Two-point correlation function of an exclusion process with hole-dependent rates

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We consider an exclusion process on a ring in which a particle hops to an empty neighboring site with a rate that depends on the number of vacancies n in front of it. In the steady state, using the well-known mapping of this model to the zero-range process, we write down an exact formula for the partition function and the particle-particle correlation function in the canonical ensemble. In the thermodynamic limit, we find a simple analytical expression for the generating function of the correlation function. This result is applied to the hop rate $u(n) = 1 + (b/n)$ for which a phase transition between high-density laminar phase and low-density jammed phase occurs for $b > 2$. For these rates, we find that at the critical density, the correlation function decays algebraically with a continuously varying exponent $b - 2$. We also calculate the two-point correlation function above the critical density and find that the correlation length diverges with a critical exponent $\nu = 1/(b-2)$ for $b < 3$ and 1 for $b > 3$. These results are compared with those obtained using an exact series expansion for finite systems.

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I. INTRODUCTION

Nonequilibrium steady states, which are characterized by a lack of detailed balance, have the important property that they can exhibit phase transitions even in one dimension [1]. The condensation transition [2] is an important example of such a transition in which particles are distributed homogeneously over the system at low densities, but above a critical density, a macroscopic number of particles form a cluster. This transition has been studied using several different models, including aggregation-diffusion models and zero-range processes (ZRP) for homogeneous systems [2,3] and for systems with quenched disorder [4,5]. The ZRP, in which a particle hops to a neighboring site with a rate that depends only on the properties of the departure site, has the attractive feature that its steady-state distribution can be found exactly [6]. This model has been generalized in various directions in recent years [7–9] and has been used to model clustering phenomena in traffic flow [10], granular gases [11] and networks [12], avalanche dynamics in sandpiles [13], slow dynamics in glasses [14], and to understand phase separation in nonequilibrium systems [15].

The jamming transition is an avatar of the condensation transition and has been studied in an exclusion process (EP) in which a hard-core particle hops to an empty nearest neighbor with a rate that depends on the vacant sites in front of it. An exclusion process with hole-dependent rate mimics traffic flow on a highway where a driver increases (decreases) its speed if the headway in front of it is large (small) [16,17]. In such traffic models, the hop rate is an increasing function of the vacancies with appropriate lower and upper bounds on the speed of the car. However, it has been shown that if the hop rates decay sufficiently slowly with the increasing number of vacancies, as the total density of the system is decreased in a closed one-dimensional system, a transition occurs from a laminar phase with typical interparticle spacing of order unity to a jammed phase in which a macroscopic headway forms in front of a particle [2]. Since the EP with hole-dependent rates can be exactly mapped to a ZRP, a lot is known about its steady-state properties; however, a basic question regarding the spatial correlation functions in the EP has not been addressed in previous studies. In this article, we are interested in calculating the particle-particle correlation function in the steady state of this model in the canonical ensemble.

Analytical formulas for the two-point correlation functions are hard to come by. For the one-dimensional totally asymmetric simple exclusion process (TASEP) on a ring, which is a special case of the exclusion model studied here, this is trivial because all configurations are equally likely in the steady state. A nontrivial exact formula for the TASEP with open boundaries (entrance rate α and exit rate β) was given in Ref. [18] for arbitrary system size. Although the exact formula was determined in the latter case, their limiting behavior has not been calculated to the best of our knowledge, especially at the critical phase line $\alpha = \beta < \frac{1}{2}$. Recently, the particle-particle correlation function for the EP with hole-dependent rates was calculated in the laminar phase for certain special choices of hopping rates in the grand canonical ensemble [19]. Here we study the same model in the canonical ensemble and find a simple analytical formula for the generating function of the two-point correlation function with arbitrary hop rates in the thermodynamic limit. This result is applied to the hop rate $u(n) = 1 + (b/n)$, which decays to a nonzero constant with the number of holes n in front of the particle. For this choice, a jamming transition occurs when $b > 2$ at a critical density $\rho_c = (b-2)/(b-1)$ [2], and here we calculate the two-point correlation function at the critical point and in the laminar phase.

The plan of the article is as follows: in Sec. II, we define the model and briefly review its steady-state properties. In Sec. III, we focus on the canonical partition function and give a formula for it in terms of integer partitions. We then turn to a calculation of the steady-state particle-particle correlation function in the canonical ensemble in Sec. IV and obtain an exact expression for it for any system size. We then find an exact expression for the generating function of the correlation function in the thermodynamic limit. In Sec. V, for $u(n) = 1 + (b/n)$, we show that at the critical density, the correlation function decays as a power law with continuously varying exponent. The behavior of the correlation function in the laminar phase is also studied. We finally conclude with a summary of our results and discussion in Sec. VI.

II. MODEL AND ITS STEADY STATE

We consider an exclusion process defined on a ring with L lattice sites and N particles in which each site can be occupied by at most one particle. A particle hops to its right empty neighbor with a rate $u(n)$, where n is the number of holes in front of the particle. This EP can be mapped to a one-dimensional ZRP with periodic boundary conditions in which a site can support any number of particles and a particle hops to its left neighbor with a rate $u(n)$, where n now is the number of particles at the departure site [2]. As a result of this mapping, the density ρ in the EP with L sites and N particles is related to the density ϱ in ZRP with $l(=N)$ sites and $n(=L-N)$ particles as $\varrho = (1-\rho)/\rho$.

As we will be exploiting the connection of EP to ZRP in the following sections, we briefly review the steady-state properties of the ZRP below and refer the reader to Ref. [2] for details. The ZRP has the important property that the single site weights factorize. More precisely, the distribution of a configuration $C \equiv \{m_1,\ldots,m_l\}$, where m_i is the number of particles at the ith site and $\sum_i m_i = n$, is given by

$$P(C) = \tilde{Z}_{l,n}^{-1} \prod_{i=1}^{l} f(m_i), \tag{1}$$

where the single site weight $f(m)$ is

$$f(m) = (1-\delta_{m,0}) \prod_{i=1}^{m} \frac{1}{u(i)} + \delta_{m,0}, \tag{2}$$

and $\tilde{Z}_{l,n}$ is the partition function of the ZRP in the canonical ensemble given by

$$\tilde{Z}_{l,n} = \sum_{0 \leqslant m_1,\ldots,m_l \leqslant n} \prod_{i=1}^{l} f(m_i) \delta_{\sum_{k=1}^{l} m_k, n}. \tag{3}$$

In general, it is difficult to obtain results in the canonical ensemble (see, however, Refs. [17,20,21]), but the grand canonical partition function $\tilde{\mathcal{Z}}_l$ can be readily obtained. Using Eq. (3), we can write

$$\tilde{\mathcal{Z}}_l(z) = \sum_{n=0}^{\infty} \tilde{Z}_{l,n} z^n = g^l(z), \tag{4}$$

where $g(z)$ is the generating function of $f(m)$ defined as

$$g(z) = \sum_{m=0}^{\infty} z^m f(m), \tag{5}$$

with a radius of convergence z^*. The number distribution at a site is given by $p(m) = z^m f(m)/g(z)$, where the fugacity $z(\leqslant z^*)$ is determined by

$$\varrho = \frac{1}{\rho} - 1 = \frac{z}{l} \frac{\partial \ln \tilde{\mathcal{Z}}_l(z)}{\partial z} = z \frac{\partial \ln g(z)}{\partial z}. \tag{6}$$

The fugacity z is an increasing function of the ZRP density ϱ. But as it is bounded above, it may happen that z reaches its maximum value at a finite critical density ϱ_c. In such a case, the distribution $p(m) = z^{*m} f(m)/g(z^*)$ for all $\varrho \geqslant \varrho_c$. But this implies that the average density in the system is $l^{-1} \sum_{m=1}^{\infty} m p(m) = \varrho_c$. The excess mass $\varrho - \varrho_c$ is then said to be condensed into a single cluster.

In Sec. V, we will consider hop rates for which the jamming transition occurs. We will work with

$$u(n) = 1 + \frac{b}{n}, \quad b > 0, \tag{7}$$

where b is a constant and n is the number of vacant sites in front of a particle in the EP picture. When $b = 0$, we arrive at the TASEP on a ring in which a particle hops to the right empty site irrespective of the vacancies in front of it (see also Appendix A). For the above choice of hop rates, the weight $f(n)$ is given by

$$f(n) = \frac{n!}{(b+1)_n}, \tag{8}$$

where $(a)_n = a(a+1)\ldots(a+n-1)$ is the Pochhammer symbol or rising factorial. Its generating function $g(z)$ is writable as

$$g(z) = {}_2F_1(1,1;1+b;z), \tag{9}$$

where the Gauss hypergeometric function is defined as [22]

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}. \tag{10}$$

It is easy to see that the radius of convergence of $g(z)$ in Eq. (9) is $z^* = 1$. Then Eq. (6) and the discussion following it shows that a jamming transition occurs at the critical density

$$\varrho_c = \frac{1}{b-2} \tag{11}$$

for $b > 2$. Although we will focus on the rate Eq. (7), which models "attractive interactions" between particles in the ZRP, we also consider the case of free particles in Appendix B for which $u(n) = n$ [4]. In the latter case, each particle is endowed with an exponential clock that ticks at rate one, but since the particles are free and act independently, the total hopout rate is equal to the number of particles at the site.

III. PARTITION FUNCTION IN THE CANONICAL ENSEMBLE

A. Exact recursions

Consider a system of N particles on L sites. If τ is a configuration in this system, let $W(\tau)$ denote the stationary weight of such a configuration. Let $Z_{L,N}$ denote the partition function of the EP in this system. That is to say,

$$Z_{L,N} = \sum_{\tau} W(\tau).$$

We will first give two different recurrence relations for $Z_{L,N}$.

Note that any configuration can be written in the form $0^{k_0} \tau 1 0^{k_N}$, where τ is a configuration in the system with $L - k_0 - k_N - 1$ sites and $N-1$ particles. Since we want this representation to be unique, τ_1 has to be 1. Thus,

$$\begin{aligned} Z_{L,N} &= \sum_{0 \leqslant k_0 + k_N \leqslant L-N} \sum_{\tau} W(0^{k_0} \tau 1 0^{k_N}) \\ &= \sum_{0 \leqslant k_0 + k_N \leqslant L-N} \sum_{\tau} f(k_0 + k_N) W(\tau) \\ &= \sum_{k=0}^{L-N} (k+1) f(k) \sum_{\tau} W(\tau), \end{aligned}$$

where we set $k = k_0 + k_N$ in the last line and the factor of $k+1$ counts for the number of ways one can split k in this way. The sum over τ now gives the partition function for a system with $L-k-1$ sites and $N-1$ particles where the first site is occupied. Since the system is translation-invariant, this gives the formula

$$Z_{L,N} = \sum_{k=0}^{L-N} f(k)(k+1)\frac{N-1}{L-k-1} Z_{L-k-1,N-1}. \quad (12)$$

Another recurrence relation for the ZRP partition function $\tilde{Z}_{l,n}$ with l sites and n particles has been obtained [21] and is given by

$$\tilde{Z}_{l,n} = \sum_{k=0}^{n} f(k)\tilde{Z}_{l-1,n-k}, \quad (13)$$

with $\tilde{Z}_{0,n} = \delta_{n,0}$ since $\tilde{Z}_{1,n} = f(n)$. Since a ZRP can be mapped to EP by regarding the N particles in EP as l sites in ZRP and $L-N$ holes in EP as n particles in ZRP, the two partition functions can be related as

$$Z_{L,N} = \frac{L}{N}\tilde{Z}_{N,L-N}. \quad (14)$$

The prefactor on the right-hand side of the above equation arises due to the fact that the mapping described above between ZRP and EP assumes that an EP configuration begins with an occupied site. The EP configurations that begin with an empty site are taken care of by the factor L/N using the argument described above.

Therefore, on using the last two equations, we get

$$Z_{L,N} = \frac{L}{N}\sum_{k=0}^{L-N} f(k)\frac{N-1}{L-k-1} Z_{L-k-1,N-1}. \quad (15)$$

We have thus shown that both recurrence relation Eqs. (12) and (15) with the initial conditions $Z_{L,0} = Z_{L,L} = 1$ give rise to the same formula. Although we have proved this result, we have no deeper understanding of this equivalence.

B. Exact formula for the partition function

It turns out that one can express $Z_{L,N}$ exactly using integer partitions. To state the result, we need some definitions. An integer partition of a positive integer n is a representation of n in terms of other positive integers which sum up to n. For convenience, the parts are written in weakly decreasing order. For example, (5,3,3,2,1) is a partition of 14. If λ is a partition of n, we denote this as $\lambda \vdash n$.

Another way of expressing a partition is in the so-called frequency representation, $1^{a_1}2^{a_2}\ldots$, where a_i represents the multiplicity of i in the partition. This information can be encoded as a vector $\bar{a} = (a_1,a_2,\ldots)$. For example, the same partition of 14 above can be written as $1^1 2^1 3^2 4^0 5^1 \equiv (1,1,2,0,1)$ followed by an infinite string of zeros, which we omit. We will write $\bar{a} \vdash n$ to mean a partition of n in this notation.

The number of parts of a partition, denoted by $|\bar{a}|$, is given by $\sum_i a_i$. Given a function f defined on the positive integers, we will denote

$$f(\bar{a}) = f(1)^{a_1} f(2)^{a_2} \ldots .$$

In the same vein, let $\bar{a}! = \prod_i a_i!$. Finally, recall that the Pochhammer symbol or rising factorial $(m)_n$ for nonnegative integer n, is given by the product $m(m+1)\cdots(m+n-1)$ if n is positive and by m if $n=0$.

The partition function of the EP can be written as

$$Z_{N+M,N} = (N+M)\sum_{\bar{a}\vdash M}\frac{(N-|\bar{a}|+1)_{|\bar{a}|-1}}{\bar{a}!} f(\bar{a}), \quad (16)$$

where the length of the system is $L = N+M$ and $(m)_n$ is the Pochhammer symbol defined after Eq. (8). We will prove this by equating both representation Eqs. (12) and (15). Doing so for $Z_{N+M,N}$ shows

$$\sum_{k=0}^{M}\frac{(N-1)f(k)}{N+M-k-1} Z_{N+M-k-1,N-1}\left(\frac{M-Nk}{N}\right) = 0.$$

Isolating the $k=0$ term and replacing $N-1$ by N gives a recurrence

$$\frac{M}{N+M} Z_{N+M,N} = \sum_{k=1}^{M}\frac{(N+1)k-M}{N+M-k} f(k)\, Z_{N+M-k,N}.$$

Define $\hat{Z}_{N+M,N} = \frac{Z_{N+M,N}}{N+M}$ to get a recurrence for $\hat{Z}$'s,

$$\hat{Z}_{N+M,N} = \sum_{k=1}^{M}\frac{(N+1)k-M}{M} f(k)\, \hat{Z}_{N+M-k,N}. \quad (17)$$

We will now prove the formula for $\hat{Z}_{N+M,N}$ equivalent to Eq. (16) by induction on M. When $M=1$, there is a single term in the sum corresponding to $\bar{a} = (1,0,\ldots)$. Thus, $\hat{Z}_{N+1,N} = f(1)$. This is correct since there is a single vacancy and a factor of $f(1)$ for the particle preceding it.

Now, we assume that Eq. (16) is true for the number of vacancies being any of $1,\ldots,M-1$. Using Eq. (17) and the induction assumption, we can write

$$\hat{Z}_{M+N,N} = \sum_{k=1}^{M}\frac{(N+1)k-M}{N+M-k} f(k) \times \sum_{\bar{a}\vdash M-k}\frac{(N-|\bar{a}|+1)_{|\bar{a}|-1}}{\bar{a}!} f(\bar{a}). \quad (18)$$

Notice that each term in the above equation contains the factor $f(k)f(\bar{a})$ where $a \vdash M-k$. We can thus replace $\bar{a}$ in the sum by $\overline{a'}$, where $\overline{a'} = \bar{a} \oplus (k)$. Then $f(k)f(\bar{a})$ can be replaced by $f(\overline{a'})$. Therefore, the sum above can be reinterpreted as a sum over partitions of M. We have to compute the coefficient of $f(\overline{a'})$ in such a term.

Suppose $\overline{a'}$ can be written as $(i_1^{a'_1},\ldots,i_j^{a'_j})$, where each $a'_k \neq 0$. Since there are j distinct parts in $\overline{a'}$, we can express $\overline{a'} = (i_k) \oplus \overline{a'}_k$, where $\overline{a'}_k = (i_1^{a'_1},\ldots,i_k^{a'_k-1},\ldots i_j^{a'_j})$ for $k = 1,\ldots,j$. There are, thus, exactly j terms that contribute to the partition $\overline{a'}$. Note that

$$|\overline{a'_k}| = |\overline{a'}| - 1, \quad f(\overline{a'}) = f(\overline{a'_k})f(i_k) \quad \text{and} \quad \overline{a'}! = \overline{a'_k}!\, a'_{i_k}.$$

The terms contributing to $\overline{a'}$ are

$$\sum_{k=1}^{j} \frac{(N+1)i_k - M}{M} f(i_k) \frac{(N-|\overline{a'_k}|+1)_{|\overline{a'_k}|-1}}{\overline{a'}_k!} f(\overline{a'_k})$$

$$= \sum_{k=1}^{j} \frac{(N+1)i_k - M}{M} \frac{(N-|\overline{a'}|+2)_{|\overline{a'}|-2}}{\overline{a'}!} a'_{i_k} \; f(\overline{a'})$$

$$= \frac{(N-|\overline{a'}|+2)_{|\overline{a'}|-2}}{\overline{a'}!} f(\overline{a'}) \sum_{k=1}^{j} \frac{(N+1)i_k a'_{i_k} - M a'_{i_k}}{M}$$

$$= \frac{(N-|\overline{a'}|+2)_{|\overline{a'}|-2}}{\overline{a'}!} f(\overline{a'})(N+1-|\overline{a'}|)$$

$$= \frac{(N-|\overline{a'}|+1)_{|\overline{a'}|-1}}{\overline{a'}!} f(\overline{a'}).$$

This is precisely what we wanted to show.

IV. TWO-POINT CORRELATION FUNCTION IN CANONICAL ENSEMBLE

A. Exact formula for finite system

We wish to calculate the two-point connected correlation function

$$C(r) = \langle n_i n_{i+r}\rangle - \rho^2, \quad r > 0 \tag{19}$$

in a system of L sites with N particles. Let us consider a set of configurations in which the r sites from i to $i+r-1$ contain k holes. Then the contribution to the correlation function $\langle n_i n_{i+r}\rangle$ comes from only those configurations in which both the ith and $(i+r)$th site are occupied. Using the mapping between EP and ZRP described in Sec. II and summing over all the particle configurations in front of the ith and $(i+r)$th particle, we get

$$\langle n_i n_{i+r}\rangle = \sum_{k=k_{\min}}^{k_{\max}} \frac{\tilde{Z}_{r-k,k} Z_{N-r+k,L-N-k}}{Z_{L,N}}, \tag{20}$$

$$= \rho \sum_{k=k_{\min}}^{k_{\max}} \frac{\tilde{Z}_{r-k,k} \tilde{Z}_{N-r+k,L-N-k}}{\tilde{Z}_{N,L-N}}, \tag{21}$$

where we have used Eq. (14) to arrive at the last expression. As the total number of particles is conserved, the maximum number of particles in the first cluster can be $N-1$. In other words, $r-k \leqslant N-1$, which gives $k_{\min} = \max(0, r-N+1)$, as the lower limit $k_{\min}$ can not be below zero. Also the local conservation in the first cluster with r sites requires that $k \leqslant r-1$. Thus we find that $k_{\max} = \min(L-N, r-1)$, since $k_{\max}$ cannot exceed the total number of holes in the system.

B. Exact expression for infinitely large system

It is evident from Eq. (21) that the partition function at all densities is required to evaluate the correlation function. However, barring some special cases that are discussed in Appendices A and B, it does not seem possible to calculate the exact partition function $\tilde{Z}_{l,n}$ for all densities. In the following subsections, we will calculate the two-point correlation function in the thermodynamic limit as the problem

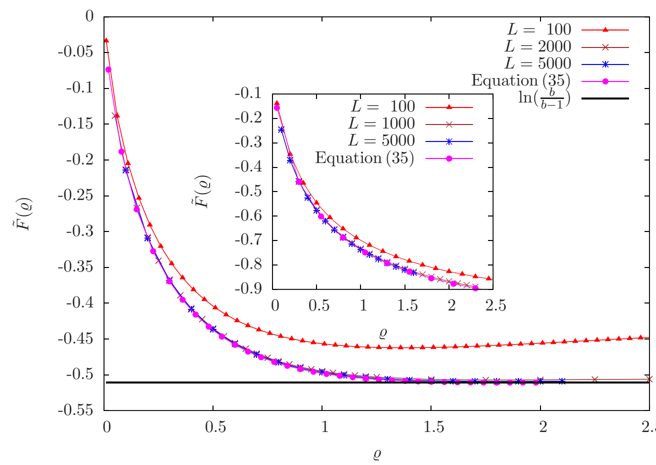

FIG. 1. (Color online) Free energy $\tilde{F}(\varrho)$ as a function of density ϱ for $b = 3/2$ (inset) and $b = 5/2$ (main) for different system sizes. The data for finite-sized systems is obtained by numerically solving the recursion Eq. (13) and is compared with the result Eq. (35) for infinitely large system.

is analytically tractable in this limit. For $L \to \infty$ and finite r, we first note that the limits in the sum appearing in Eq. (21) simplify to $k_{\min} = 0$ and $k_{\max} = r-1$. Furthermore, inspired by equilibrium statistical mechanics, we conjecture that there exists a "free energy" $\tilde{F}(\varrho)$ defined as

$$\tilde{F}(\varrho) = -\lim_{l\to\infty} \frac{\ln \tilde{Z}_{l,n}}{l}. \tag{22}$$

For the hop rate Eq. (7), using the recursion Eq. (13), we calculated the partition function $\tilde{Z}_{l,n}$ as a function of density for various system sizes. Figure 1 shows that the scaled logarithmic partition function indeed approaches a limiting function with increasing system size.

Thus, for large L, using Eq. (22), we can write [23]

$$\ln\left(\frac{\tilde{Z}_{N-r+k,L-N-k}}{\tilde{Z}_{N,L-N}}\right) = k\mu - (r-k)P, \tag{23}$$

where the chemical potential μ and the pressure P are given by

$$\mu = \left.\frac{\partial(l\tilde{F})}{\partial n}\right|_l = \tilde{F}'(\varrho), \tag{24a}$$

$$P = -\left.\frac{\partial(l\tilde{F})}{\partial l}\right|_n = -\tilde{F}(\varrho) + \varrho\tilde{F}'(\varrho), \tag{24b}$$

and the prime stands for derivative with respect to ϱ. Using Eq. (23) in Eq. (21) for correlation function $\langle n_i n_{i+r}\rangle$ and the boundary condition $\tilde{Z}_{0,n} = \delta_{n,0}$ [refer to the discussion after Eq. (13)], we get

$$\langle n_i n_{i+r}\rangle = \rho e^{-rP} \sum_{k=0}^{r} \tilde{Z}_{r-k,k} e^{k(\mu+P)}, \quad r \geqslant 0, \tag{25}$$

$$= \rho e^{r\mu} \sum_{k=0}^{r} \tilde{Z}_{k,r-k} e^{-k(\mu+P)}, \tag{26}$$

$$= \rho e^{r\mu} \sum_{k=0}^{r} \frac{k}{r} Z_{r,k} e^{-k(\mu+P)}, \tag{27}$$

$$= -\frac{\rho}{r} e^{r\mu} \frac{d}{d(\mu+P)} \sum_{k=0}^{r} Z_{r,k} e^{-k(\mu+P)}. \tag{28}$$

Thus, the correlation function is related to the *grand canonical* partition function of the EP with r sites, which is not known.

However, as explained in Sec. II, the grand canonical partition function for ZRP is known. We therefore define the generating function of the correlation function as $G(y) = \sum_{r=0}^{\infty} y^r C(r)$, which, on using Eq. (26), works out to be

$$G(y) = \rho \sum_{l=0}^{\infty} (ye^{-P})^l \sum_{n=0}^{\infty} \tilde{Z}_{l,n} (ye^{\mu})^n - \frac{\rho^2}{1-y}, \tag{29}$$

$$= \frac{\rho}{1 - ye^{-P} g(yz)} - \frac{\rho^2}{1-y}, \tag{30}$$

where $z = e^{\mu}$. Furthermore, we recall that the equation of state in the grand canonical ensemble is given by [23]

$$Pl = \ln(\tilde{\mathcal{Z}}_l(z)) = l \ln g(z), \tag{31}$$

which thus gives $e^{-P} = 1/g(z)$. Thus, we arrive at our main result, namely

$$G(y) = \frac{\rho g(z)}{g(z) - yg(yz)} - \frac{\rho^2}{1-y}, \tag{32}$$

where the fugacity $z(\rho)$ is determined by Eq. (6). The correlation function is then given by

$$C(r) = \frac{1}{r!} \frac{d^r G(y)}{dy^r}\bigg|_{y=0}, \tag{33}$$

$$= \oint_C \frac{dy}{2\pi i} \frac{G(y)}{y^{r+1}}, \tag{34}$$

where the integral in the last expression is along the closed curve C around the origin [24]. We check that $C(0)=\rho(1-\rho)$ is obtained from the above expression. The behavior at $r \to \infty$ is obtained by taking the limit $y \to 1$. Expanding Eq. (32) close to $y = 1$ and using Eq. (6), we see that $G(y)$ [and hence $C(r)$] vanishes as $r \to \infty$.

Before proceeding further, we note that due to Eqs. (24) and (31), the free energy can be written as

$$\tilde{F}(\varrho) = \varrho\mu - \ln g(z). \tag{35}$$

This expression is also plotted in Fig. 1 for hop rate Eq. (7) with $b = 3/2$ and $5/2$, and we see that it matches well with the results for large systems. We note that $\tilde{F}(\varrho)$ is a decreasing function of the density for $b < 2$, but it saturates to $-\ln g(1)$ at high density for $b > 2$.

V. CORRELATION FUNCTION FOR HOP RATE EQ. (7)

We now apply the general result Eq. (32) for the generating function of the correlation function to the choice Eq. (7) of the hop rates. The correlation function can be easily obtained numerically from Eq. (32) for an infinitely large system, and these results are shown along with those obtained using the exact result Eq. (21) for a finite system in Figs. 2–4, and we

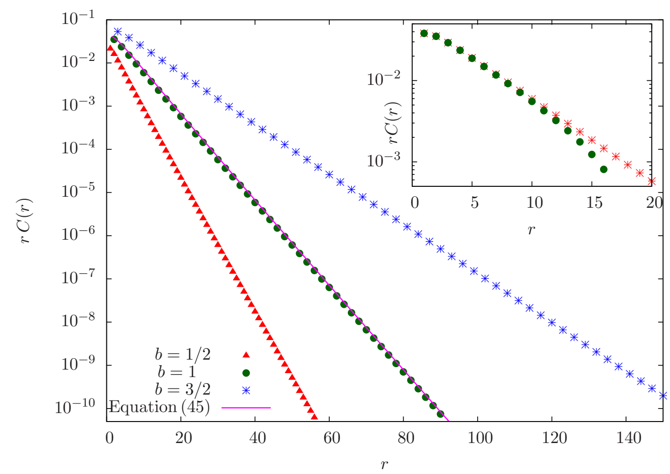

FIG. 2. (Color online) Decay of the correlation function in the laminar phase at $\rho = 0.4$ for various $b < 2$ in the infinite system. The analytical result Eq. (45) for $b = 1$ is also shown. The inset compares the correlation function for $b = 1$ obtained using Eq. (21) for $L = 10^4$ and Eq. (32) for infinite system.

see that the latter approaches the result obtained from Eq. (32) with increasing system size. In the following subsections, we obtain analytical results for $C(r)$ using Eq. (32).

A. Laminar phase: $0 < b < 2$

When $b = 0$, we obtain the well-known TASEP [6] on a ring for which the steady state is known exactly. This case is discussed briefly in Appendix A using Eq. (32). For $b = 1$, the generating function $g(z)$ given by Eq. (9) takes a particularly

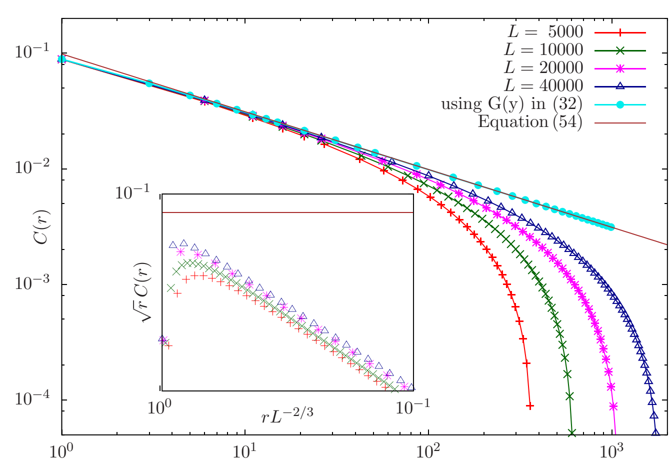

FIG. 3. (Color online) Decay of spatial correlation function with distance at the critical density for $b = 5/2$. The data for finite-sized systems is obtained by numerically solving Eqs. (13) and (21), while the result in the thermodynamic limit is obtained using Eq. (32). The analytical result Eq. (54) valid for large interparticle distances is also shown. The inset shows the data collapse of $C(r,L)$ for different system sizes using Eq. (65).

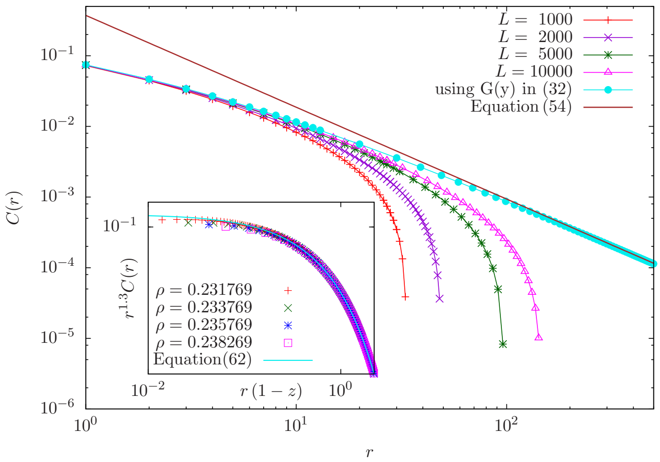

FIG. 4. (Color online) Decay of spatial correlation function with distance at the critical density for $b = 3.3$. The data for finite-sized systems is obtained by numerically solving Eqs. (13) and (21), while the result in the thermodynamic limit is obtained using Eq. (32). The analytical result Eq. (54) valid for large interparticle distances is also shown. The inset shows the data collapse of the correlation function for different densities close to the critical point in the laminar phase for infinite system using Eq. (62).

simple form:

$$g(z) = -\frac{\ln(1-z)}{z}. \tag{36}$$

Therefore, from Eq. (32), we get the generating function of the correlation function as

$$G(y) = \frac{\rho \ln(1-z)}{\ln(y_0 - 1) - \ln(y_0 - y)} - \frac{\rho^2}{1-y}, \tag{37}$$

where $y_0 = 1/z > 1$. The density-fugacity relation Eq. (6) is given by

$$\rho = -\frac{1-z}{z}\ln(1-z), \quad z < 1. \tag{38}$$

To calculate the correlation function, we consider the following integral in the complex-y plane along a closed contour C' wrapped around the branch cut at y_0, which consists of a large circle of radius R about the origin and a small circle of radius ϵ about y_0:

$$I_1 = \frac{1}{2\pi i}\oint_{C'} \frac{dy}{y^{r+1}} \frac{\rho \ln(1-z)}{\ln(y_0 - 1) - \ln(y_0 - y)}. \tag{39}$$

As the integrand has a simple pole at $y = 1$ and poles of order $r + 1$ at $y = 0$, due to Eq. (34), the residue at these poles immediately gives $C(r)$. It is easy to check that the contribution from the integrals over the large and the small circle vanishes when $R \to \infty$ and $\epsilon \to 0$. Since the integrand in Eq. (39) also has a branch cut singularity at $y = y_0$, we finally obtain

$$C(r) = \frac{1}{2\pi i}\left[\int_{AB} \frac{dy}{y^{r+1}} \frac{\rho \ln(1-z)}{\ln(y_0 - 1) - \ln(y_0 - y)} + \int_{B'A'} \frac{dy}{y^{r+1}} \frac{\rho \ln(1-z)}{\ln(y_0 - 1) - \ln(y_0 - y)}\right], \tag{40}$$

where $y - y_0 = x$ along AB and $y - y_0 = xe^{i2\pi}$ along $B'A'$. Since the correlation function is real, writing $-x = xe^{-i\pi}$, we get

$$C(r) = \frac{\rho \ln(1-z)}{2\pi i}\int_0^\infty \frac{dx}{(y_0 + x)^{r+1}}\left[\frac{1}{\ln(y_0 - 1) - \ln x + i\pi} - \frac{1}{\ln(y_0 - 1) - \ln x - i\pi}\right] \tag{41}$$

$$= \frac{-\rho \ln(1-z)(1-z)}{y_0^r}\int_0^\infty \frac{dx}{(1 + x(1-z))^{r+1}} \frac{1}{(\ln x)^2 + \pi^2}. \tag{42}$$

We are not able to perform the above integral exactly. But an approximate expression can be found for large r as follows:

$$C(r) \approx \frac{-\rho \ln(1-z)(1-z)}{y_0^r}\int_0^\infty dx \frac{e^{-rx(1-z)}}{(\ln x)^2 + \pi^2}, \tag{43}$$

$$\approx \frac{-\rho \ln(1-z)(1-z)}{y_0^r}\int_0^{\frac{1}{r(1-z)}} \frac{dx}{(\ln x)^2 + \pi^2}, \tag{44}$$

$$\approx \frac{\rho|\ln(1-z)|}{r} \frac{e^{-r|\ln z|}}{[\ln(r(1-z))]^2 + \pi^2}, \tag{45}$$

where the last expression is obtained after an integration by parts and the fugacity is determined in terms of density from Eq. (38). The last result is plotted against that obtained by solving Eq. (32) numerically, and we see an excellent agreement. Like the $b = 1$ case, in general for $0 < b < 2$, the correlation function shows an exponential decay (with power law correction), as can be seen in Fig. 2.

B. At the critical density: $b > 2$

We now calculate the correlation function at the critical density Eq. (11) using Eq. (32). At the critical density ϱ_c, as the fugacity $z = 1$, we get

$$G(y) = \frac{\rho_c g(1)}{g(1) - yg(y)} - \frac{\rho_c^2}{1-y}. \tag{46}$$

We first consider the case when b is not an integer. For large r, we can expand $g(y)$ given by Eq. (9) about $y = 1$. Using Eq. (15.3.6) of Ref. [22], we obtain

$$g(y) = g(1) - sg'(1) + \frac{s^2}{2!}g''(1) + \cdots + \frac{(-s)^n}{n!}g^{(n)}(1) + \alpha s^{b-1} + \mathcal{O}(s^b), \tag{47}$$

where $s = 1 - y$. Here we have retained analytic terms in the Taylor series expansion up to nth order where n is the integer part of $b - 1$ and the leading nonanalytic term. In the above expression, $\alpha = b\pi \csc(b\pi)$ and $g(1) = b/(b-1)$. Then we have

$$G(s) = \int_0^\infty dr e^{-sr} C(r), \tag{48}$$

$$= \frac{\rho_c}{\frac{s}{\rho_c} + c_2 s^2 + \cdots + c_n s^n - \frac{\alpha s^{b-1}}{g(1)}} - \frac{\rho_c^2}{s}, \tag{49}$$

$$= \frac{\rho_c^2}{s}\left[\frac{\alpha \rho_c s^{b-2}}{g(1)} - \rho_c(c_2 s + \cdots + c_n s^{n-1})\right], \tag{50}$$

where the coefficients c_i are writable in terms of the derivatives of $g(z)$ evaluated at $z = 1$. Since $G(s)$ has a branch cut singularity at $s = 0$, its inverse Laplace transform is given by [24]

$$C(r) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds e^{sr} G(s), \tag{51}$$

$$= -\frac{\rho_c^3}{2\pi i}\int_{-i\infty}^{i\infty} ds e^{sr}\left(c_2 + c_3 s + \cdots + c_n s^{n-2} - \frac{\alpha s^{b-3}}{g(1)}\right). \tag{52}$$

An integral similar to the one above also appears in the calculation of the canonical partition function of the ZRP [20] and we can use those results here. In the above expression, the first integral is $\delta(r)$ and all the integrals (barring the last one) are the derivatives of the δ function. Therefore, for large r, these integrals vanish, and we are left with

$$C(r) = \frac{\alpha\rho_c^3}{g(1)}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i} e^{sr} s^{b-3}. \tag{53}$$

The above integral can be obtained from the integral I_2 calculated in Appendix C by setting $c = t = 0$, and we obtain

$$C(r) = \frac{\rho_c^2\Gamma(b-1)}{r^{b-2}}. \tag{54}$$

This result is compared against that obtained using Eq. (32), and we see an excellent match at large r.

When b is an integer, as before, we expand $g(y)$ about $y = 1$ and using Eq. (15.3.11) of Ref. [22], we obtain

$$g(1-s) = g(1) - s g'(1) + \frac{s^2}{2!} g''(1) + \cdots + \frac{(-s)^n}{n!} g^{(n)}(1) + \beta s^{b-1}\ln s, \tag{55}$$

where $\beta = (-1)^b b$. Following the same steps as described above, we get

$$C(r) = \frac{\beta\rho_c^3}{g(1)}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i} e^{sr} s^{b-3}\ln s, \tag{56}$$

$$= \frac{\rho_c^3\beta}{g(1) r^{b-2}}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i} e^{s} s^{b-3}(\ln s - \ln r), \tag{57}$$

$$= \frac{\rho_c^3\beta}{g(1) r^{b-2}}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i} e^{s} s^{b-3}\ln s, \tag{58}$$

where we have used that b is an integer to arrive at the last equation. As the above integrand has a branch cut at $s = 0$, proceeding in a manner similar to that described in Appendix C with $c = t = 0$, we find the above integral to be $(-1)^{b-2}\Gamma(b-2)$, which shows that Eq. (54) is valid for integer b as well.

C. Above the critical density: $b > 2$

We now consider the behavior of the correlation function in the laminar phase at a density close to the critical point. Since the fugacity is below one here, we write $t = 1 - z$ and expand Eq. (6) about $z = 1$ to find the relationship between t and ρ. We find that

$$\frac{1}{\rho} = \begin{cases} \frac{1}{\rho_c} - \frac{\alpha(b-1)g'(1)}{g^2(1)} t^{b-2}, & 2 < b < 3 \\ \frac{1}{\rho_c} + \frac{g'(1)}{g(1)}\left(\frac{g'(1)}{g(1)} - \frac{g''(1)}{g'(1)} - 1\right)t, & b > 3. \end{cases} \tag{59}$$

The next-order corrections to the above expression can also be worked out, and turn out to be of the order t for $2 < b < 3$, t^{b-2} for $3 < b < 4$ and t^2 for $b > 4$.

For large distances and densities close to the critical density, we now expand the generating function $G(y,z)$ in Eq. (32) about $y = 1$ and $z = 1$. For $b > 3$, on using Eq. (59), we obtain

$$G(s,t) = \frac{\alpha\rho^3}{g(1)}\frac{(s+t)^{b-1} - t^{b-1}}{s^2}, \tag{60}$$

where, as before, $s = 1 - y$ and we have dropped the analytic terms as they do not contribute to $C(r,z)$ for the same reasons as described in the last subsection. We then have

$$C(r,z) = \frac{\alpha\rho^3}{g(1)}\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i} e^{sr}\frac{(s+t)^{b-1} - t^{b-1}}{s^2}, \tag{61}$$

where c is a positive real number. The above integral is calculated in Appendix C, and we find that in the limit $t = 1 - z \to 0, r \to \infty$ with rt finite, the correlation function is of the following scaling form:

$$C(r,z) = r^{2-b}\mathcal{H}[r(1-z)], \tag{62}$$

where the scaling function

$$\mathcal{H}(x) = (b-2)\Gamma(b-1)\rho_c^2 e^{-x}[(x+b-1)e^x E_{b-1}(x) - 1] \tag{63}$$

is a decreasing function of x. In the above expression, $E_n(x) = \int_1^\infty dt e^{-xt} t^{-n}$ is the exponential integral. By carrying out a calculation similar to above, it can be checked that the results Eqs. (62) and (63) hold for $2 < b < 3$ and integer b as well. The inset of Fig. 4 shows the data collapse for the correlation function for various densities close to the critical point and the scaling function.

Using the asymptotic properties of the exponential integral $E_n(x)$ [22], we find that the scaling function $\mathcal{H}(x) \overset{x\to 0}{\sim} \Gamma(b-1)\rho_c^2$, which thus reproduces the result at the critical point obtained in the last subsection. At large x, as the scaling function $\mathcal{H}(x) \overset{x\to\infty}{\sim} \Gamma(b)(b-2)\rho_c^2 e^{-x}/x^2$, the correlation function decays exponentially fast with interparticle distance r. This analysis yields the correlation length defined by $x = r/\xi$ to be

$$\xi \sim (1-z)^{-1} \sim (\rho - \rho_c)^{-\nu}, \tag{64}$$

which, by virtue of Eq. (59), gives $\nu = 1/(b-2)$ for $b < 3$ and 1 for $b > 3$.

VI. DISCUSSION

In this article, we studied an exclusion process on a ring in which a particle hops to a right empty neighbor with a rate that depends on the number of empty sites in front of it. Although we assumed that the hops are totally asymmetric, the results obtained here hold for the general case also in which a particle may hop to either left or right empty neighbor with nonzero rate. This is because the general exclusion model maps to a

ZRP whose partition function is independent of the bias in the hop rates [2]. Then our exact Eq. (21) for the correlation function, which holds for any bias in the hop rates, gives the same solution as obtained in the previous sections.

Although most of the results for the ZRP and hence the exclusion process have been obtained in the grand canonical ensemble [2], some studies in the canonical ensemble have also been carried out [17,20,21,25]. In particular, an expression for the partition function $\tilde{Z}_{l,n}$ in the canonical ensemble at and in the vicinity of the critical point has been calculated for finite systems [20], and it has been shown that for the weight $f(m)$ with the same asymptotic behavior as Eq. (9), $\tilde{Z}_{l,n}$ depends exponentially on system size for $\varrho < \varrho_c$, but sublinearly on l for $\varrho \geqslant \varrho_c$. This implies that the free energy Eq. (22) changes with the density ϱ in the homogeneous phase but becomes a constant equal to $-\ln g(1)$ (which is chosen to be zero in Ref. [20]) for all $\varrho \geqslant \varrho_c$, as seen here in Fig. 1. Due to the latter property, our analysis cannot be carried over to the jammed phase. However, since we are mainly concerned with critical exponents here, it suffices to consider the system in the infinite size limit.

For an infinitely large system, we have derived an exact expression Eq. (32) for the generating function of the steady-state two-point correlation function in the canonical ensemble. This result was applied to the hop rate Eq. (7) for $\rho \geqslant \rho_c$ to find the relevant critical exponents. Interestingly, we find that at the critical point, the exponent characterizing the power law decay of the two-point correlation function changes continuously with the parameter b in the hop rate Eq. (7). Equilibrium systems in two dimensions that show continuously varying exponents at the critical point are known [26], and their behavior is understood in terms of conformal field theories with central charge one [27]. We do not know if the behavior found here has any such deeper significance. The correlation length exponent ν in Eq. (64) also changes continuously for $2 < b < 3$, whereas it is constant for $b > 3$. This scaling for the correlation length has been obtained in a previous work [28] as well. In addition, we have also derived the scaling function for the correlation function in the high-density phase here. The case of $b = 1$, where the system is in laminar phase for all densities, has been considered in Ref. [19], but an explicit expression for the correlation function was not provided.

From the numerical data shown in Figs. 3 and 4 at the critical point, we note that the finite size effects set in early on. For example, in Fig. 3 for $b = 5/2$ and a system size $L = 10^4$, a power law is seen for about a decade only. This makes a numerical determination of the correlation function exponent difficult. Here we have given an expression, Eq. (32), for the generating function of the two-point correlation function for an infinite system, which can easily generate several decades of data. For a finite system with L sites, we expect the correlation function to be of the following scaling form:

$$C(r,L) = \frac{1}{r^{b-2}} \mathcal{F}(r L^{-z}), \tag{65}$$

where the scaling function $\mathcal{F}(x)$ is a constant for $x \ll 1$ and decays for $x \gg 1$. In the ZRP, the average mass cluster at the critical point scales as $l^{1/(b-1)}$, $b < 3$, and $\sqrt{l}$, $b > 3$ [20]. If we make the reasonable assumption that at the critical density there is a single length scale in the system under consideration and is set by the typical headway, we expect $z = 1/(b-1)$ for $b < 3$. This expectation is consistent with the data shown in the inset of Fig. 3 for $b = 5/2$, where we see that the data collapse gets better with increasing L.

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APPENDIX A: SIMPLE EXCLUSION PROCESS

For $u(n) = 1, n > 0$, as all configurations are equally likely [29], the steady-state partition function is given by $Z_{L,N} = \binom{L}{N}$. The two-point correlation function $C(r) = \binom{L-2}{N-2}/\binom{L}{N} = N(N-1)/[L(L-1)], r > 0$ vanishes in the limit $L \to \infty$. It can be easily checked that Eq. (21) also gives this result. For large systems, the free energy defined in Eq. (22) works out to be

$$\tilde{F}(\varrho) = (1+\varrho)\ln(1+\varrho) - \varrho \ln \varrho, \tag{A1}$$

which is an increasing function of the density ϱ. Furthermore, since $f(m) = 1$, we have $g(z) = 1/(1-z)$, and therefore

$$G(y) = \rho \frac{1-(1-\rho)y}{1-y} - \frac{\rho^2}{1-y}, \tag{A2}$$

which immediately yields $C(r) = 0, r > 0$, as expected in the thermodynamic limit.

APPENDIX B: FREE PARTICLE CASE

When the particles jump independently (in the ZRP picture), the hop-out rate is proportional to the number of particles at the site, $u(n) = n$. Therefore, from Eq. (3), the ZRP partition function is easily seen to be

$$\tilde{Z}_{l,n} = \frac{l^n}{n!}.$$

Using this in Eq. (21), we obtain the exact expression for the two-point correlation function as

$$\langle n_i n_{i+r} \rangle = \frac{\rho}{N^{L-N}} \sum_{k=k_{\min}}^{k_{\max}} \frac{(r-k)^k}{k!} (N - r + k)^{L-N-k} \times \frac{(L-N)!}{(L-N-k)!}. \tag{B1}$$

In the thermodynamic limit, the above expression gives

$$\langle n_i n_{i+r} \rangle = \rho \sum_{k=0}^{r-1} \frac{(r-k)^k}{k!} \varrho^k e^{-(r-k)\varrho}. \tag{B2}$$

For this case, we have $g(z) = e^z$ and $z = \varrho$. As a result, Eq. (32) gives

$$G(y) = \frac{\rho}{1 - y e^{z(y-1)}} - \frac{\rho^2}{1-y}. \tag{B3}$$

It can be checked that the correlation function in Eq. (B2) matches that obtained from the series expansion of Eq. (B3).

To obtain an explicit expression for the correlation function $C(r)$, we use the Euler-Maclaurin formula given by [30]

$$\sum_{k=0}^{r} f(k) \approx \int_0^r dx f(x) + \frac{1}{2}[f(0)+f(r)] - \int_0^r dx f'(x) \sum_{j=1}^{\infty} \frac{\sin(2j\pi x)}{\pi j}, \quad \text{(B4)}$$

$$= \int_0^r dx f(x) + \frac{f(0)}{2} + 2\sum_{j=1}^{\infty} \int_0^r dx \cos(2j\pi x) f(x), \quad \text{(B5)}$$

$$= \int_0^r dx f(x) + \frac{f(0)}{2} + 2\sum_{j=1}^{\infty} \mathrm{Re}\left[\int_0^r dx e^{i2j\pi x} f(x)\right], \quad \text{(B6)}$$

where $f(k)$ is the summand in Eq. (B2). Our main task is to calculate the integral on the right-hand side of the last equation, which can be carried out using the saddle point method for large r. We find that

$$\int_0^r dx e^{i2j\pi x} f(x) \approx \frac{e^{r(x_0-\varrho)}}{1+x_0}, \quad \text{(B7)}$$

where x_0 is the solution of the saddle point equation

$$\varrho - x_0 + \ln(\varrho/x_0) + i2\pi j = 0. \quad \text{(B8)}$$

Writing $x_0 = \varrho\alpha e^{i\theta}$, we find that α and θ obey the following equations:

$$\varrho = \frac{2\pi j - \theta}{\tan\theta} + \ln\left(\frac{2\pi j - \theta}{\varrho \sin\theta}\right), \quad \text{(B9a)}$$

$$\alpha = \frac{2\pi j - \theta}{\varrho\sin\theta}. \quad \text{(B9b)}$$

For $j = 0$, the saddle point $x_0 = \varrho$, which immediately gives

$$C(r) = 2\rho \sum_{j=1}^{\infty} \int_0^r dx \cos(2j\pi x) f(x), \quad \text{(B10)}$$

where the summand is given by

$$e^{r(\frac{2\pi j-\theta}{\tan\theta}-\varrho)} \times \frac{\cos[r(2\pi j-\theta)]\left(1+\frac{2\pi j-\theta}{\tan\theta}\right)+\sin[r(2\pi j-\theta)](2\pi j-\theta)}{\left(1+\frac{2\pi j-\theta}{\tan\theta}\right)^2+(2\pi j-\theta)^2}. \quad \text{(B11)}$$

Since the contribution of the successive terms in the sum decreases with increasing j, we estimate only the $j = 1$ term here. Also, numerical analysis of Eq. (B9a) shows that θ increases with j and therefore we work within small-θ approximation. These considerations finally yield

$$\theta = \frac{2\pi}{W(\varrho e^{1+\varrho})}, \quad \text{(B12)}$$

where W is the Lambert function that satisfies $W(z)e^{W(z)} = z$ [31], and

$$C(r) = 2\rho e^{-r\left(\frac{1}{\rho}-\frac{2\pi}{\theta}\right)} \frac{\cos(r\theta)\left(\frac{2\pi}{\theta}\right) - \sin(r\theta)(2\pi-\theta)}{\left(\frac{2\pi}{\theta}\right)^2 + (2\pi-\theta)^2}, \quad \text{(B13)}$$

$$\approx \rho e^{-r(\frac{1}{\rho}-\frac{2\pi}{\theta})} \frac{\theta\cos(r\theta)}{\pi}, \quad \text{(B14)}$$

which is an oscillatory function with decaying amplitude.

APPENDIX C: EVALUATION OF THE INTEGRAL EQ. (61)

Consider the following integral:

$$I_2 = \frac{1}{2\pi i}\oint_{C'} ds e^{sr} \frac{(s+t)^{b-1} - t^{b-1}}{s^2}, \quad t \geqslant 0, \quad \text{(C1)}$$

where the contour C' around the branch cut at $-t$ includes the Bromwich contour along the line $s = c$, c being real and nonnegative. The residue from the second-order pole at $s = 0$ gives $I_2 = (b-1)t^{b-2}$. The integral along the large semicircle with radius R decays exponentially fast with increasing R, and the one along the small semicircle with radius ϵ is proportional to ϵ^{b-2} and therefore vanishes as $\epsilon \to 0$. Thus, we get

$$I_2 = \frac{1}{2\pi i}\left[\int_{c-i\infty}^{c+i\infty} + \int_{AB} + \int_{B'A'} ds e^{sr} \frac{(s+t)^{b-1} - t^{b-1}}{s^2}\right]. \quad \text{(C2)}$$

Since $s = -t + xe^{\pm i\pi}$ along the upper (lower) branch $AB(B'A')$, we get

$$\int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} e^{sr} \frac{(s+t)^{b-1} - t^{b-1}}{s^2}, \quad \text{(C3)}$$

$$= \frac{\sin(b\pi)}{\pi} e^{-tr} \int_0^{\infty} dx e^{-xr} \frac{x^{b-1}}{(x+t)^2} + (b-1)t^{b-2}, \quad \text{(C4)}$$

$$= \frac{\sin(b\pi)}{\pi}\Gamma(b-1)e^{-tr} \frac{(rt+b-1)E_{b-1}(tr)e^{tr} - 1}{r^{b-2}} + (b-1)t^{b-2}. \quad \text{(C5)}$$

For $t = c = 0$, using that $e^x E_{b-1}(x) \overset{x\to 0}{\sim} (b-2)^{-1} + \mathcal{O}(x^{b-2})$ [22], the above integral reduces to

$$\int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} e^{sr} s^{b-3} = \frac{\sin(b\pi)}{\pi} \frac{\Gamma(b-2)}{r^{b-2}}. \quad \text{(C6)}$$

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