_{\text{suc}}\langle e^{-\beta W}\rangle_{\longrightarrow 1} + (1 - P_{\text{suc}})\langle e^{-\beta W}\rangle_{\longrightarrow 0}. \tag{3.10}$$

The substitution of $\langle e^{-\beta W}\rangle_{\longrightarrow 1}$ and $\langle e^{-\beta W}\rangle_{\longrightarrow 0}$ according to Eqs. (3.7) and (3.8) gives rise to the expected value of $\langle e^{-\beta W}\rangle$ equal to 1. The validity of the classical Jarzynski equality has been checked numerically in our present study.

The variation of A_{01} and A_{11} with τ may be clarified with the help of the detailed Jarzynski equality applied to the two subprocesses corresponding to the act of successful erasure of entropic memory, namely transfer of an entropic bit of information from state 0 to state 1 and from state 1 to state 1. It has been shown [14] by Kawai *et al.* that

$$\langle e^{-\beta W}\rangle_j = \frac{\overline{\rho}_j(t)}{\rho_j(t)} e^{-\beta\Delta F_{\text{effective}}}, \tag{3.11}$$

where the subscript j corresponds to the index of the nonoverlapping subsets constituting the overall phase space. ρ_j and $\overline{\rho}_j$ stand for the phase-space density evaluated at the same but otherwise arbitrary intermediate instant of time for the forward and reverse processes, respectively. The subspace may be formed on the basis of any kind of distinction. In the present study, there are two subsets, i.e., $j = (01, 11)$, depending upon the initial position of the trajectory involved in a successful erasing process. If we consider $t = 0$, it is evident that $\rho_{01} = \rho_{11} = 1/2$ and $\overline{\rho}_{01}$ (or $\overline{\rho}_{11}$) is equivalent to the proportion of trajectories that return to state 0 ($\overline{P}_{10}$) [or state 1 ($\overline{P}_{11}$)] starting from state 1 under the time-reversed process. Taking into consideration the value of $e^{-\beta\Delta F_{\text{effective}}}$ equal to 1/2 and the phase-space densities under the forward and the reverse subprocesses individually, new descriptions of A_{01} and A_{11} are at hand,

$$\begin{aligned}\langle e^{-\beta W}\rangle_{0\longrightarrow 1} &= A_{01} = \overline{P}_{10},\\ \langle e^{-\beta W}\rangle_{1\longrightarrow 1} &= A_{11} = \overline{P}_{11}.\end{aligned} \tag{3.12}$$

To examine this relation, we run the entropic memory erasure process in the reverse direction and calculate $\overline{P}_{10}$ and $\overline{P}_{11}$, varying τ numerically over a wide range. Next, we compare this variation with A_{01} and A_{11}, respectively. This has been represented in Fig. 6. It is observed that the agreement holds quite well in the case of the entropic memory

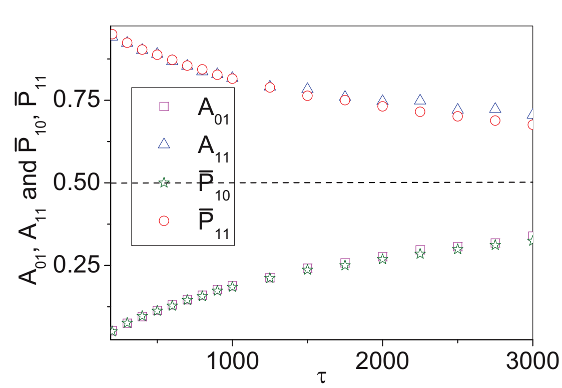

FIG. 6. (Color online) Variation of A_{01}, A_{11}, $\overline{P}_{10}$, and $\overline{P}_{11}$ with τ. The parameter set used is $a = 0.25$, $b = 0.5$, $c = 0.02$ and $D = 0.1$, $B_{\max} = 1.2$, $t_{\text{therm}} = 400$, $\tau_1 = 1900$, $\tau_2 = 100$, and $G = 0.0001$.

erasure phenomenon. A_{01}, i.e., $\overline{P}_{10}$, increases with increasing τ. This is because a long time duration of the reverse protocol promotes transition from one lobe to another. For a very long time period, $\overline{P}_{10}$ and $\overline{P}_{11}$ tend to equalize with each other. This is quite expected according to the above explanation.

One pertinent point must be clarified here. In previous studies related to the Landauer limit in an energetic domain, the erasure cycle consists of two steps, the first step being the symmetric lowering of the potential energy barrier and the second involving tilting of the potential. The lowering of the barrier does not have much significance in the evaluation of the work done [5,13]. This is even more conspicuous in systems having physical confinement with varying width. To modulate the depth or size of the lobes, one has to tune the system boundary. Therefore, the barrier lowering part does not affect directly the numerical calculation of the work done. Consequently, this barrier lowering step may be omitted without any alteration in the work distribution, and the erasure cycle consists of only the later step involving external forcing driving the particle to the desired lobe. As we do not deal with symmetric modulation of the wall function (equivalent to the symmetric lowering of the potential energy barrier) in the present study, we do not maintain a very high entropic barrier height, so that barrier crossing becomes almost improbable under the given condition. We actually assume that barrier lowering has already taken place at the beginning of the erasure cycle, and the cycle terminates instantly after the bias force is switched off. Thus soon after, the system regains its initial state.

IV. VALIDITY OF THE CLASSICAL AND DETAILED JARZYNSKI EQUALITY FOR ENTROPIC TRANSPORT: AN ANALYTICAL SCHEME

So far, we have concentrated on the numerical study associated with the entropic memory erasure process and analyzed the numerical results. The findings suggest that both the detailed and the classical Jarzynski equality hold well for the system with an entropic barrier. However, in previous studies the Jarzynski equality has been verified analytically for the Brownian movement of particles in the presence of an energetic potential barrier [16,17], but not in the case of physical confinement having varying width. Here in this section, we try to investigate analytically the validity of the Jarzynski equality when the movement of Brownian particles is guided by an effective entropic potential in reduced dimension.

We consider a parameter-dependent Hamiltonian $H_B(x,y)$, which depends on the value of the work parameter $B(t)$ and accounts for the total energy of the system in state $(x(t),y(t))$. With given $H_B(x,y)$, the partition function and free energy are defined as

$$Z_B(\beta) \equiv \int_{-x_r}^{+x_r} dx \int_{\omega_l(x)}^{\omega_u(x)} dy \exp[-\beta H_B(x,y)], \quad (4.1)$$

where $-x_r$ and $+x_r$ correspond to the extreme left and extreme right end of the confining system and

$$F_B(\beta) \equiv -\beta^{-1} \ln Z_B(\beta), \quad (4.2)$$

respectively. Again, β must be considered as a real, positive, dimensionless constant that corresponds to the inverse of the thermal energy of the system in equilibrium with the heat bath. Here also we deal with dimensionless quantities and dynamics as they are defined in terms of properly scaled variables and parameters introduced previously.

To proceed further, we define the work done along a particular trajectory due to the evolution of the work parameter [16–18] as follows:

$$W \equiv \int_{t_{\text{therm}}}^{t_{\text{therm}}+\tau} dt \;\; \dot{B}\frac{\partial H_B}{\partial B}(x(t),y(t)). \quad (4.3)$$

For a cyclic process, this definition of W and the expression used in Eq. (3.1) to denote the same are equivalent [18]. $\dot{B}(t)$ may be immediately evaluated using Eqs. (2.8). As discussed earlier, the system gets thermalized with the bath before switching on the external bias. Therefore, the probability distribution function at time t_{therm} is a canonical distribution defined as

$$P^{\text{eq}}(x,y,t_{\text{therm}}) = Z_0^{-1}\exp[-\beta H_0(x,y)]. \quad (4.4)$$

To have an idea about the analytic form of the distribution function, we return to the two-dimensional Fokker-Planck equation [Eq. (2.9)] described in Sec. II. In the presence of the external bias $B(t)$ acting along the x direction and constant bias G acting along the negative y direction, the potential function takes the form $u(x,y,t) = Gy - B(t)x$. In the presence of these two bias forces, Eq. (2.9) may be written in an alternative form described as

$$\frac{\partial P(x,y,t)}{\partial t} = -\frac{\partial}{\partial x}[B(t)]P(x,y,t) + D\frac{\partial^2}{\partial x^2}P(x,y,t) + \frac{\partial}{\partial y}[G]P(x,y,t) + D\frac{\partial^2}{\partial y^2}P(x,y,t). \quad (4.5)$$

The operator acting on the distribution function $P(x,y,t)$ on the right-hand side of the above equation is a linear operator. It can be abbreviated as

$$\frac{\partial P(x,y,t)}{\partial t} = \widehat{R}_B P(x,y,t), \quad (4.6)$$

where $\widehat{R}_B$ is a linear operator that has a parametric dependence on B. As stated before, initially the system is in thermal equilibrium. When $B(t)$ is swept infinitely slowly, the system remains in quasistatic equilibrium with the bath in all steps, and the distribution function matches with the canonical distribution. However, for finite time switching the distribution function $P(x,y,t)$ lags behind the equilibrium distribution [16–18]. The point to note here is that for a fixed value of B, the process actually describes a steady-state Markov process and the distribution function ultimately collapses to a canonical distribution corresponding to that particular value of B. This has been verified with the canonical distribution function $P^{\text{eq}}(x,y) = C\exp[-\frac{Gy-Bx}{k_BT}]$ that the linear operator $\widehat{R}_B$ nullifies the canonical distribution, i.e.,

$$\widehat{R}_B P^{\text{eq}}(x,y) = 0. \quad (4.7)$$

To explore the validity and proper applicability of the Jarzynski equality for the system having varying width, we define some important quantities and functions. For example, the work done up to any arbitrary time t due to the action of the external bias along a trajectory $(x(t),y(t))$ is described as follows:

$$W(t)=\int_{t_{\text{therm}}}^{t}dt'\dot{B}\frac{\partial H_B}{\partial B}(x(t'),y(t')). \tag{4.8}$$

At this point, we focus on the processes where the trajectories pass through a particular x value (as the state of the system is determined by the x value of the position coordinate of the particle) and not on the processes involving other trajectories. This has been inspired by the fact that ultimately the ensemble average of the exponential function of work done for the trajectories ending up in the desired state (i.e., state 1 or the right lobe) at the end of the erasure process gives the effective free-energy change for the entropic memory erasure process, as discussed earlier. A function $M(x,t)$ is defined as the average of $\exp[-\beta W(t)]$ for the trajectories that traverse through x at time t, i.e.,

$$M(x,t)=\langle\exp[-\beta W(t)]\rangle_{(x,t)}. \tag{4.9}$$

Another important function $f(x,y,t)$ constituted with earlier defined functions is considered. This is expressed as

$$f(x,y,t)=P(x,y,t)M(x,t). \tag{4.10}$$

The time evolution of this function can be described by the following equation:

$$\frac{\partial f(x,y,t)}{\partial t}=\left[\widehat{R}_B-\beta\dot{B}\frac{\partial H_B}{\partial B}\right]f(x,y,t). \tag{4.11}$$

This equation arises by taking the time derivative of Eq. (4.10) and substituting the value of $\dot{W}$, which follows from Eq. (4.8). The above equation [Eq. (4.11)] is solved to give [17]

$$f(x,y,t)=\frac{1}{Z_0}\exp[-\beta H_B], \tag{4.12}$$

where at time t, $B(t)=B$. The solution follows from the two properties of the operator $\widehat{R}_B$, namely the linearity of the operator and the ability to annihilate the canonical distribution of the system after acting on it [17]. Another consideration, which leads to the above expression for $f(x,y,t)$, is the initial condition, $f(x,y,t_{\text{therm}})=P(x,y,t_{\text{therm}})=\frac{1}{Z_0}\exp[-\beta H_0]$, as $W(t_{\text{therm}})=0$ for all the trajectories. Now, as we are interested in the trajectories terminating their motion at the right lobe at the end of the cycle and not particularly on the y value of their position coordinate, we integrate Eq. (4.12) along the transverse direction, i.e.,

$$\int dy\, f(x,y,t)=\frac{1}{Z_0}\int dy\exp[-\beta H_B]. \tag{4.13}$$

$f(x,y,t)$ is integrated over y to give $f_1(x,t)$, which contains the marginal probability distribution function $C(x,t)$ along the x direction and is defined as follows:

$$\begin{aligned}\int dy\, f(x,y,t)&=\left[\int dy\, P(x,y,t)\right]M(x,t)\\&=C(x,t)M(x,t)\\&=f_1(x,t).\end{aligned} \tag{4.14}$$

The integral of $\exp[-\beta H_B]$ along the y direction is defined following the description of the effective entropic potential discussed in Sec. II. Similarly as the effective entropic potential, we consider an effective canonical distribution for a fixed value of B as follows:

$$C^{\text{eq}}(x)=\frac{1}{Z_B}\int dy\exp[-\beta H_B(x,y)]=\frac{1}{Z_B}\exp\big[-\beta H_B^1(x)\big]. \tag{4.15}$$

This may be understood from the relation [22] $\int dy\exp[-\beta u(x,y)]=\exp[-\beta A(x)]$. H_B^1 stands for an effective Hamiltonian in reduced dimension. The partition function for the exact two-dimensional system and that in the reduced dimension has the same form,

$$\begin{aligned}Z_B&=\int_{-x_r}^{+x_r}dx\int_{\omega_l(x)}^{\omega_u(x)}dy\exp[-\beta H_B(x,y)]\\&=\int_{-x_r}^{+x_r}dx\exp\big[-\beta H_B^1(x)\big]\\&=Z_B^1.\end{aligned} \tag{4.16}$$

The partition function with superscript 1 in Eq. (4.16) corresponds to the one-dimensional partition function. The above considerations [Eqs. (4.14)–(4.16)] when applied to Eq. (4.13) yield

$$\begin{aligned}f_1(x,t)&=\frac{1}{Z_0^1}\exp\big[-\beta H_B^1(x)\big]\\&=\frac{Z_B^1}{Z_0^1}C^{\text{eq}}(x).\end{aligned} \tag{4.17}$$

Substituting the partition functions in Eq. (4.17) with their expressions in terms of the corresponding free energy of the system ($F_B=-\beta^{-1}\ln Z_B^1$), we get

$$C(x,t)\langle\exp[-\beta W(t)]\rangle_{(x,t)}=C^{\text{eq}}(x)\exp(-\beta\Delta F), \tag{4.18}$$

where $\Delta F=F_B-F_0$. So finally we arrive at the detailed Jarzynski equality [Eq. (4.18)]. The marginal probability distribution function is equivalent to the probability density $\rho(x,t)$ defined earlier. $\rho(x,t)$ is supposed to account for the total probability of finding a particle along the entire y range for a given value of x. This reconsideration makes Eq. (4.18) identical with Eq. (3.6). Now, if one wishes to calculate the effective free-energy change for the entropic memory erasure process, the mean value of the function $\exp[-\beta W(t)]$ is evaluated over the ensemble of trajectories that pass through $x>0$ at time $t_{\text{therm}}+\tau$. To get this, Eq. (4.18) may be rewritten as

$$\begin{aligned}\langle e^{-\beta W(t_{\text{therm}}+\tau)}\rangle_{(x>0,t_{\text{therm}}+\tau)}&=\frac{C^{\text{eq}}(x>0)}{C(x>0,t_{\text{therm}}+\tau)}e^{-\beta\Delta F}\\&=e^{-\beta\Delta F_{\text{effective}}}.\end{aligned} \tag{4.19}$$

The classical Jarzynski equality can be easily proved for the system with an entropic barrier in reduced dimension if we consider the relation

$$\langle e^{-\beta W}\rangle=\int dx\, f_1(x,t_{\text{final}}), \tag{4.20}$$

i.e., here we average over all the trajectories. Now, substituting the expression of $f_1(x,t_{\text{final}})$ in Eq. (4.20) from Eq. (4.17), we finally get

$$\langle e^{-\beta W}\rangle = \frac{1}{Z_0}\int dx \exp\big[-\beta H^1_{B(t_{\rm final})}(x)\big] = \frac{Z_B}{Z_0} = \exp[-\beta\Delta F]. \quad (4.21)$$

The above discussions indicate that the effectiveness of the detailed and the classical Jarzynski equality for the systems with confining geometry with varying width can also be recovered from the theoretical study, which corroborates our numerical results presented here. Another important point should be mentioned. We start our analytical derivation with the two-dimensional Fokker-Planck equation [Eq. (2.9)], and final results are given in terms of the one-dimensional description of the system considering the dimension reduction scheme [22] due to Zwanzig discussed earlier. On the other hand, our numerical study deals with the exact two-dimensional dynamics of the system along with the appropriate boundary conditions. Our present study reveals that analytical and numerical findings regarding the entropic memory erasure process support each other. This validates the dimension reduction approximation for the erasure of an entropic bit of information.

V. CONCLUSION

In conclusion, we have presented a comprehensive analysis about the work done and the free-energy change for an entropic memory erasure process. We have considered the states of a Brownian particle in two lobes of a bilobal enclosure as two binary digits. An ensemble of such system represents bits of information that are entropic in nature. The particles initially get thermalized with the bath, allowing both memory states to be occupied equally. An external bias drives all the particles selectively to a given lobe, which leads to erasure of a particular type of bits of information. This entropic memory erasure process is associated with irreversible logic operations involving entropic memory. The work done and the effective free-energy cost for this entropic memory erasure process are of significant importance. We have carried out a detailed numerical simulation on the work done and the free-energy change of the erasure protocol. All the work effect and the free-energy change for the erasure protocol arise as a nontrivial boundary effect. Then we suggest an analytical scheme that might be considered as a derivation of the Jarzynski equality for the system subjected to an entropic barrier in reduced dimension. The results can be summarized as follows:

(i) The numerical results suggest that we can recapture the Landauer limit in terms of free-energy change irrespective of the duration of the erasure cycle, whereas in terms of the average work done this limit is approached only for a very long erasure protocol.

(ii) The observations can be explained with the help of the detailed Jarzynski equality. The applicability of the general and the detailed Jarzynski equality to the combined process and individual subprocesses has also been demonstrated.

(iii) The average exponential of the work function for the subprocesses can be linked with the probability of residence at the corresponding lobe under the reverse protocol.

(iv) The analytical study presented in this paper suggests that the application of the detailed and the classical Jarzynski equality is justified for the confined system with irregular width because they appear through theoretical calculations also for such systems.

Our study demonstrates a close connection between logic operations and thermodynamically inspired quantities in the absence of any true energetic potential in physical systems where geometric constraints play the role of a major guiding factor.

ACKNOWLEDGMENTS

M.D. would like to thank Professor D. S. Ray for helpful discussions and a thorough reading of the manuscript. Thanks are due to the Council of Scientific & Industrial Research, Government of India, for partial financial support.

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