# Virial coefficients in the $(\tilde{\mu},q)$-deformed Bose gas model related to compositeness of particles and their interaction: Temperature-dependence problem

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We establish the relation of the second virial coefficient of a recently proposed $(\tilde{\mu},q)$-deformed Bose gas model [A. M. Gavrilik and Yu. A. Mishchenko, Ukr. J. Phys. **58**, 1171 (2013)] to the interaction and compositeness parameters when either of these factors is taken into account separately. When the interaction is dealt with, the deformation parameter becomes linked directly to the scattering length and the effective radius of interaction (in general, to scattering phases). The additionally arising temperature dependence is a feature absent in the deformed Bose gas model within the adopted interpretation of the deformation parameters $\tilde{\mu}$ and $q$. Here the problem of the temperature dependence is analyzed in detail and its possible solution is proposed.

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## I. INTRODUCTION

Nonlinear physical systems involving either nonideality or nonlinearity factors are often effectively described by means of some deformed (e.g., algebraic or phenomenological) models as counterpart to their corresponding ideal prototype. Deformed Bose gas models (DBGMs), along with deformed oscillators, deformed quantum mechanics, and some other extensions, having evolved since the end of 1980s until now, belong to such models of an effective description. Generally speaking, the DBGMs may or may not be directly linked with deformed oscillators, which in many cases are taken as the structural object underlying the former. Soon after the introduction of deformed oscillators or deformed bosons [1–3], their use for elaborating respective deformed analogs of the Bose gas model became popular. The early instances of DBGMs (for a few examples see [4–9]) had already witnessed the appearance of a very important new direction with a long-term perspective and with good potential for useful (and realistic) applications. The latter range from the $^4$He system [10] to, e.g., objects of high-energy physics such as two- and three-particle correlations of pions generated in relativistic collisions of nuclei [11]. These applications yielding a good effective description stimulated the study of deeper reasons of the applicability of DBGMs to real physical systems. Helpful, from this viewpoint, appears the idea that deformation of the ideal Bose gas model could and should provide an efficient effective description of the properties of more realistic (i.e., nonideal) gases of Bose-like particles. Moreover, the deviation from strict ideality may originate for several reasons (nonideality factors). It was demonstrated [12] that the nonpointlike form of particles may serve as the first and most obvious such factor and it is possible to link the parameter $q$ of deformation directly with the ratio of the excluded volume (the sum of nonzero proper volumes of the particles) to the whole volume. The next factor of nonideality is the interaction between the particles and as shown in [13] it can be naturally taken into account by a version of deformation.

More recently it became clear that the possibility of realization (of operator algebra) of *composite* bosons by deformed bosons proven in [14] is naturally promoted to the elaboration of the DBGM able to effectively account for the compositeness of Bose particles (the compositeness makes them quasibosons, differing from strict bosons). Finally, let us mention recent work [15] that shows how to incorporate simultaneously two different factors of nonideality: the compositeness of particles and their interaction. That work motivated our present study.

Let us mention several versions of DBGMs and their application to physical systems in different contexts. The thermodynamics of the $q$-DBGMs was studied, e.g., in [16,17]; for the Bose condensation of the deformed gases see [18]. The DBGMs and many-body systems of $q$ bosons were applied to phonon gas in $^4$He [10], to excitons in [19], and to a study of pairing correlations in nuclei [20]. Some of the DBGMs were applied when studying two-particle correlation functions [11,21–24]. The extent or strength of deformation of the models mentioned usually is characterized by one or more deformation parameters. Until now, the question about the relation between the deformation parameters and the microscopic nonideality factors and their parameters had remained opened and the microscopic analysis of the correspondence between a physical system and its deformed counterpart had still been absent.

In this work, similarly to [14], where the deformation parameter for the realization of bifermionic composites (quasibosons) by deformed bosons was related to the wave functions of bifermionic states being realized, we establish the relation between the deformation of a special (class of) DBGM and the characteristics of the interaction along with compositeness of particles of a gas, which the DBGM is implied to effectively incorporate jointly. As the criterion of the effective description (or realization) in the former case [14], the realizability of quasibosonic operator relations was taken. In the present case the proximity of the virial expansion of the equation of state for the nonideal quantum gas to that of the DBGM is chosen as such a criterion. The structure function characterizing the DBGM of the effective description for the concreteness is taken to be of the same special form as in [15], however, the analogous consideration given below may concern a more general situation. This structure function is the functional

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composition of those corresponding to effectively taking the compositeness and interaction into account separately. The most optimal form of such a functional dependence is also an open question.

Among central issues of this paper is the temperature dependence of virial coefficients. According to [4,13,15–18,25,26], the virial coefficients within the DBGMs studied therein depend only on the deformation parameter(s), which in our interpretation are interrelated with nonideality factors and thus should not depend on the temperature. On the other hand, the virial coefficients for a gas with interaction manifest a temperature dependence [27]. Just this problem is the focus of the present work.

## II. RELATION OF DEFORMATION PARAMETERS TO THE INTERACTION BETWEEN QUASIBOSONS AND TO THEIR COMPOSITENESS

We start with the recently obtained [15] deformed virial expansion for the $(\tilde{\mu},q)$-deformed Bose gas whose thermodynamics or statistical physics is given through the structure function $\varphi_{\tilde{\mu},q}(N)$ (with the definition $[N]_q \equiv \frac{1-q^N}{1-q}$),

$$\varphi_{\tilde{\mu},q}(N) = \varphi_{\tilde{\mu}}([N]_q) = (1+\tilde{\mu})[N]_q - \tilde{\mu}[N]_q^2. \qquad (1)$$

The structure function determines quantitatively how the thermodynamics or statistical physics is deformed for that or another system. Namely, in [15] the $(\tilde{\mu},q)$-DBGM is constructed so that $\varphi_{\tilde{\mu},q}(z\frac{d}{dz})$ replaces the derivative $z\frac{d}{dz}$ in the known relation for the total number of particles given through the partition function, i.e., $N = z\frac{d}{dz}\ln Z$, yielding the definition for the deformed total number of particles in terms of the nondeformed partition function

$$\tilde{N} \equiv N^{(\tilde{\mu},q)}(z,V,T) \equiv \varphi_{\tilde{\mu},q}\left(z\frac{d}{dz}\right)\ln Z. \qquad (2)$$

All other deformed physical quantities are recovered using the nondeformed version of the relations of the ideal quantum Bose gas. For instance, for the second virial coefficient, which is of interest for us herein, within the $(\tilde{\mu},q)$-DBGM we have obtained [15]

$$V_2^{(\tilde{\mu},q)} = -\frac{\varphi_{\tilde{\mu},q}(2)}{2^{7/2}} = -\frac{(1+q)(1-\tilde{\mu}q)}{2^{7/2}}. \qquad (3)$$

In our treatment, the parameter $q$ of $\varphi_{\tilde{\mu},q}(N)$ corresponds to effectively taking the interparticle interaction into account and $\tilde{\mu}$ to composite-structure effects. Somewhat earlier, the Arik-Coon structure function $[N]_q$ was used to effectively incorporate [13] the interaction between the particles of a gas of elementary bosons.

Note that if, in addition to deformed thermodynamic relations, the structure function $\varphi_{\tilde{\mu},q}(N)$ describes some deformed boson algebra related to the $(\tilde{\mu},q)$-DBGM studied herein, certain ranges of admissible $\tilde{\mu}$ and $q$ hold. These can be deduced from the condition $\varphi_{\tilde{\mu},q}(n) \geqslant 0,\ n = 1,\ldots,N_{\max}$, which corresponds to non-negativity of the norm of deformed boson Fock states ($N_{\max}$ is the maximum occupation number). In particular, the non-negativity of $\varphi_{\tilde{\mu},q}(2)$ yields $\tilde{\mu}q \leqslant 1$ and $q \geqslant -1$. However, we do not appeal to the relation with a deformed boson algebra.

Besides $\varphi_{\tilde{\mu},q}(N)$, one can take yet another version of combining the two structure functions $\varphi_{\tilde{\mu}}(N)$ and $[N]_q$, e.g., in the form $\varphi_{q,\tilde{\mu}}(N) = [\varphi_{\tilde{\mu}}(N)]_q$, or as a family with one more parameter: $t\varphi_{\tilde{\mu},q}(N) + (1-t)\varphi_{q,\tilde{\mu}}$ for $0 \leqslant t \leqslant 1$. We remark that the treatment below can be extended to the case of an even more general structure function $\varphi(N)$ when some of the deformation parameters are responsible for interparticle interaction and the others for the composite structure of particles in the effective description.

### A. Effective account for the particle-particle interaction to $(\lambda^3/v)^2$ terms

As known, the deviation (from the ideal or noninteracting case) of the second virial coefficient $V_2$ due to the two-particle interaction is expressed through the partial wave phase shifts $\delta_l(k)$ and the bound state (if any) energies $\varepsilon_B$ as follows [27]:

$$V_2 - V_2^{(0)} = -8^{1/2}\sum\nolimits_B e^{-\beta\varepsilon_B} - \frac{8^{1/2}}{\pi}\sum\nolimits_l'(2l+1)\int_0^\infty e^{-\beta\hbar^2k^2/m}\frac{\partial\delta_l(k)}{\partial k}dk. \qquad (4)$$

Here $B$ runs over bound states, $l$ is the angular momentum quantum number, and the summation is performed over even $l$ in the bosonic case and over odd $l$ in the fermionic case. In the low-energy approximation we retain in (4) only the $l=0$ summand ($s$-wave approximation). The corresponding phase shift $\delta_0(k)$ generally can be determined by solving the Schrödinger equation for a specified interaction potential. However, in the low-energy limit (when $l=1$ effects are negligible) the following expansion, known as the effective range approximation, holds [28–30]:

$$k\cot\delta_0 = -\frac{1}{a} + \frac{1}{2}r_0k^2 + \cdots,$$
$$r_0 = 2\int_0^\infty dr\left[\left(1-\frac{r}{a}\right)^2 - \chi_0^2(r)\right], \qquad (5)$$

where $a$ is the scattering length, $r_0$ is the effective range (radius), and $\chi_0(r)$ is the radial wave function of the lowest state multiplied by $r$. Since for some typical potentials $r_0$ depends only on the range and depth of the potential, this expansion is sometimes called the shape-independent approximation. For the shape-independent approximation we find $\frac{\partial\delta_0}{\partial k} = -a + (a - 3r_0/2)a^2k^2 + O(k^4)$. Putting this derivative in (4) and performing integration, within the $s$-wave approximation we obtain

$$V_2 - V_2^{(0)} = -8^{1/2}\sum_B e^{-\beta\varepsilon_B} + 2\frac{a}{\lambda_T} - 2\pi^2\left(1-\frac{3}{2}\frac{r_0}{a}\right)\times\left(\frac{a}{\lambda_T}\right)^3 + O((a/\lambda_T)^5), \qquad (6)$$

where $\lambda_T \equiv \lambda = h/\sqrt{2\pi mk_BT}$ is the thermal wavelength. Below we give the explicit expressions for $V_2 - V_2^{(0)}$, or for the pair $a$ and $r_0$ through which it is expressed in (6), for a number of potentials (their definitions and some details are relegated to the Appendix). For the hard-sphere interaction potential (A1) we have [27]

$$V_2 - V_2^{(0)} = 2\frac{D}{\lambda_T} + \frac{10\pi^2}{3}\left(\frac{D}{\lambda_T}\right)^5 + \cdots, \quad l = 0,2. \quad (7)$$

For the constant repulsive potential (A3) and subsequent potentials the corresponding quantities are given using [29]. So, we have

$$a = R\left(1 - \frac{\tanh K_0 R}{K_0 R}\right), \quad r_0 = 0. \quad (8)$$

For the square-well potential (A5)

$$a = -R\left(\frac{\tan K_0 R}{K_0 R} - 1\right),$$
$$r_0 = R\left(1 - \frac{1}{K_0^2 R a} - \frac{R^2}{3a^2}\right). \quad (9)$$

For the anomalous scattering potential (A7)

$$a = R - \frac{\tanh K_0(R - r_1) + (K_0/K_1)\tan K_1 r_1}{K_0[1 + (K_0/K_1)\tan(K_1 r_1)\tanh K_0(R - r_1)]}. \quad (10)$$

A somewhat awkward expression for $r_0$ is omitted. For scattering resonances (A9)

$$a = \frac{\Omega}{\Omega + 1}R, \quad r_0 = \frac{2}{3}\frac{\Omega - 1}{\Omega}R. \quad (11)$$

For the modified Pöschl-Teller potential (A11), at integer $\lambda$,

$$a = \frac{1}{\alpha}\sum_{n=1}^{\lambda-1}\frac{1}{n}, \quad r_0 = \frac{2}{3\alpha}\frac{\left(\sum_{n=1}^{\lambda-1} n^{-1}\right)^3 - \sum_{n=1}^{\lambda-1} n^{-3}}{\left(\sum_{n=1}^{\lambda-1} n^{-1}\right)^2}. \quad (12)$$

For the inverse power repulsive potential (A13)

$$a = r_0\frac{\Gamma(1 - 1/2\eta)}{\Gamma(1 + 1/2\eta)}\left(\frac{g}{2\eta}\right)^{1/\eta}, \quad \eta = \frac{n - 2}{2}, \quad (13)$$

and $r_0$ can be found from (5) using (A15). With the data given above, we have the deviation of the second virial coefficient in Eq. (6) from that of the ideal Bose gas, for each of the considered potentials (A1) and (A3)–(A13).

On the other hand, within the $(\varphi_{\tilde{\mu},q})$-deformed Bose gas model we have [15] [see also Eq. (3)]

$$V_2^{(\tilde{\mu},q)} - V_2^{(0)}\big|_{\tilde{\mu}=0} = \frac{2 - \varphi_{\tilde{\mu},q}(2)}{2^{7/2}}\bigg|_{\tilde{\mu}=0} = \frac{1 - q}{2^{7/2}}. \quad (14)$$

By juxtaposing this with (6), we obtain

$$q = q(a,r_0,T) = 1 - 2^{9/2}\frac{a}{\lambda_T} + 2^{9/2}\pi^2\left(1 - \frac{3}{2}\frac{r_0}{a}\right)\left(\frac{a}{\lambda_T}\right)^3 + \cdots + 2^5\sum_B e^{-\beta\varepsilon_B}, \quad (15)$$

which constitutes one of our main results. Of course, this formula should be appended with $a$ and $r_0$ taken, e.g., for the chosen cases from (8)–(13) or for any other desired case.

The temperature dependence of the deformation parameter in (15) appears somewhat unexpected since, in our interpretation, the deformation parameter characterizes the nonideality of the deformed Bose gas model as a whole and $T$ is its internal parameter. One of the approaches to resolve this issue consists in a modification of the very deformation in the deformed Bose gas model. For instance, we can use the extended deformed derivative (here $z = e^{\beta\mu}$ is the fugacity and $\mu$ the chemical potential)

$$z\frac{\partial}{\partial z} \to z\tilde{\mathcal{D}}_z \equiv \varphi\left(z\frac{\partial}{\partial z}\right) + \chi\left(z\frac{\partial}{\partial z}\right)\frac{\partial}{\partial\beta} + g(\beta)\rho\left(z\frac{\partial}{\partial z}\right), \quad (16)$$

with structure functions $\varphi$, $\chi$, and $\rho$, in the relation

$$\tilde{N} = z\tilde{\mathcal{D}}_z \ln Z^{(0)} = \varphi\left(z\frac{\partial}{\partial z}\right)\ln Z^{(0)} - \chi\left(z\frac{\partial}{\partial z}\right)U^{(0)} - \beta g(\beta)\rho\left(z\frac{\partial}{\partial z}\right)\Phi_G^{(0)}.$$

Here $Z^{(0)}$, $U^{(0)}$, and $\Phi_G^{(0)}$ are the nondeformed partition function, internal energy, and Gibbs thermodynamic potential, respectively; $(z,V,T)$ serve as independent variables. Thus, on the thermodynamics level, $\chi$ and $\rho$ reflect the effect on the total number of particles of the internal energy and the Gibbs thermodynamic potential, which now appear on the same footing as the logarithm of the grand partition function. The corresponding analysis will be carried out in Sec. III below.

*Remark.* It is worth estimating the relative magnitude of the terms $-8^{1/2}\sum_B e^{-\beta\varepsilon_B}$ and $2\frac{a}{\lambda_T}$ in (6) at low-energy scattering when bound states do exist. According to [29] we have the following estimate for the binding energy in terms of scattering data:

$$\varepsilon_B \simeq -\frac{\hbar^2}{2ma^2}\left(1 + \frac{r_0}{a}\right).$$

Using this we arrive at

$$-8^{1/2}e^{-\beta\varepsilon_B} + 2\frac{a}{\lambda_T} \simeq -8^{1/2}\exp\left[\frac{\hbar^2}{2ma^2 k_B T}\left(1 + \frac{r_0}{a}\right)\right] + 2\frac{a}{\lambda_T} = -8^{1/2}\exp\left(\frac{1}{4\pi}\frac{\lambda_T^2}{a^2}(1 + r_0/a)\right) + 2\frac{a}{\lambda_T} < 0 \quad \text{for } a/\lambda_T < 1. \quad (17)$$

Thus, the binding energy term in (6) (if a bound state exists) is dominating over $2\frac{a}{\lambda_T}$ for small $a/\lambda_T$.

### B. Effective account for the compositeness of particles up to $(\lambda^3/v)^2$ terms

Let us now evaluate the second virial coefficient in the absence of an explicit interaction between quasibosons (composite bosons). Note that the partition function from which the second virial coefficient can be extracted, for the system of composite bosons within a general framework, was considered in [31]. Within our approach (which is both effective and efficient), however, the task of obtaining the virial coefficient(s) is completely tractable, leading, for the deformed Bose gas, to exact results.

The two-component quasibosons concerned here have the following creation and annihilation operators [12,14]

$$A_\alpha^\dagger = \sum_{\mu\nu}\Phi_\alpha^{\mu\nu}a_\mu^\dagger b_\nu^\dagger, \quad A_\alpha = \sum_{\mu\nu}\overline{\Phi}_\alpha^{\mu\nu}b_\nu a_\mu, \quad (18)$$

where $a_\mu^\dagger, b_\nu^\dagger$ and $a_\mu, b_\nu$ are, respectively, the creation and annihilation operators for the constituent fermions and the set of matrices $\Phi_\alpha^{\mu\nu}$ determine the quasiboson wave function. As a starting point we take the known general expression for the second virial coefficient [27]

$$V_2 = \frac{1}{2!V}[(\mathrm{Tr}_1 e^{-\beta H_1})^2 - \mathrm{Tr}_2 e^{-\beta H_2}]. \quad (19)$$

Here $\mathrm{Tr}_1$ denotes the trace over one-quasiboson states and $\mathrm{Tr}_2$ over the states of two quasibosons; $H_1$ and $H_2$ are, respectively, one- and two-quasiboson Hamiltonians. The distinction between the second virial coefficients for the ideal Bose and ideal Fermi gases is caused by the nilpotency of the fermionic creation operators and consequently by the nullification of the respective terms in $\mathrm{Tr}_2 e^{-\beta H_2}$ from (19). Analogously, in the case of bifermionic quasibosons the nonzero summands from $\mathrm{Tr}_2 e^{-\beta H_2}$ are determined by the condition $|(A_\alpha^\dagger)^2|0\rangle|^2 \neq 0$. Let us calculate $|(A_\alpha^\dagger)^2|0\rangle|^2$:

$$\begin{aligned} &|(A_\alpha^\dagger)^2|0\rangle|^2 \\ &= \sum_{\mu_1\mu_2\cdots\nu_1'\nu_2'} \langle 0|b_{\nu_2'}a_{\mu_2'}b_{\nu_1'}a_{\mu_1'}\overline{\Phi_\alpha^{\mu_1'\nu_1'}}\,\overline{\Phi_\alpha^{\mu_2'\nu_2'}}\Phi_\alpha^{\mu_1\nu_1}\Phi_\alpha^{\mu_2\nu_2}a_{\mu_1}^\dagger b_{\nu_1}^\dagger \\ &\quad \times a_{\mu_2}^\dagger b_{\nu_2}^\dagger|0\rangle \\ &= 2\sum_{\mu_1\neq\mu_2,\nu_1\neq\nu_2}\left(\left|\Phi_\alpha^{\mu_1\nu_1}\right|^2\left|\Phi_\alpha^{\mu_2\nu_2}\right|^2 - \overline{\Phi_\alpha^{\mu_1\nu_2}\Phi_\alpha^{\mu_2\nu_1}}\Phi_\alpha^{\mu_1\nu_1}\Phi_\alpha^{\mu_2\nu_2}\right) \\ &= 2[1-\mathrm{Tr}(\Phi_\alpha\Phi_\alpha^\dagger\Phi_\alpha\Phi_\alpha^\dagger)]. \end{aligned} \quad (20)$$

The traces in (19) are calculated as follows:

$$\mathrm{Tr}_1 e^{-\beta H_1} = \sum_{\mathbf{k}_1 n_1}\langle 0|A_{\mathbf{k}_1 n_1}e^{-\beta H_1}A_{\mathbf{k}_1 n_1}^\dagger|0\rangle = \sum_{\mathbf{k}_1 n_1} e^{-\beta\varepsilon_{\mathbf{k}_1 n_1}}, \quad (21)$$

$$\begin{aligned} &\mathrm{Tr}_2 e^{-\beta H_2} \\ &= \frac{1}{2}\sum_{(\mathbf{k}_1 n_1)\neq(\mathbf{k}_2 n_2)}\langle 0|A_{\mathbf{k}_2 n_2}A_{\mathbf{k}_1 n_1}e^{-\beta H_2}A_{\mathbf{k}_1 n_1}^\dagger A_{\mathbf{k}_2 n_2}^\dagger|0\rangle \\ &\quad + \frac{1}{|(A_{\mathbf{k}_1 n_1}^\dagger)^2|0\rangle|^2}\sum_{\mathbf{k}_1 n_1}{}'\langle 0|(A_{\mathbf{k}_1 n_1})^2 e^{-\beta H_2}(A_{\mathbf{k}_1 n_1}^\dagger)^2|0\rangle \\ &= \frac{1}{2}\left(\sum_{\mathbf{k}_1 n_1} e^{-\beta\varepsilon_{\mathbf{k}_1 n_1}}\right)^2 - \frac{1}{2}\sum_{\mathbf{k}_1 n_1} e^{-2\beta\varepsilon_{\mathbf{k}_1 n_1}} + \sum_{\mathbf{k}_1 n_1}{}' e^{-2\beta\varepsilon_{\mathbf{k}_1 n_1}}. \end{aligned} \quad (22)$$

Here $\mathbf{k}_{1,2}$ is the momentum quantum number, $n_{1,2}$ contains all the other quasibosonic quantum numbers, $\varepsilon_{\mathbf{k}_1 n_1}$ is the energy of the quasiboson in the state $|\mathbf{k}_1 n_1\rangle$, and the prime in $\sum'$ implies the summation over all the modes $(\mathbf{k},n)$ for which $(A_{\mathbf{k},n}^\dagger)^2|0\rangle \neq 0$. Substituting (21) and (22) in (19) and splitting $\varepsilon_{\mathbf{k}n}$ into kinetic energy $\frac{\hbar^2\mathbf{k}^2}{2m}$ and internal energy $\varepsilon_n^{\mathrm{int}}$ as $\varepsilon_{\mathbf{k}n} = \frac{\hbar^2\mathbf{k}^2}{2m} + \varepsilon_n^{\mathrm{int}}$, we obtain

$$\begin{aligned} V_2(T) &= \frac{1}{2^{5/2}}\sum_n e^{-2\beta\varepsilon_n^{\mathrm{int}}} \\ &\quad - \frac{\lambda_T^3}{V}\sum_{\mathbf{k}n}{}'\exp\left[-2\beta\left(\frac{\hbar^2\mathbf{k}^2}{2m}+\varepsilon_n^{\mathrm{int}}\right)\right]. \end{aligned} \quad (23)$$

If for all the $(\mathbf{k},n)$ modes $(A_{\mathbf{k},n}^\dagger)^2|0\rangle \neq 0$, then performing the summation over $\mathbf{k}$ according to $\sum_{\mathbf{k}} e^{-2\beta(\hbar^2\mathbf{k}^2/2m)} = 2^{-3/2}V/\lambda_T^3$ we obtain

$$V_2(T) - V_2^{(0)} = -\frac{1}{2^{5/2}}\left(\sum_n e^{-2\beta\varepsilon_n^{\mathrm{int}}} - 1\right). \quad (24)$$

On the other hand, in the deformed case we have the (exact) result [15] [see also Eq. (3)], i.e.,

$$V_2^{(\tilde\mu,q)} - V_2^{(0)}\Big|_{q=1} = \frac{2-\varphi_{\tilde\mu,q}(2)}{2^{7/2}}\Bigg|_{q=1} = \frac{\tilde\mu}{2^{5/2}} \quad (25)$$

from which, after juxtaposing, according to our interpretation, with (24) we arrive at

$$\tilde\mu = \tilde\mu\left(\varepsilon_n^{\mathrm{int}},\Phi_\alpha^{\mu\nu},T\right) = 1 - \sum_n e^{-2\beta\varepsilon_n^{\mathrm{int}}} \quad (26)$$

(the dependence on $\Phi_\alpha^{\mu\nu}$ is retained for the general case). As can be seen now, the obtained difference (24) is mainly related to the internal energy of a quasiboson, not to its (nonbosonic) commutation relations.

The structure function $\varphi_{\tilde\mu,q}(N)$ with $q = q(a,r_0,T)$ and $\tilde\mu = \tilde\mu(\varepsilon_n^{\mathrm{int}},\Phi_\alpha^{\mu\nu},T)$ is chosen for the goal of the effective account (in certain approximations) for the factors of interaction and of the composite structure of particles of a gas. Let us emphasize that the direct microscopic treatment may lead to a quite different relation between the second virial coefficient incorporating both factors (interaction and compositeness) and the virial coefficients involving only one nontrivial factor. The functional composition as in (1) may not already hold, nevertheless, the linear part of the Taylor expansion of $V_2^{(\tilde\mu,q)}$ in small $\epsilon = q-1$ and $\tilde\mu$ may coincide with the corresponding part found from the microscopic treatment. It is clear that the modification of deformation according to (16) may lead to a quite different dependence of deformation parameters on the characteristics of the interaction and compositeness.

Let us note that the major deformation structure function $\varphi$ in (16) is a general one. The choice $\varphi(z\frac{\partial}{\partial z}) = \varphi_{\tilde\mu,q}(z\frac{\partial}{\partial z})$ results in the formulas (14) and (25) for the virial coefficient $V_2$. Clearly, other choices for $\varphi$ in (16) will result in another form of respective virial coefficient $V_2$ and the respective temperature dependence.

## III. MODIFICATION OF THE DERIVATIVE $z\frac{d}{dz}$ AIMED TO YIELD TEMPERATURE-DEPENDENT VIRIAL COEFFICIENTS

As already mentioned, we can obtain temperature-dependent deformed (i.e., within the deformation-based approach) virial coefficients, say, by performing the extension of the deformed derivative [see (16)]. The functions $\varphi$, $\chi$, and $\rho$ should not depend on the temperature. The term $g(\beta)\rho(z\frac{\partial}{\partial z})$ is introduced in order to reflect the ambiguity in the (left or right) position of $\partial/\partial\beta$, i.e., to cover the terms such as $\frac{\partial}{\partial\beta}\chi(z\frac{\partial}{\partial z})$. This can be verified by means of the commutation relation

$$[\partial/\partial\beta, f(z\partial/\partial z)] = -\beta^{-1}(z\partial/\partial z)f'(z\partial/\partial z). \quad (27)$$

The noncommutativity of the derivatives $\partial/\partial\beta$ and $z\partial/\partial z$ is observed after presenting $z\partial/\partial z$ as $\beta^{-1}\partial/\partial\mu$, where $\mu$ is the chemical potential (recall that $z = e^{\beta\mu}$).

Applying the deformed derivative (16) to the known expansion for the partition function

$$\ln Z^{(0)}(z,V,T)=\frac{V}{\lambda_T^3}\sum_{n=1}^{\infty}\frac{z^n}{n^{5/2}},$$

we obtain the series for the deformed (represented by a tilde) total number of particles

$$\begin{aligned}\tilde{N}&=z\tilde{\mathcal{D}}_z\ln Z^{(0)}(z,V,T)\\&=\frac{V}{\lambda_T^3}\sum_{n=1}^{\infty}\{\varphi(n)+\beta^{-1}[n\chi(n)\ln z-3/2\chi(n)+n\chi'(n)]\\&\quad+g(\beta)\rho(n)\}\frac{z^n}{n^{5/2}}.\end{aligned}\tag{28}$$

The deformed partition function is then recovered as

$$\begin{aligned}\ln\tilde{Z}&=(d/dz)^{-1}\tilde{N}\\&=\frac{V}{\lambda_T^3}\sum_{n=1}^{\infty}\{\varphi(n)+\beta^{-1}[n\chi(n)\ln z-5/2\chi(n)+n\chi'(n)]\\&\quad+g(\beta)\rho(n)\}\frac{z^n}{n^{7/2}}.\end{aligned}\tag{29}$$

Expanding fugacity as $z=z_0+z_1\frac{\lambda_T^3}{\tilde{v}}+z_2(\frac{\lambda_T^3}{\tilde{v}})^2+\cdots$, denoting by $\tilde{v}=\frac{\tilde{N}}{V}$ the deformed specific volume, and remembering that $z_i=z_i(T)$, $i=0,1,\ldots$, after substituting the resulting expansion into (28) we obtain the relation

$$\frac{\lambda_T^3}{\tilde{v}}=\sum_{n=0}^{\infty}R_n(T;\varphi,\chi,\rho)\left(\frac{\lambda_T^3}{\tilde{v}}\right)^n,\tag{30}$$

where the coefficients at the same powers of $\frac{\lambda_T^3}{\tilde{v}}$ on the left- and right-hand sides should be

$$\begin{aligned}R_0&\equiv\sum_{n=1}^{\infty}\{\varphi(n)+\beta^{-1}[n\chi(n)\ln z_0-3/2\chi(n)+n\chi'(n)]\\&\quad+g(\beta)\rho(n)\}\frac{z_0^n}{n^{5/2}}=0,\end{aligned}\tag{31}$$

$$\begin{aligned}R_1&\equiv z_1\sum_{n=1}^{\infty}\{\varphi(n)+\beta^{-1}[n\chi(n)\ln z_0-1/2\chi(n)+n\chi'(n)]\\&\quad+g(\beta)\rho(n)\}\frac{z_0^{n-1}}{n^{3/2}}=1,\end{aligned}\tag{32}$$

$$\begin{aligned}R_2&\equiv\left(\frac{z_2}{z_1}-\frac{1}{2}\frac{z_1}{z_0}\right)P_1+\frac{1}{2}\frac{z_1^2}{z_0}\sum_{n=1}^{\infty}\{\varphi(n)+\beta^{-1}[n\chi(n)\ln z_0\\&\quad+1/2\chi(n)+n\chi'(n)]+g(\beta)\rho(n)\}\frac{z_0^{n-1}}{n^{1/2}}=0,\end{aligned}\tag{33}$$

etc. Similarly, for the deformed equation of state $\frac{\tilde{P}V}{k_BT}=\ln\tilde{Z}$, using (29) we find the virial $\lambda_T^3/\tilde{v}$ expansion

$$\frac{\tilde{P}}{k_BT}=\frac{1}{\lambda_T^3}\tilde{V}_0(T;\varphi,\chi,\rho)+\tilde{v}^{-1}\sum_{n=1}^{\infty}\tilde{V}_n(T;\varphi,\chi,\rho)\left(\frac{\lambda_T^3}{\tilde{v}}\right)^{n-1}\tag{34}$$

with virial coefficients

$$\begin{aligned}\tilde{V}_0&\equiv\sum_{n=1}^{\infty}\{\varphi(n)+\beta^{-1}[n\chi(n)\ln z_0-5/2\chi(n)+n\chi'(n)]\\&\quad+g(\beta)\rho(n)\}\frac{z_0^n}{n^{7/2}}=0,\end{aligned}\tag{35}$$

$$\begin{aligned}\tilde{V}_1&\equiv z_1\sum_{n=1}^{\infty}\{\varphi(n)+\beta^{-1}[n\chi(n)\ln z_0-3/2\chi(n)+n\chi'(n)]\\&\quad+g(\beta)\rho(n)\}\frac{z_0^{n-1}}{n^{5/2}}=1,\end{aligned}\tag{36}$$

$$\begin{aligned}\tilde{V}_2&\equiv\left(\frac{z_2}{z_1}-\frac{1}{2}\frac{z_1}{z_0}\right)\tilde{V}_1+\frac{1}{2}\frac{z_1^2}{z_0}\sum_{n=1}^{\infty}\{\varphi(n)+\beta^{-1}[n\chi(n)\ln z_0\\&\quad-1/2\chi(n)+n\chi'(n)]+g(\beta)\rho(n)\}\frac{z_0^{n-1}}{n^{3/2}},\end{aligned}\tag{37}$$

etc. The equalities in (35) and (36) are imposed in order that the virial expansion (34) reproduces the corresponding limit of the classical ideal gas. One of the solutions of (31) and (35) is $z_0=0$. Let us dwell on this case. The deformed equation of state $\tilde{P}=\tilde{P}(\lambda_T^3/\tilde{v})$ [see (34)] can be written in the implicit parametric form [see (28) and (29)]

$$\begin{aligned}\frac{\lambda_T^3}{\tilde{v}}&=\sum_{n=1}^{\infty}\{\varphi(n)+\beta^{-1}[n\chi(n)\ln z-3/2\chi(n)+n\chi'(n)]\\&\quad+g(\beta)\rho(n)\}\frac{z^n}{n^{5/2}},\end{aligned}\tag{38}$$

$$\begin{aligned}\frac{\tilde{P}}{k_BT}&=\frac{1}{\lambda_T^3}\sum_{n=1}^{\infty}\{\varphi(n)+\beta^{-1}[n\chi(n)\ln z-5/2\chi(n)+n\chi'(n)]\\&\quad+g(\beta)\rho(n)\}\frac{z^n}{n^{7/2}}.\end{aligned}\tag{39}$$

The value $z=z_0=0$ corresponds to $\lambda_T^3/\tilde{v}|_{z=0}=0$ as $z\ln z\to 0$ at $z\to 0$ in (38). Consider the first derivative of (39) by $\lambda_T^3/\tilde{v}$, namely,

$$\begin{aligned}\frac{\partial(\tilde{P}/k_BT)}{\partial(\lambda_T^3/\tilde{v})}&=\frac{\partial(\tilde{P}/k_BT)/\partial z}{\partial(\lambda_T^3/\tilde{v})/\partial z}=\frac{1}{\lambda_T^3}\frac{\sum_{n=1}^{\infty}\{\varphi(n)+\beta^{-1}[(n\ln z-\frac{3}{2})\chi(n)+n\chi'(n)]+g(\beta)\rho(n)\}(z^n/n^{5/2})}{\sum_{n=1}^{\infty}\{\varphi(n)+\beta^{-1}[(n\ln z-\frac{1}{2})\chi(n)+n\chi'(n)]+g(\beta)\rho(n)\}(z^n/n^{3/2})}\\&\underset{z\to 0}{\longrightarrow}\frac{1}{\lambda_T^3}\frac{\varphi(1)+\beta^{-1}[\chi(1)(\ln z-\frac{3}{2})+\chi'(1)]+g(\beta)\rho(1)}{\varphi(1)+\beta^{-1}[\chi(1)(\ln z-\frac{1}{2})+\chi'(1)]+g(\beta)\rho(1)}\underset{z\to 0}{\longrightarrow}\frac{1}{\lambda_T^3}.\end{aligned}$$

For the second derivative we obtain

$$\frac{\partial^2(\tilde{P}/k_BT)}{\partial(\lambda_T^3/\tilde{v})^2} = \left(\frac{\partial^2(\tilde{P}/k_BT)}{\partial z^2}\frac{\partial(\lambda_T^3/\tilde{v})}{\partial z} - \frac{\partial(\tilde{P}/k_BT)}{\partial z}\frac{\partial^2(\lambda_T^3/\tilde{v})}{\partial z^2}\right)\Big/\left[\partial(\lambda_T^3/\tilde{v})/\partial z\right]^3$$
$$= \lambda_T^{-3}\{\varphi(1)+\beta^{-1}[\chi(1)(\ln z-1/2)+\chi'(1)]+g(\beta)\rho(1)\}^{-3}\{\beta^{-2}\chi^2(1)z^{-1}-2^{-3/2}\beta^{-2}\chi(2)\chi(1)\ln^2 z$$
$$+2^{-5/2}\beta^{-1}[-\varphi(2)\chi(1)+5/2\beta^{-1}\chi(2)\chi(1)-g(\beta)\chi(1)\rho(2)-2\varphi(1)\chi(2)-2g(\beta)\chi(2)\rho(1)-2\beta^{-1}\chi(1)\chi'(2)$$
$$-2\beta^{-1}\chi'(1)\chi(2)]\ln z-2^{-5/2}[\varphi(2)\varphi(1)-5/2\beta^{-1}\varphi(2)\chi(1)+g(\beta)\varphi(2)\rho(1)+5/2\beta^{-1}\varphi(1)\chi(2)$$
$$+g(\beta)\varphi(1)\rho(2)-17/4\beta^{-2}\chi(1)\chi(2)+5/2\beta^{-1}g(\beta)\chi(2)\rho(1)-5/2\beta^{-1}g(\beta)\chi(1)\rho(2)+g^2(\beta)\rho(1)\rho(2)$$
$$+\beta^{-1}\varphi(2)\chi'(1)+2\beta^{-1}\varphi(1)\chi'(2)+2\beta^{-2}\chi'(1)\chi'(2)+5/2\beta^{-2}\chi(2)\chi'(1)-4\beta^{-2}\chi'(2)\chi(1)$$
$$+2\beta^{-1}g(\beta)\chi'(2)\rho(1)+\beta^{-1}g(\beta)\chi'(1)\rho(2)]\}. \tag{40}$$

As it is seen from this last expression, for the second deformed virial coefficient $\tilde{V}_2 = \frac{1}{2}\lambda_T^3\frac{\partial^2(\tilde{P}/k_BT)}{\partial(\lambda_T^3/\tilde{v})^2}$ to be finite at $z \to z_0 = 0$ we have to require $\chi(1) = 0$. Then

$$\tilde{V}_2 = -\frac{2\beta^{-1}\chi(2)\ln z+\varphi(2)+\beta^{-1}\left[\frac{5}{2}\chi(2)+2\chi'(2)\right]+g(\beta)\rho(2)}{2^{7/2}[\varphi(1)+\beta^{-1}\chi'(1)+g(\beta)\rho(1)]^2}.$$

Likewise, the requirement of finiteness leads to $\chi(2) = 0$ and thus to

$$\tilde{V}_2 = -\frac{1}{2^{7/2}}\frac{\varphi(2)+2\beta^{-1}\chi'(2)+g(\beta)\rho(2)}{[\varphi(1)+\beta^{-1}\chi'(1)+g(\beta)\rho(1)]^2}. \tag{41}$$

The obtained general formula involves dependence on the choice of deformation [through the values $\varphi(k)$, $\chi'(k)$, and $\rho(k)$, $k=\overline{1,2}$, of the structure functions $\varphi$, $\chi$, and $\rho$ from (16)].

In some situations it may be more convenient to deal with virial $z$ expansions. Say, for the total number of particles we have

$$N(z,V,T) = \frac{V}{\lambda_T^3}\left[z+2\left(\frac{1}{2^{5/2}}-2\frac{a}{\lambda_T}\right)z^2+\cdots\right]$$
$$\simeq z\tilde{\mathcal{D}}_z\left\{\frac{V}{\lambda_T^3}\left(z+\frac{1}{2^{5/2}}z^2+\cdots\right)\right\}. \tag{42}$$

Using this last expression we can compare the result of the microscopic treatment with the action of the deformation. On the right-hand side of (42) we have exactly the right-hand side of (28). Taking there $\chi(n) = 0$, $n = 1,2,\ldots$ (as the simplest variant to exclude singularity at $z \to z_0 = 0$), and comparing the first two terms with the corresponding ones on the left-hand side of (42) we arrive at the relations

$$\varphi(1)+g(\beta)\rho(1) = 1, \tag{43}$$

$$\varphi(2)+g(\beta)\rho(2) = 2(1-2^{7/2}a/\lambda_T). \tag{44}$$

From these we find

$$g(\beta) = 2^{9/2}(a/\lambda_{T_0}-a/\lambda_T)\rho^{-1}(2),$$
$$\varphi(1) = 1, \quad \rho(1) = 0, \tag{45}$$

where $T_0$ is defined from $\varphi(2) = 2(1-2^{7/2}a/\lambda_{T_0})$.

The first example of the respective deformed derivative is

$$\left[z\frac{d}{dz}\right]_q + (\lambda_{T_0}/\lambda_T - 1)(q-1)\left(z\frac{d}{dz}-1\right), \tag{46}$$

where

$$q = 1-2^{9/2}a/\lambda_{T_0}. \tag{47}$$

Note that the form of the latter is very natural: It shows that the extent (magnitude) $1-q$ of deformation is just proportional to the scattering length $a$ divided by $\lambda_{T_0}$.

A more general case is the $(\tilde{\mu},q)$-deformed one, for which

$$z\tilde{\mathcal{D}}_z = \varphi_{\tilde{\mu},q}\left(z\frac{\partial}{\partial z}\right)+(\lambda_{T_0}/\lambda_T-1)(q-1)\left(z\frac{d}{dz}-1\right)$$
$$-2\frac{\sum_n e^{-2\beta_{T_0'}\varepsilon_n^{\rm int}}-e^{-2\beta_T\varepsilon_n^{\rm int}}}{1-\sum_n e^{-2\beta_{T_0'}\varepsilon_n^{\rm int}}}\tilde{\mu}\left(z\frac{d}{dz}-1\right), \tag{48}$$

$$q = 1-2^{9/2}a/\lambda_{T_0}, \quad \tilde{\mu} = 1-\sum_n e^{-2\beta_{T_0'}\varepsilon_n^{\rm int}}. \tag{49}$$

The comparison of Eqs. (47) and (49) shows the difference between the two situations. In the former, only the interaction is effectively taken into account, while in the latter, more general, case both factors, the interaction and compositeness, are involved.

We remark that besides the modified deformed derivative (16), its further extensions may be considered, e.g.,

$$z\tilde{\mathcal{D}}_z \equiv \varphi\left(z\frac{\partial}{\partial z}\right)+\chi\left(z\frac{\partial}{\partial z}\right)h\left(\frac{\partial}{\partial\beta}\right)+g(\beta)\rho\left(z\frac{\partial}{\partial z}\right)+\cdots. \tag{50}$$

For the commutator $[h(\partial/\partial\beta), f(z\partial/\partial z)]$ we obtain

$$\left[h\left(\frac{\partial}{\partial\beta}\right), f\left(z\frac{\partial}{\partial z}\right)\right] = \sum_{k=1}^{\infty}\frac{\beta^{-k}}{k!}Q_k\left(z\frac{\partial}{\partial z}\right)h^{(k)}\left(\frac{\partial}{\partial\beta}\right), \tag{51}$$

where $Q_k(x) \equiv (-1)^k x^{-k}(x^2\frac{d}{dx})^k f(x)$. So, since the ambiguity in the position of $h(\frac{\partial}{\partial\beta})$ is still present, the terms with higher derivatives $h^{(i)}(\frac{\partial}{\partial\beta})$ may enter the ellipsis in (50).

## IV. CONCLUSION

In the analysis of the second virial coefficient of the nonideal Bose gas from the viewpoint of the role of important factors of nonideality such as the interaction of particles and their compositeness, we have found an explicit expression for $V_2 - V_2^{(0)}$ given through the scattering length $a$ and the effective radius $r_0$ of the interaction. The latter result, when compared to the virial coefficient (14) of the deformed Bose gas, has led us to one of our main formulas (15). Though the dependence of the deformation parameter on $a$ and $r_0$ is rather expected, the $T$ dependence encountered is somewhat of a surprise. With the goal of finding a reasonable explanation and to justify the appearance of the $T$ dependence in $q$, we developed the appropriate extension of the very starting point of the procedure to deform the thermodynamics. The main step consists in adopting the modified derivative (16) or its more general version (50) involving additional structure functions. In a similar way, when we deal with compositeness and derive the relation (26) for $\tilde{\mu} = \tilde{\mu}(\varepsilon_n^{\rm int},\Phi_\alpha^{\mu\nu},T)$, the appearance of the temperature dependence in the effective description can be described by the use of Eq. (16) as well, however with different structure functions involved. Besides the above-considered extension of the deformation, the possibility remains to obtain another consistent deformation of a Bose gas with temperature-dependent deformed virial coefficients.

As a further direction of research let us point out the unified microscopic treatment of the second virial coefficient when both factors of the compositeness and interaction are present simultaneously. Note that the corresponding dependence $V_2 = V_2(a,r_0,\varepsilon_n^{\rm int},\ldots,T)$ may be different from that obtained above and thus leads to some other structure functions of deformation.


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## APPENDIX: EXAMPLES OF THE INTERACTION POTENTIAL

Here we present a number of examples of the interaction potentials and respective phase shifts or scattering length or effective radius through which the second virial coefficient in (4) or (6) is expressed.

### 1. Hard-sphere interaction potential

The hard-sphere interaction potential is given by

$$U(r) = \begin{cases} +\infty, & r < D \\ 0, & r > D. \end{cases} \tag{A1}$$

Direct calculation using (4) yields (see, e.g., [27])

$$V_2 - V_2^{(0)} = \begin{cases} 2\frac{D}{\lambda_T} + \frac{10\pi^2}{3}\left(\frac{D}{\lambda_T}\right)^5 + \cdots, & l = 0,2 \quad \text{(Bose case)} \\ 6\pi\left(\frac{D}{\lambda_T}\right)^3 - 18\pi^2\left(\frac{D}{\lambda_T}\right)^5 + \cdots, & l = 1 \quad \text{(Fermi case).} \end{cases} \tag{A2}$$

### 2. Constant repulsive potential

The constant repulsive potential is defined by

$$U(r) = \begin{cases} U_0 > 0, & r < R \\ 0, & r > R. \end{cases} \tag{A3}$$

The respective $l = 0$ phase shift and scattering length are then given as (for this and further examples see, e.g., [29])

$$\delta_0 = kR\left(\frac{\tanh K_0R}{K_0R} - 1\right),$$
$$a = R\left(1 - \frac{\tanh K_0R}{K_0R}\right), \tag{A4}$$

where $K_0^2 = \frac{2mU_0}{\hbar^2}$ and $r_0 = 0$ [see also Eq. (8)]. The difference $V_2 - V_2^{(0)}$ can be calculated using (6) as in the previous case (this concerns also the rest of the examples).

### 3. Square-well potential

The square-well potential has the definition

$$U(r) = \begin{cases} -U_0 < 0, & r < R \\ 0, & r > R. \end{cases} \tag{A5}$$

The scattering length and effective radius are equal to

$$a = -R\left(\frac{\tan K_0R}{K_0R} - 1\right),$$
$$r_0 = R\left(1 - \frac{1}{K_0^2Ra} - \frac{R^2}{3a^2}\right), \tag{A6}$$

with $K_0$ defined as in the previous example.

### 4. Anomalous scattering potential

In this case

$$\frac{2m}{\hbar^2}U(r) = \begin{cases} -K_1^2, & 0 \leqslant r < r_1 \\ +K_0^2, & r_1 \leqslant r \leqslant R \\ 0, & R \leqslant r. \end{cases} \tag{A7}$$

For the $l = 0$ phase shift we have

$$\delta_0 = -kR + \arctan\left\{\frac{kR}{\kappa R}\,\frac{\tanh\kappa(R-r_1) + \kappa r_1(\tan Kr_1/Kr_1)}{1 + \kappa r_1(\tan Kr_1/Kr_1)\tanh\kappa(R-r_1)}\right\}, \tag{A8}$$

where $\kappa^2 = K_0^2 - k^2$ and $K^2 = K_1^2 + k^2$.

#### 5. Scattering resonances

The corresponding potential is

$$U(r) = \frac{\hbar^2}{2m}\frac{\Omega}{R}\delta(r-R). \quad \text{(A9)}$$

The phase shift $\delta_0$ is given as

$$\tan(kR+\delta_0) = \frac{\tan kR}{1+\Omega(\tan kR/kR)}. \quad \text{(A10)}$$

#### 6. Modified Pöschl-Teller potential

The potential is

$$U(r) = -\frac{\hbar^2\alpha^2}{2m}\frac{\lambda(\lambda-1)}{\cosh^2\alpha r}. \quad \text{(A11)}$$

The respective phase shift $\delta_0$ reads

$$\delta_0 = \arctan\frac{2\tilde{k}}{\lambda} - \arctan\left(\cot\frac{\pi\lambda}{2}\text{th}\,\pi\tilde{k}\right) + \sum_{n=1}^{\infty}\left\{\arctan\frac{2\tilde{k}}{\lambda+n} - \arctan\frac{2\tilde{k}}{n}\right\}, \quad \tilde{k} = \frac{k}{2\alpha}. \quad \text{(A12)}$$

#### 7. Inverse power repulsive potential

The inverse power repulsive potential

$$U(r) = \frac{\hbar^2}{2m}\frac{g^2}{r_0^2}\left(\frac{r_0}{r}\right)^n, \quad \text{(A13)}$$

for which the scattering length and the wave function of the lowest state (through which the effective radius is expressed) are respectively given as

$$a = r_0\frac{\Gamma(1-1/2\eta)}{\Gamma(1+1/2\eta)}\left(\frac{g}{2\eta}\right)^{1/\eta}, \quad \eta = \frac{n-2}{2}, \quad \text{(A14)}$$

$$\chi_0 = C\sqrt{r/r_0}K_{1/2\eta}\left(\frac{g}{\eta}(r/r_0)^{-\eta}\right), \quad \text{(A15)}$$

where $K_\nu(z)$ is the modified Hankel function.

#### 8. First Born approximation

Finally, let us present the expression for the $l$th phase shift in the first Born approximation

$$\delta_l \simeq -\frac{2mk}{\hbar^2}\int_0^\infty U(r)[j_l(kr)]^2 r^2 dr, \quad \text{(A16)}$$

where $j_l$ is the spherical Bessel function. However, its applicability is quite restricted and the respective validity conditions reduce to the ones on $U(r)$ of the first Born approximation.


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