Generalization of the extended optical theorem for scalar arbitrary-shape acoustical beams in spherical coordinates

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The extended optical theorem is generalized for scalar acoustical beams of arbitrary character with any angle of incidence interacting with an object of arbitrary geometric shape and size, and placed randomly in the beam's path with any scattering angle. Analytical expressions for the extinction, absorption, and scattering cross sections are derived, and the connections with the axial (i.e., along the direction of wave propagation) torque and radiation force calculations are discussed. As examples to illustrate the analysis for a viscoelastic object, the extinction, absorption, and scattering cross sections are provided for an infinite plane progressive wave, infinite nondiffracting Bessel beams, a zero-order spherical quasi-Gaussian beam, and a Bessel-Gauss vortex beam emanating from a finite circular aperture, which reduces to a finite high-order Bessel beam, a finite zero-order Bessel beam, and a finite piston radiator vibrating uniformly with appropriate selection of beam parameters. The similarity with the asymptotic quantum inelastic cross sections is also mentioned.

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I. INTRODUCTION

The interaction of waves and scattering by a particle is an important topic in various fields including nuclear physics [1], quantum mechanics [2], optics [3–5], and acoustics [6,7]. In this process, the power [8] of the total (i.e., incident + scattered) wavefield [9] is extinguished [10,11] both by scattering [12] and by absorption inside the particle; thus, a specific cross section (denoted by $\sigma_{\mathrm{ext,sc,abs}}$) has been defined with each of these processes [3]. The extinction cross section σ_{ext} corresponds to the extinguished power normalized by the power per unit area incident upon the scatterer [8]. This phenomenon constitutes a general law in scattering theory, known as the optical theorem [13,14] or alternatively the extinction theorem [3], which relates the extinction cross section of an object of arbitrary geometry placed in the field of *monochromatic plane waves* to its forward scattering amplitude, which is the scattered wave amplitude measured in the far field along the forward direction of wave propagation. The application of the optical theorem has been generalized (originally in quantum mechanics [15,16]) to a form that involves an angular integral of a product of scattering amplitudes (of plane waves) to obtain a condition on scattering amplitudes in an arbitrary direction (instead of just the forward amplitude), which was later discussed (pp. 135–138 in [17]) and extended in the context of electron diffraction theory [18], optical evanescent waves [19], surface waves and layered elastic media [20], and acoustic backscattering by elastic targets (with no internal dissipation) having inversion symmetry [21].

The statement of conservation of energy applied to scattering and the associated cross section definitions is written as (p. 13 in [3])

$$\sigma_{\mathrm{ext}} = \sigma_{\mathrm{sca}} + \sigma_{\mathrm{abs}}. \tag{1}$$

The immediate application of the optical theorem is the numerical predictions of cross sections rather than direct integration procedures [14]. Moreover, the optical theorem finds useful applications in the theory of dispersion [22,23], near-field diffraction tomography [24], and inverse scattering [25]. Note that the standard optical theorem, which has been essentially established for plane waves, is not applicable for beams that possess some degree of amplitude roll-off in the transverse direction [26,27], such as Gaussian and Bessel beams. When extending the usual plane-wave form of the optical theorem (involving only forward scattering) to an angular superposition of amplitudes for nondiffracting beams, the convention has been to describe that as an *extended* optical theorem [28], which can be connected with radiation force and torque.

Motivated by the important applications of the (extended) optical theorem for scalar beams (which satisfy the Helmholtz equation), a generalized formulation applicable to any beam of arbitrary character with any angle of incidence is developed for a scatterer of arbitrary geometric shape and size, and placed on or off the beam's axis with any scattering angle. The generalization of the extended optical theorem gives generalized partial-wave series expansions for the extinction, absorption, and scattering cross sections in terms of the beam-shape and scattering coefficients of the target. Particular examples are considered for cases where a viscoelastic sphere is centered on the axis of wave propagation of the incident beam. Though the present analysis treats the case of acoustical beams, the similarity with the inelastic cross sections for asymptotic quantum arbitrary beams [29] is also mentioned.

II. THEORY

Consider an acoustic beam of angular frequency ω incident along an arbitrary direction on a viscoelastic object of arbitrary geometric shape immersed in a nonviscous fluid of density ρ_0 (Fig. 1). The density and the speed of sound inside the object are denoted by ρ_s and c_s, respectively. The origin of the coordinate system is chosen to be the center of the object. The incident and scattered first-order pressure amplitudes (the

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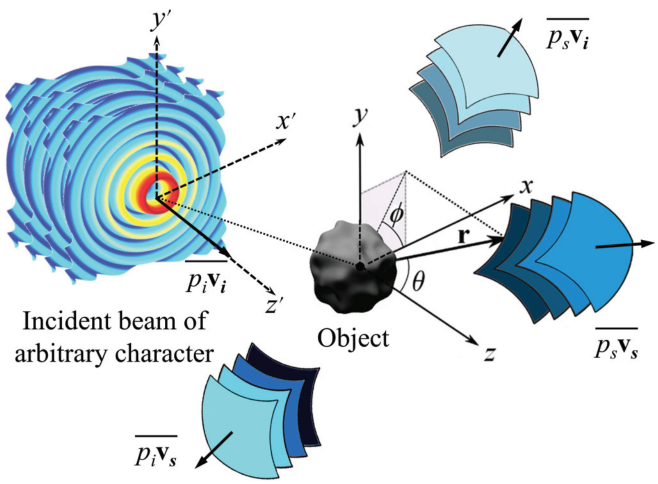

FIG. 1. (Color online) An object of arbitrary geometrical shape placed in the field of an incident acoustical beam of arbitrary character. The primed coordinate system has its origin at the center of the beam, while the unprimed coordinate system is referenced to the object. The parameters θ and φ are the polar and azimuthal angles, respectively. The time-averaged intensity vectors associated with the incident ($\overline{p_i\mathbf{v}_i}$), interacting ($\overline{p_i\mathbf{v}_s},\overline{p_s\mathbf{v}_i}$), and scattered ($\overline{p_s\mathbf{v}_s}$) waves are used to derive the specific cross sections.

time dependence $e^{-i\omega t}$ is suppressed for convenience) can be expanded into a partial-wave series expansion (PWSE) in the spherical coordinates system (r,θ,ϕ) as follows [30,31]:

$$p_i = p_0 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{n,m}\, j_n(kr) Y_n^m(\theta,\phi), \tag{2}$$

$$p_s = p_0 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{n,m} s_n h_n^{(1)}(kr)\, Y_n^m(\theta,\phi), \tag{3}$$

where p_0 is the pressure amplitude in the absence of the waves, $k = \omega/c_0$ is the wave number and c_0 the speed of sound in the surrounding nonviscous fluid, the function $j_n(\cdot)$ is the spherical Bessel function, $h_n^{(1)}(\cdot)$ is the spherical Hankel function of the first kind, and $Y_n^m(\cdot)$ are the Laplace spherical harmonics. $a_{n,m}$ are the beam-shape coefficients (BSCs—given explicitly by Eq. (7) in [31]) and s_n are the scaled scattering coefficients.

In a nonviscous fluid, it is convenient to use the far-field expressions for the incident and scattered acoustic fields [given by Eqs. (2) and (3)] by using the asymptotic expressions for the spherical Bessel and Hankel functions for large arguments. Thus,

$$p_i|_{kr\to\infty} \approx \frac{p_0}{kr} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{n,m} \sin\left(kr - \frac{n\pi}{2}\right) Y_n^m(\theta,\phi), \tag{4}$$

$$p_s|_{kr\to\infty} \approx p_0 f_\infty(\theta,\phi)\left(\frac{e^{ikr}}{r}\right), \tag{5}$$

where

$$f_\infty(\theta,\phi) = \frac{1}{k} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} i^{-n-1} a_{n,m} s_n Y_n^m(\theta,\phi). \tag{6}$$

The scaled scattering coefficients s_n are known for various sphere materials [7] (i.e., elastic, viscoelastic, layered spherical shells, etc.), and are obtained using appropriate boundary conditions at the interface of the solid and outer fluid.

The analysis is started by evaluating the absorption, scattering, and extinction cross sections of an arbitrary-shaped object. The cross sections are obtained from the ratio [8] of the time-averaged acoustic power W to the acoustic intensity, $I_0 = |p_0|^2/2\rho_0 c_0$.

The absorption cross section is therefore obtained from the ratio of the time-averaged absorbed acoustic power to the acoustic intensity, where the absorbed power is evaluated by the surface integral of the time-averaged energy flux of the total field over a fixed spherical surface [32], such that

$$\sigma_{\rm abs} = \frac{W_{\rm abs}}{I_0} = -\frac{1}{I_0}\iint_S (\overline{p_i\mathbf{v}_i} + \overline{p_i\mathbf{v}_s} + \overline{p_s\mathbf{v}_i} + \overline{p_s\mathbf{v}_s})\cdot d\mathbf{S}, \tag{7}$$

where S is a spherical surface that encloses the scatterer, $d\mathbf{S} = \mathbf{n}r^2 \sin\theta d\theta d\phi$ is the differential vector surface element and $\mathbf{n}$ the outward normal (the overbar denotes time averaging over the period of the wave), and $\mathbf{v}_i$ and $\mathbf{v}_s$ are the incident and scattered fluid particle velocities, respectively.

In the absence of the object in the acoustical field, there is no absorbed (or extinct) power. Therefore, the first term in Eq. (7) involving incident fields vanishes, and thus does not contribute in the evaluation of Eq. (7). Moreover, if the object is nonabsorptive, $\sigma_{\rm abs} = 0$ leading to $\sigma_{\rm ext} = \sigma_{\rm sc}$, according to Eq. (1).

When the object is absorptive, the incident beam is extinguished both by scattering and by absorption according to Eq. (1). Noticing that the scattering surface cross section, obtained from the ratio of the time-averaged scattered acoustic power [32] to the acoustic intensity, can be expressed as

$$\sigma_{\rm sca} = \frac{W_{\rm sca}}{I_0} = \frac{1}{I_0}\iint_S \overline{p_s\mathbf{v}_s} d\mathbf{S}, \tag{8}$$

the expression for $\sigma_{\rm ext}$ is therefore deduced from Eqs. (1), (7), and (8), such that

$$\sigma_{\rm ext} = \frac{W_{\rm ext}}{I_0} = -\frac{1}{I_0}\iint_S (\overline{p_i\mathbf{v}_s} + \overline{p_s\mathbf{v}_i}) d\mathbf{S}, \tag{9}$$

where $W_{\rm ext}$ is the power extracted from the incident beam.

In the far-field region, the expressions for the incident and scattered waves can be approximated to the following expressions, such that

$$\mathbf{v}_i \approx \boldsymbol{\nabla} p_i/(i\omega\rho_0), \tag{10}$$

and

$$(\mathbf{n}\cdot\mathbf{v}_s) \approx p_s/(\rho_0 c_0). \tag{11}$$

After some algebraic manipulation using the angular integrals of the Laplace spherical harmonics (see the Appendix in Ref. [33]), it follows that the substitution of Eqs. (4)–(6) with Eqs. (10) and (11) into Eqs. (7)–(9) gives the expressions for the absorption, scattering, and extinct cross sections as

$$\sigma_{\text{abs}} = -\frac{1}{2k^2}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(s_n + s_n^* + 2|s_n|^2)|a_{n,m}|^2, \tag{12}$$

$$\sigma_{\text{sca}} = \iint_S |f_\infty(\theta,\phi)|^2 dS = \frac{1}{k^2}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}|s_n|^2|a_{n,m}|^2, \tag{13}$$

and

$$\sigma_{\text{ext}} = -\frac{1}{k|p_0|^2}\text{Im}\left\{\iint_S p_s^*\left(ikp_i + \frac{\partial p_i}{\partial r}\right)dS\right\} = -\frac{1}{2k^2}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(s_n + s_n^*)|a_{n,m}|^2, \tag{14}$$

where the superscript * denotes a complex conjugate. Note that the double integral for the extinction cross section, given in the first line of Eq. (14), was given in equivalent forms in terms of the extinction efficiency factor for a *sphere* (Eq. (15) in [32]), or the extinction power (Eq. (9) in [28]).

III. RESULTS

The most general law of acoustic wave scattering theory for any beam of arbitrary character scattered by an arbitrary-shaped object, relating the extinction cross section of the scatterer to the interacting incident and scattered pressures, is given by Eq. (14). Moreover, Eqs. (12) and (13) are the generalized expressions for the absorption and scattering cross sections written in terms of the BSCs, a_{nm}, and the scaled scattering coefficients of the object, s_n. Note that Eq. (14) equals the scattering cross section, Eq. (13), if the object is nonabsorptive. Equivalent forms for Eqs. (12)–(14) have been provided in the context of quantum beams (Sec. iv in [29]). Though the cross sections have the dimensions of area, in quantum mechanics, this concept is used to express the *probability* of absorption, scattering, or extinction (i.e., annihilation) between particles, while in the acoustical context, the (classical) cross sections determine the strength of absorption, scattering, and extinction of mechanical waves from the object.

With the cross sections expressed in generalized partial-wave series expansions, generalized efficiency factors [3,32,34] for an arbitrary-shaped object of cross-sectional surface S_c can be expressed as

$$Q_{\text{ext,abs,sca}} = \sigma_{\text{ext,abs,sca}}/S_c. \tag{15}$$

It is also important to note the close connection of the absorption cross section given by Eq. (12) with the *axial* component τ_z of the dimensionless torque of arbitrary acoustical waves (Eq. (11) in [35]). Denoting by V the volume of the object, the expression for the axial component of the dimensionless torque given previously in [35] is generalized to

$$\tau_z = -\frac{2}{3Vk^3}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}m(s_n + s_n^* + 2|s_n|^2)|a_{n,m}|^2, = -\frac{2}{3Vk^3}\sum_{n=0}^{\infty}\sum_{m=1}^{n}m(s_n + s_n^* + 2|s_n|^2)(|a_{n,m}|^2 - |a_{n,-m}|^2). \tag{16}$$

Moreover, it should be noted that the factor $(s_n + s_n^* + 2|s_n|^2)$ in Eqs. (12) and (16), [and Eqs. (17) and (18), following], can be rewritten in terms of a scattering function S_n [7] such that $\text{S}_n = 2s_n + 1$, leading to $(|\text{S}_n|^2 - 1)/2 = (s_n + s_n^* + 2|s_n|^2)$. For an elastic material, the scattering function S_n is unimodular [7], i.e., $|\text{S}_n| = 1$. Consequently, the factor $(s_n + s_n^* + 2|s_n|^2)$ is zero, and there is no *axial* torque on the object in the ideal case of no absorption. See also an equivalent form in Eq. (18) of [36] in which the scattering coefficients were denoted by A_n therein—a misprint occurred in which Eq. (18) in [36] should have been printed as

$$(|\text{S}_n|^2 - 1) = 4(\text{Re}\{A_n\} + |A_n|^2) = 4\Gamma_n = 0.$$

As particular examples, simplified partial-wave series expansions can be obtained for the cross sections given by Eqs. (12)–(14) in the case of a viscoelastic arbitrary-shaped object centered on the axis of a known incident beam.

In the particular case where the axis of wave propagation of the incident beam coincides with one of the axes of the coordinates system centered on the object, the expression for the BSCs reduces to a simplified form. In that case, those are defined as the *axial* BSCs. Generally, when the axial BSCs satisfy the condition $a_{n,-m} = 0$ (which is typical for high-order Bessel-vortex beams, plane waves, or quasi-Gaussian beams centered on a sphere), Eqs. (12) and (16) are reduced to

$$\sigma_{\text{abs}}^{\text{axial}} = -\frac{1}{2k^2}\sum_{n=|m|}^{\infty}(s_n + s_n^* + 2|s_n|^2)\left|a_{n,m}^{\text{axial}}\right|^2, \tag{17}$$

$$\tau_z^{\text{axial}} = -\frac{2m}{3Vk^3}\sum_{n=|m|}^{\infty}(s_n + s_n^* + 2|s_n|^2)\left|a_{n,m}^{\text{axial}}\right|^2; \tag{18}$$

hence, the axial component of the dimensionless torque can be expressed as

$$\tau_z^{\text{axial}} = \frac{4m}{3Vk}\sigma_{\text{abs}}^{\text{axial}}. \tag{19}$$

For the case of a *sphere* of radius a, where $V = \frac{4}{3}\pi a^3$, Eq. (19) indicates that in the axial configuration, two different spheres with the same radius and having the same absorption cross section will exhibit exactly the same dimensionless torque for a given m. This has also been observed in the context of the axial optical torque of circularly polarized light [37].

For arbitrary plane waves, the BSCs are given by [5]

$$a_{n,m}^{pw} = 4\pi i^n Y_n^{m*}(\beta,\alpha)\delta_{m0}. \tag{20}$$

Assuming the axis of wave propagation coincides with the one centered on the object, the angles $\alpha = \beta = 0$, and Eq. (14) reduces to the well-known optical theorem for plane waves (involving forward scattering only),

$$\sigma_{\text{ext}}^{pw} = \frac{4\pi}{k}\text{Im}\left\{\frac{1}{ik}\sum_{n=0}^{\infty}(2n+1)\,s_n\right\}. \tag{21}$$

Similarly, the absorption [Eq. (12)] and scattering [Eq. (13)] cross sections can be expressed, respectively, as

$$\sigma_{\text{abs}}^{pw} = -\frac{4\pi}{k^2}\sum_{n=0}^{\infty}(2n+1)\left(\text{Re}\,[s_n]+|s_n|^2\right), \tag{22}$$

$$\sigma_{\text{sca}}^{pw} = \frac{4\pi}{k^2}\sum_{n=0}^{\infty}(2n+1)|s_n|^2. \tag{23}$$

Moreover, the optical theorem can be extended to evaluate the cross sections for other types of beams centered on an arbitrary-shaped object, such as Bessel-vortex, or trigonometric beams [38]. The *axial* BSCs for a Bessel-vortex (BV) beam are given by Eq. (33) in Ref. [38] as

$$a_{n,m}^{\text{BV}} = 4\pi i^{n-m} Y_n^{m*}(\beta,0)\,H\,(n-m)\,\delta_{\ell m}, \tag{24}$$

where ℓ is the order (known also as topological charge) of the beam, $H(\cdot)$ is the Heaviside step function, and δ_{ij} is the Kronecker delta function. Substituting Eq. (24) into Eqs. (12)–(14), the extinction, absorption, and scattering cross sections become

$$\sigma_{\text{ext}}^{\text{BV}} = \frac{4\pi}{k}\text{Im}\left\{\frac{1}{ik}\sum_{n=|\ell|}^{\infty}(2n+1)\frac{(n-\ell)!}{(n+\ell)!}\left[P_n^{\ell}(\cos\beta)\right]^2 s_n\right\}, \tag{25}$$

$$\sigma_{\text{abs}}^{\text{BV}} = -\frac{4\pi}{k^2}\sum_{n=|\ell|}^{\infty}(2n+1)\,\frac{(n-\ell)!}{(n+\ell)!}\left[P_n^{\ell}(\cos\beta)\right]^2 \times (\text{Re}[s_n]+|s_n|^2), \tag{26}$$

$$\sigma_{\text{sca}}^{\text{BV}} = \frac{4\pi}{k^2}\sum_{n=|\ell|}^{\infty}(2n+1)\,\frac{(n-\ell)!}{(n+\ell)!}\left[P_n^{\ell}(\cos\beta)\right]^2|s_n|^2. \tag{27}$$

Thus, from Eq. (19), the axial torque can be evaluated. For the *sphere* case with a cross-sectional surface $S_c = \pi a^2$, the axial torque $\tau_z^{\text{BV,axial}} = \ell\sigma_{\text{abs}}^{\text{BV}}/(kaS_c)$, in agreement with the result provided previously in [32,36,39,40].

For a Bessel trigonometric (BT) beam, which may be expressed as a combination of two vortex beams of opposite helicity [41,42], the axial BSCs are expressed as

$$a_{n,m}^{\text{BT}} = 2\pi i^{n-m} Y_n^{m*}(\beta,0)[H(n-m)\delta_{\ell m} + (-1)^m H(n+m)\delta_{-\ell m}]. \tag{28}$$

[Equation (28) should replace the mathematical expressions given previously by Eq. (34) in [38], and Eq. (15) in [43], since the BSCs describe the incident beam in the spherical coordinates system, and should be expressed independently of the angular coordinates (after integration)]. From Eq. (28), the following property can be deduced; that is,

$$a_{n,-m}^{\text{BT}} = (-1)^m a_{n,m}^{\text{BT}}. \tag{29}$$

Equation (29) indicates that for a BT beam of order ℓ, the BSCs $a_{n,-m}^{\text{BT}} \neq 0$. Thus Eqs. (18) and (19) are not applicable to BT beams. In addition, the substitution of Eq. (29) into Eq. (16) shows that the *axial* dimensionless torque for a BT beam vanishes, in agreement with previous investigations showing that it does not carry a vortex [41,42]. Furthermore, it is found that the extinction, absorption, and scattering cross sections for the BT beam equal half those obtained for the BV beam [Eqs. (25)–(27)]. A similar result was previously obtained for the axial radiation force of a BT beam (on a sphere), which also equals half the force of a BV beam [41].

Another exact solution to the Helmholtz equation, known as the lowest-order spherical quasi-Gaussian beam [44–47], is considered, for which the axial BSCs can be expressed as

$$a_{n,m}^{qG_{00}} = i^n\sqrt{4\pi\,(2n+1)}g_n\,(kz_R)\,\delta_{m0}, \tag{30}$$

where the function $g_n(kz_R)$ is given explicitly by Eq. (4) in [45], and kz_R is the dimensionless beam waist. Substituting Eq. (30) into Eqs. (12)–(14), the extinction, absorption, and scattering cross sections for a viscoelastic object centered on the focus of a zeroth-order quasi-Gaussian beam are expressed as

$$\sigma_{\text{ext}}^{qG_{00}} = \frac{4\pi}{k}\text{Im}\left\{\frac{1}{ik}\sum_{n=0}^{\infty}(2n+1)\,[g_n\,(kz_R)]^2 s_n\right\}. \tag{31}$$

$$\sigma_{\text{abs}}^{qG_{00}} = -\frac{4\pi}{k^2}\sum_{n=0}^{\infty}(2n+1)[g_n(kz_R)]^2(\text{Re}[s_n]+|s_n|^2), \tag{32}$$

$$\sigma_{\text{sca}}^{qG_{00}} = \frac{4\pi}{k^2}\sum_{n=0}^{\infty}(2n+1)\,[g_n\,(kz_R)]^2|s_n|^2. \tag{33}$$

An additional important example is considered here, which concerns the derivation of the extinction, absorption, and scattering cross sections for a viscoelastic object centered on a high-order Bessel-Gauss vortex beam of order ℓ, emanating from a *finite* circular aperture. In practice, every acoustic source (except a point source radiating omnidirectional waves) produces a finite propagating beam, as opposed to waves of infinite extent considered previously.

Following Eqs. (6), (7), and (11) in Ref. [48], the axial BSCs for a high-order Bessel-Gauss vortex beam, which satisfy the Helmholtz equation, can be expressed as

$$a_{n,m}^{\text{HOBGVB}} = (-1)^{n+m}\sqrt{4\pi(2n+1)\frac{(n-m)!}{(n+m)!}} \times f_{\text{HOBGVB}}\,H(n-m)\delta_{\ell m}, \tag{34}$$

where the function f_{FBGV} is given by the following integral:

$$f_{\text{HOBGVB}} = \int_{kr_0}^{kr_a}\left[(kr_1)J_m\left(k_\rho\sqrt{r_1^2-r_0^2}\right)e^{-[(r_1^2-r_0^2)/w_0^2]}h_n^{(1)}(kr_1)P_n^m\left(\frac{r_0}{r_1}\right)\right]d(kr_1). \tag{35}$$

The parameters in Eq. (35) are the distance r_a from the edge of the radiator to the object, the distance r_1 from a point on the surface of the radiator to the object, the distance r_0 from the center of the radiator to the object, the half-cone angle β with $k_\rho = k \sin\beta$, the beam's waist w_0, the cylindrical Bessel function $J_m(\cdot)$ of the first kind of order m, and the associated Legendre functions $P_n^m(\cdot)$ of degree and order n and m, respectively.

Substituting Eq. (34) into Eqs. (12)–(14) and manipulating the results, the extinction, absorption, and scattering cross sections become

$$\sigma_{\text{ext}}^{\text{HOBGVB}} = \frac{4\pi}{k}\text{Im}\left\{\frac{1}{ik}\sum_{n=|\ell|}^{\infty}(2n+1)\frac{(n-\ell)!}{(n+\ell)!}^2 |f_{\text{HOBGVB}}|^2 s_n\right\}, \tag{36}$$

$$\sigma_{\text{abs}}^{\text{HOBGVB}} = -\frac{4\pi}{k^2}\sum_{n=|\ell|}^{\infty}(2n+1)\frac{(n-\ell)!}{(n+\ell)!}|f_{\text{HOBGVB}}|^2 \times (\text{Re}[s_n] + |s_n|^2), \tag{37}$$

$$\sigma_{\text{sca}}^{\text{HOBGVB}} = \frac{4\pi}{k^2}\sum_{n=|\ell|}^{\infty}(2n+1)\frac{(n-\ell)!}{(n+\ell)!}|f_{\text{HOBGVB}}|^2|s_n|^2. \tag{38}$$

Particularly interesting cases can be obtained from Eqs. (34)–(38) for other beams; for example, by letting $w_0 \to \infty$, which corresponds to the case of a collimated beam, the axial BSCs and related cross sections can be derived for a finite high-order Bessel-vortex beam of order ℓ. Moreover, if ℓ is set to zero, the axial BSCs and related cross sections for a finite zero-order Bessel-Gauss, or a zero-order Bessel beam (i.e., $\ell = 0$, $w_0 \to \infty$) can be obtained. Note that the axial BSCs for a zero-order Bessel beam, deduced from Eqs. (34)–(38) when $\ell = 0$ and $w_0 \to \infty$, can be accurately recovered from Eqs. (6) and (7) of Ref. [49]. In addition, the case of a finite piston circular radiator vibrating uniformly can be also deduced by setting $w_0 \to \infty$, $\ell = \beta = 0$ in Eqs. (34)–(38). In that case, the integral given by Eq. (34) has an exact closed-form solution [50,51].

As the axial dimensionless radiation torque can be linked to the absorption cross section as shown by Eq. (19) for beams centered on an object and satisfying the condition $a_{n,-m} = 0$, it is instructive to note also that previous analyses have demonstrated the connection between the *axial* radiation force with the scattering cross section for the case of plane waves on a rigid sphere (see Sec. V in [52]), both the scattering and absoption cross sections for the case of plane progressive waves [53,54], both the scattering and extinction cross sections for plane waves on an object with arbitrary oscillating surface [55], and for Bessel-vortex beams [32] on a sphere. An analogy has been also noted for optical beams [56]. Nevertheless, considering the generalized expression for the axial component of the radiation force (i.e., Eq. (13) in Ref. [57], or Eq. (2) in Ref. [58]), there has been no simplified expression describing the *direct* connection with the generalized cross sections as given by Eqs. (12)–(14) for the axial case.

IV. DISCUSSION

The results presented here provide a generalized solution and insight into the wave interference phenomena in acoustical scattering. In particular, the PWSE in Eq. (14) provides the most general relationship describing the extinction of acoustical *beams* [as opposed to (plane) waves of infinite extent] by the scattering and absorption phenomena, related to the presence of an arbitrary-shaped object. Precisely, the results show that the extinction, scattering, and absorption cross sections (or power) are meaningful measures of the object scattering and absorption properties for certain forms of the incident field. Essentially, all applications and experimental methods, processes, and operational devices involving the scattering of acoustical waves can benefit from this analysis. For example, ultrasonic attenuation spectroscopy (UAS) [59] currently used in practice for a wealth of applications in materials science, powder technology, and multiphase flow metering to name a few; computerized ultrasound tomography (CUT) [60], which is accomplished through measurements of the ultrasonic field extinction [due to the (multiple) scattering and attenuation] throughout the object under analysis; and ultrasound absorption microscopy (UAM) [61,62], in which the transmitted acoustic field is detected in a confocal setting via the extinguished signal, are all fundamentally based on this approach. The present generalized theoretical formalism should therefore assist in the design of improved acoustical systems using the above-mentioned methods and taking into account the beam shape and character of the incident wave fronts.

Further extension of the analytical formalism presented here may be obtained for the extinction of acoustical beam by a collection of acoustically interacting arbitrary-shaped objects, and this analysis should assist along that direction of research so as to include the attenuation and extinction effects due to multiple scattering phenomena.

V. CONCLUSION

In summary, a generalization for the extended optical theorem for arbitrary acoustical beams that satisfy the source-free Helmholtz equation is presented, and generalized partial-wave series expansions (PWSEs) for the extinction, absorption, and scattering cross sections [Eqs. (13)–(15)] are derived stemming from the general law that the power of the incident beam is extinguished both by scattering and by absorption inside a viscoelastic object of arbitrary shape. PWSE for the extinction, absorption, and scattering cross sections in plane waves, infinite, and finite Bessel-Gauss, Bessel-vortex, and Bessel trigonometric beams, as well as quasi-Gaussian beams centered on the object, are provided.

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