Detection of weak signals in memory thermal baths

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The nonlinear relaxation time and the statistics of the first passage time distribution in connection with the quasideterministic approach are used to detect weak signals in the decay process of the unstable state of a Brownian particle embedded in memory thermal baths. The study is performed in the overdamped approximation of a generalized Langevin equation characterized by an exponential decay in the friction memory kernel. A detection criterion for each time scale is studied: The first one is referred to as the receiver output, which is given as a function of the nonlinear relaxation time, and the second one is related to the statistics of the first passage time distribution.

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I. INTRODUCTION

Since its introduction by Benzi *et al.* in 1981, stochastic resonance (SR) has been recognized as a paradigm for noise-induced effects in driven nonlinear dynamic systems. This phenomenon has been widely studied, from both the experimental and the theoretical points of view, in a variety of systems in fields as diverse as physics, chemistry, biology, and medicine [1–14]. For any of these systems operating in noisy environments and subject to a periodic modulating signal—so weak as to be normally undetectable—the mechanism of SR appears when both the weak signal and noise enter in resonance, increasing the detectability of the weak signal and the transmission efficiency of information. Hence, SR is a phenomenon capable of detecting and transmitting efficiently information embedded in weak signals stemming from nonlinear systems due to the presence of random noise. Practically all of the above mentioned phenomena are essentially described in terms of the one-dimensional Brownian motion in a potential field within the high damping (diffusive) regime, thus admitting a description in terms of the Fokker-Planck, master, or Langevin-type equations. The phenomenon of SR has also been generalized to the Lévy flights [15,16] and recently observed in systems characterized by a multiplicative noise with long jumps in the context of generalized Langevin equation [17].

Alternative to SR, there exists another physical mechanism capable of detecting weak signals in the study of relaxation of nonequilibrium phenomena: It consists in the amplification of a weak external signal by means of the decay of an unstable state driven by stochastic fluctuations. To the best of the authors knowledge, the study of the detection of weak signals in the decay of unstable states was initiated in 1989 by Vemuri and Roy [18], who proposed that weak optical signals can be detected via the transient dynamics of a laser, much in the same way as the superregenerative detection in radar receivers. The physical idea behind the detection of weak signals in the decay of unstable states is that weak signals are greatly amplified when used to trigger the decay process. The criterion proposed by Vemuri and Roy is referred to as receiver output, which has been shown to be sensitive to the presence of the weak signal. The authors' idea was immediately corroborated experimentally by measurements of the statistics of the initiation time of an argon laser under the influence of an attenuated He-Ne laser which produces the injected signal [19]. Alternatively to those works, another theoretical criterion to detect weak optical signals in the switch-on process of a laser was given in [20] in terms of the statistics of the first passage time distribution. The criterion allows us to find a critical value of certain parameter denoted as β for which the reduction of the mean first passage time (MFPT) is greater or of the same order that the maximum variance. Furthermore, the study given by Vemuri and Roy was connected with the nonlinear relaxation times (NLRTs) to detect also weak optical signals in the switch-on process of a laser [21]. Recently, an alternative physical mechanism to detect weak electrical signals was given. It deals with the decay process of a charged Brownian particle under the action of constant crossed electric and magnetic fields [22]. An interesting and surprising aspect of this physical system is that its dynamic behavior is very similar to that followed by the aforementioned laser system. The mechanism for the detection process in this case is as follows. The particle is initially located on the unstable state of the potential. Once the decay process is initiated due to the internal fluctuations, a weak external signal is then injected, thus accelerating the decay process. The process suggests also the amplification of the weak signal in order to be detected. The amplitude of the external signal is less or of the same order than the noise, which is also small. After the initial cooperative effect between both the noise and the weak signal the dynamics is dominated by the potential force.

It is worth commenting that the study of relaxation processes of nonequilibrium phenomena through the MFPT has been formulated in the context of Langevin, Fokker-Planck, or master equations to Markovian as well as non-Markovian processes [20–46]. In particular, the study of the decay process of the unstable states have been focused on different descriptions,

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namely, the evolution of the statistical moments of the relevant variables in terms of Fokker-Planck equations [25,28], the inverse probability current also in the context of Fokker-Planck equation [29,30], or the study of time evolution of averages in dynamical systems [31,33,34]. The works in [29–34] provide a good description in the study of such transient stochastic dynamics. However, the one which has proved to be the most appropriate in the study of the detection of weak signals in the decay of an unstable state is the so-called quasideterministic (QD) approach [20–24], due to the following reasons. It has been used appropriately in connection with the MFPT and NLRT in the time characterization of such a decay process in the physical situation commented above. It gives a precise physical picture of the mechanism responsible for the decay of the unstable state. The physical mechanism is twofold: Small fluctuations change the initial condition in the surrounding of the unstable state, and afterwards the deterministic motion drives the system out of this state. This approach provides a simple way to deal with arbitrary nonlinear unstable potentials without using a Fokker-Planck formulation. On the other hand, all the works related to the detection of weak signals in the laser system and in the decay of Brownian particles are basically formulated using the standard Langevin equation. That is, the noise term in both cases is considered as the usual thermal Gaussian white noise, a property of the Markovian processes. Furthermore, in the case of the Brownian charged particle the friction coefficient was considered as a constant quantity. As far as we know, the study of the detection process of weak signals in the decay of unstable states in thermal baths with memory has not yet been reported elsewhere, and this is precisely our goal in the present contribution. We consider a Brownian particle in a one-dimensional bistable potential embedded in a memory thermal bath characterized by a generalized Langevin equation (GLE) with an exponential decay in the friction memory kernel (Ornstein-Uhlenbeck process). It is well known that, for all non-Markovian processes characterized by a GLE with arbitrary friction memory kernel, the fluctuation-dissipation relation holds [47,48], which allows the system to reach the equilibrium state. The detection process is studied in terms of the two aforementioned time scales, which characterize the decay process of the unstable state in the overdamped approximation of the GLE. In first place we characterize the decay process of the unstable state through the NLRT in connection with the QD approach. Then the receiver output is calculated as a function of the time scale. To apply the second criterion in terms of the MFPT the decay of the unstable state is characterized through the statistics of this passage time using also the QD approach.

Our work is structured as follows. In Sec. II we present the GLE for the Brownian particle in a one-dimensional bistable potential profile. Section III focuses on the calculation of the NLRT for arbitrary nonlinear potential profile using the QD approach. The receiver output is then calculated as a function of the NLRT for the bistable potential. In Sec. IV we calculate the statistics of the passage time distribution and apply the corresponding criterion for the detection process of weak signals. The conclusions of our work are given in Sec. V and two appendixes complement the explicit calculations.

II. GENERALIZED LANGEVIN EQUATION

We consider a Brownian particle of mass m embedded in a thermal bath of temperature T and located on the unstable state of a bistable potential $V(x) = -(a_0/2)x^2 + (b_0/4)x^4$, with $a_0, b_0 > 0$. The GLE for the Brownian particle embedded in a non-Markovian thermal bath and in the presence of a constant external force F_e can be written as

$$\dot{x} = v, \tag{1}$$

$$m\ddot{x} = a_0 x - b_0 x^3 + F_e - \int_0^t \gamma(t-t')\dot{x}(t')dt' + f(t), \tag{2}$$

where $\gamma(t-t')$ is the friction memory kernel, $f(t)$ the Gaussian fluctuating force with zero mean value $\langle f(t)\rangle = 0$ that also satisfies the fluctuation-dissipation relation with a correlation function $\langle f(t)f(t')\rangle = k_B T\gamma(t-t')$, where k_B is the Boltzmann constant and T the temperature of the thermal bath. For non-Markovian dynamics the friction memory kernel is usually modeled as $\gamma(t-t') = (\gamma_0/\tau)e^{-|t-t'|/\tau}$, and so $\langle f(t)f(t')\rangle = (\gamma_0 k_B T/\tau)e^{-|t-t'|/\tau}$, which satisfies the Ornstein-Uhlenbeck process, τ being the noise correlation time (memory time of the non-Markovian dynamics) and γ_0 a constant (friction coefficient). For $\tau = 0$ the Ornstein-Uhlenbeck process reduces to one with a Gaussian white noise. By introducing the variable [42]

$$\eta(t) = -\frac{\gamma_0}{\tau}\int_0^t e^{-\frac{(t-t')}{\tau}} v(t')dt' + f(t), \tag{3}$$

with

$$f(t) = \frac{\sqrt{\lambda}}{\tau}\int_0^t e^{-\frac{(t-t')}{\tau}}\xi(t')dt', \tag{4}$$

and $\lambda = \gamma_0 k_B T$, the above GLE transforms into a set of three coupled stochastic differential equations,

$$\dot{x} = v, \tag{5}$$

$$m\ddot{x} = a_0 x - b_0 x^3 + F_e + \eta(t), \tag{6}$$

$$\dot{\eta} = -\frac{1}{\tau}\eta - \frac{\gamma_0}{\tau}\dot{x} + \frac{\sqrt{\lambda}}{\tau}\xi(t), \tag{7}$$

where $\xi(t)$ is a Gaussian white noise with zero mean value and a correlation function $\langle\xi(t)\xi(t')\rangle = 2\delta(t-t')$. Here we are interested in the characterization of the decay process out of the unstable state in the overdamped approximation (high friction limiting case) of Eqs. (6) and (7), which in this case reduce to

$$\left(1 - \frac{a_0}{\gamma_0}\tau\right)\dot{x} + 3\frac{b_0}{\gamma_0}\tau x^2\dot{x} = \frac{a_0}{\gamma_0}x - \frac{b_0}{\gamma_0}x^3 + \frac{F_e}{\gamma_0} + \frac{\lambda}{\gamma_0}\xi(t). \tag{8}$$

We also consider the dynamics for which the condition $(1 - \frac{a_0}{\gamma_0}\tau) \gg 3\frac{b_0}{\gamma_0}\tau x^2$ holds. In this case

$$\dot{x} = \bar{a}x - \bar{b}x^3 + \bar{F}_e + \bar{\lambda}\xi(t), \tag{9}$$

where $\bar{a} = a/(1-a\tau)$, $\bar{b} = b/(1-a\tau)$, $\bar{F}_e = F_e/\gamma_0(1-a\tau)$, and $\bar{\lambda} = \sqrt{\lambda}/\gamma_0(1-a\tau)$, with $a = a_0/\gamma_0$ and $b = b_0/\gamma_0$. It should be noticed that $a_0/b_0 = \bar{a}/\bar{b}$. Equation (9) represents an equivalent Markovian dynamics for the stochastic process, in the diffusive regime, described by Eq. (2) and reduces to a Markovian one by taking $\tau = 0$.

III. NLRT AND QD APPROACH

In this section we use the NLRT criterion and focus on the dynamic relaxation of the average $\langle x^2(t)\rangle$, where $\langle\cdots\rangle$ stands for the average taken on both the noise realizations $\xi(t)$ and the initial conditions, which are considered distributed with a specific probability distribution function. In the absence of the external force F_e, the quantity $\langle x^2(t)\rangle$ evolves from an initial value $\langle x^2(0)\rangle$ to its corresponding steady-state value $\langle x^2\rangle_{st}$. The NLRT is defined as [21,22,27,28]

$$\mathcal{T} = \int_0^\infty \frac{\langle x^2(t)\rangle - \langle x^2\rangle_{st}}{\langle x^2(0)\rangle - \langle x^2\rangle_{st}} dt. \tag{10}$$

This time scale, along with the QD approach, makes it possible to characterize not only the complete dynamical relaxation of Eq. (8) but also the relaxation processes associated with arbitrary nonlinear unstable potentials, the bistable potential being just a particular case. We introduce a more general definition of a deterministic nonlinear unstable state in terms of the scalar variable $\mathfrak{r} \equiv x^2$. The deterministic dynamics for this variable then reads as

$$\frac{d\mathfrak{r}}{dt} = \mathfrak{f}(\mathfrak{r}), \qquad \mathfrak{f}(\mathfrak{r}) = \frac{\mathfrak{r}(\mathfrak{r}_{st} - \mathfrak{r})}{C_0 + \mathfrak{r}g(\mathfrak{r})}, \tag{11}$$

where $C_0 = \mathfrak{r}_{st}/2\bar{a}$ is the steady-state value and $g(\mathfrak{r}) > 0$ is a polynomial. The function $\mathfrak{f}(\mathfrak{r})$ has two roots: One is at $\mathfrak{r} = 0$, which corresponds to the unstable state such that $\mathfrak{f}'(\mathfrak{r})|_{\mathfrak{r}=0} > 0$, and the other root is at $\mathfrak{r} = \mathfrak{r}_{st}$, corresponding to the stable state and thus $\mathfrak{f}'(\mathfrak{r})|_{\mathfrak{r}=\mathfrak{r}_{st}} < 0$. The deterministic evolution of Eq. (8) without the external force must be compatible with Eq. (11) for a particular expression of $g(\mathfrak{r})$. The connection between the NLRT and the QD approach can be achieved by assuming that $\mathfrak{r}(0) \equiv x_0^2 = h^2$ is a random variable which plays the role of an effective initial condition responsible for the decay of the unstable state towards its steady state characterized by the value $\mathfrak{r}(\infty) \equiv x_{st}^2 = \mathfrak{r}_{st}$. In Fig. 1 we plot the temporal evolution of the mean value $\langle x^2(t)\rangle$ from its initial unstable state to its corresponding steady state; once this latter is attained, the process stops at $\mathcal{T}_c$ and can thus be considered a quench time which will be later employed to compute the RO. This process is presented for various τ values and it can be readily appreciated that, as τ increases, the critical value $\mathcal{T}_c$ diminishes.

After substituting Eq. (11) into Eq. (10) and assuming distributed initial conditions such that $\langle\mathfrak{r}(0)\rangle = 0$, we get, in terms of $\mathfrak{r}$,

$$\mathcal{T} = \int_0^\infty \frac{\langle\mathfrak{r}(t)\rangle - \langle\mathfrak{r}\rangle_{st}}{\langle\mathfrak{r}(0)\rangle - \langle\mathfrak{r}\rangle_{st}} dt = \frac{1}{\mathfrak{r}_{st}}\left\langle \int_{h^2}^{\mathfrak{r}_{st}} \frac{\mathfrak{r}_{st} - \mathfrak{r}}{\mathfrak{f}(\mathfrak{r})} d\mathfrak{r}\right\rangle$$

$$= \frac{1}{2\bar{a}}\left\langle \ln\left(\frac{\mathfrak{r}_{st}}{h^2}\right)\right\rangle + \frac{1}{\mathfrak{r}_{st}}\left\langle \int_{h^2}^{\mathfrak{r}_{st}} g(\mathfrak{r}) d\mathfrak{r}\right\rangle. \tag{12}$$

The logarithmic term is the universal and relevant contribution arising from the time characterization of the decay process in the linear regime of the nonlinear potential, wherein the stochastic fluctuations are dominant. The last term comes from the nonlinear contributions of the potential away from the initial unstable state. As stated by the QD approach, in the nonlinear regime the dynamical evolution of the particle is practically deterministic and the stochastic fluctuations are irrelevant; thus, $h \to 0$. Under these circumstances, the NLRT becomes

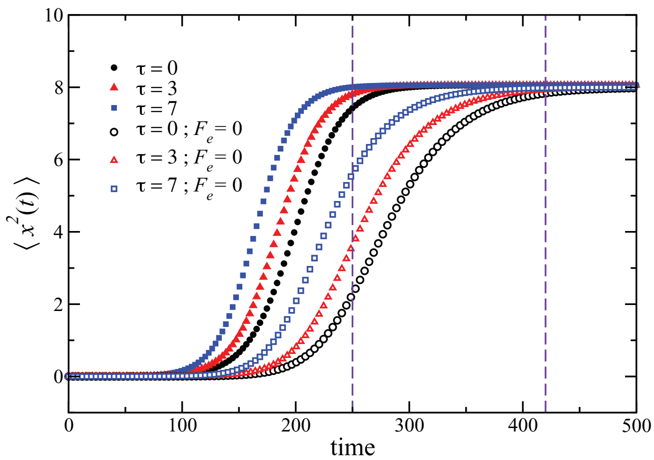

FIG. 1. (Color online) Time evolution for the mean value $\langle x^2(t)\rangle$ for different values of the correlation time τ in the presence (solid symbols) and absence (open symbols) of the external force field: $\tau = 0$ (circles), $\tau = 3$ (triangles), and $\tau = 7$ (squares). Vertical dashed lines indicate the approximate critical values $\mathcal{T}_c = 250$ and 420 corresponding to instances $\tau = 7$ with and without field, respectively (see text for more details). Simulations were performed over 10^4 initial conditions, with $a_0 = 0.5$, $b_0 = 0.0625$, $\gamma_0 = 20$, $\lambda = 10^{-5}$, and, for the corresponding point set, $F_e = 10^{-2}$.

$$\mathcal{T} = \frac{1}{2\bar{a}}\left\langle \ln\left(\frac{\mathfrak{r}_{st}}{h^2}\right)\right\rangle + C_{NL}, \tag{13}$$

where C_{NL} is a constant which is calculated through

$$C_{NL} = \lim_{h\to 0} \frac{1}{\mathfrak{r}_{st}} \int_{h^2}^{\mathfrak{r}_{st}} g(\mathfrak{r}) d\mathfrak{r}, \tag{14}$$

which accounts for nonlinear contributions and is a model-dependent quantity. The time scale given by Eq. (13) characterizes the complete dynamical relaxation of arbitrary nonlinear unstable potentials in terms of the relaxing quantity $\langle\mathfrak{r}\rangle$. The first term of Eq. (13) can be explicitly calculated using the QD approach, which relies on the linear approximation of Eq. (9) and reads as

$$\dot{x} = \bar{a}x + \bar{F}_e + \bar{\lambda}\xi(t). \tag{15}$$

The solution of this linear equation, assuming the initial condition $x(0) = 0$, is $x(t) = h(t)e^{\bar{a}t}$, where

$$h(t) = \int_0^t e^{-\bar{a}s}[\bar{F}_e + \bar{\lambda}\xi(s)]ds. \tag{16}$$

According to the QD approach [20–23], the process $h(t)$ plays the role of an effective initial condition and, as time increases, it becomes a Gaussian random variable (GRV). This is indeed the case since, for small values of both the noise intensity and $\bar{F}_e$,

$$\lim_{t\to\infty} \frac{dh(t)}{dt} = \lim_{t\to\infty} e^{-\bar{a}t}[\bar{F}_e + \bar{\lambda}\xi(t)] \to 0. \tag{17}$$

Thus, at large times $h(\infty) = h$, where h is a GRV. In this case the process $x(t)$ becomes a QD one such that

$$x^2(t) = h^2 e^{2\bar{a}t}. \quad (18)$$

In this linear approximation Eq. (18) can also be written, taking into account the whole process, as $x^2(t) = h^2 e^{2\bar{a}t}\theta(t_i - t) + x_{st}^2\theta(t - t_i)$, where $\theta(t)$ is the step function. After substitution of this expression into Eq. (13) and a time integration we obtain, in the linear approximation, that the NLRT is

$$\mathcal{T}_{_L} = \frac{1}{2\bar{a}}\left\langle \ln\left(\frac{\mathfrak{r}_{st}}{h^2}\right)\right\rangle - C_{_L}, \quad (19)$$

and $C_{_L} = (1/2\bar{a})[1 - \langle h^2\rangle/\mathfrak{r}_{st}]$. This time scale can be calculated from the marginal probability density $P(h)$, which is the Gaussian distribution function given by

$$P(h) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(h-\langle h\rangle)^2/2\sigma^2}, \quad (20)$$

with $\sigma^2 \equiv \langle h^2\rangle - \langle h\rangle^2$ being the variance. From Eq. (16) it can be shown that

$$\langle h\rangle = \int_0^\infty e^{-\bar{a}t}\tilde{F}_e dt = \frac{F_e}{a_0}, \quad (21)$$

$$\langle h^2\rangle = \langle h\rangle^2 + \bar{\lambda}^2\int_0^\infty\int_0^\infty e^{-\bar{a}(t+t')}\langle\xi(t)\xi(t')\rangle dt dt'$$
$$= \langle h\rangle^2 + \frac{\lambda/\gamma_0^2}{a(1-a\tau)}, \quad (22)$$

and therefore $\sigma^2 = D/a(1-a\tau) = D_e/a$, where $D_e = D/(1-a\tau)$, $D = \lambda/\gamma_0^2 = k_{_B}T/\gamma_0$ being Einstein's diffusion coefficient and D_e an effective diffusion coefficient. To calculate the constant $C_{_L}$ we need to evaluate the mean value $\langle h^2\rangle$. It is clear that $\langle h^2\rangle = \sigma^2 + F_e^2/a_0^2$, which can be neglected for small noise and small amplitude of the external force. Hence, the constant $C_{_L}$ can be approximated by $C_{_L} = 1/2\bar{a}$. As shown in Eq. (A7) of Appendix A, the linear time scale (19) can be written as

$$\mathcal{T}_{_L} = \mathcal{T}_{_L}^0 + \frac{1}{2\bar{a}}\psi(1/2) - \frac{1}{2\bar{a}}I, \quad (23)$$

where

$$I = e^{-\beta^2}\psi(1/2) + \frac{e^{-\beta^2}}{\sqrt{\pi}}\sum_{n=1}^\infty \psi\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)\frac{(2\beta)^{2n}}{(2n)!}, \quad (24)$$

with $\beta^2 = \langle h\rangle^2/2\sigma^2 = (F_e/\gamma_0)^2/2aD_e$ and

$$\mathcal{T}_{_L}^0 = \frac{1}{2\bar{a}}\left\{\ln\left(\frac{a x_{st}^2}{2D_e}\right) - \psi(1/2) - 1\right\} \quad (25)$$

being the linear approximation of the NLRT in the absence of the external force ($\beta = 0$) and $\psi(1/2) = -2 - \ln 2$ the digamma function [49]. On the other hand, from Eqs. (13), (19), and (23) we can conclude that the NLRT for arbitrary nonlinear unstable potentials in the one-dimensional case reads as

$$\mathcal{T}_e = \mathcal{T}_0 + \frac{1}{2\bar{a}}\psi(1/2) - \frac{1}{2\bar{a}}I, \quad (26)$$

where

$$\mathcal{T}_0 = \frac{1}{2\bar{a}}\left\{\ln\left(\frac{a x_{st}^2}{2D_e}\right) - \psi(1/2)\right\} + C_{_{NL}} \quad (27)$$

is the NLRT in the absence of the external force. In the particular case of a bistable potential, the deterministic equation associated with Eq. (11) without the external force can be written as $d\mathfrak{r}/dt = 2\bar{a}\mathfrak{r}(\mathfrak{r}_{st} - \mathfrak{r})/\mathfrak{r}_{st}$, with $\mathfrak{r}_{st} \equiv x_{st}^2 = a/b = a_0/b_0$. It is clear that $g(\mathfrak{r}) = 0$, and thus the NLRT in this case is the same as Eq. (26) and T_0 is the same as (27) with $C_{_{NL}} = 0$.

Receiver-output

Following the methodology of Ref. [22] (where the definitions of A_e and A_0 are made) we can also calculate the receiver-output ratio $\mathfrak{R} = A_e/A_0$ through the relaxation process of the bistable potential ($C_{_{NL}} = 0$) in the presence of a constant force field. According to Fig. 1, the receiver-output can be expressed in terms of the NLRT given by Eqs. (26) and (27). This relation can be achieved if the NLRT (12) is approximated by

$$\mathcal{T} = \int_0^\infty \frac{\langle\mathfrak{r}(t)\rangle - \langle\mathfrak{r}\rangle_{st}}{\langle\mathfrak{r}(0)\rangle - \langle\mathfrak{r}\rangle_{st}}dt \simeq \int_0^{\mathcal{T}_c}\frac{\langle\mathfrak{r}(t)\rangle - \langle\mathfrak{r}\rangle_{st}}{\langle\mathfrak{r}(0)\rangle - \langle\mathfrak{r}\rangle_{st}}dt, \quad (28)$$

where $\mathcal{T}_c$ is determined from the behavior of $\langle x^2(t)\rangle$, as was done in Fig. (1). This approximation makes sense if $\mathcal{T}_c \geqslant K\mathcal{T}_0$, where $\mathcal{T}_0$ is the same as Eq. (27) and $K = 1.5$. If we make $\langle\mathfrak{r}(0)\rangle = 0$, it can be shown from Fig. 1 that the RO can be written as

$$\mathfrak{R} = \frac{\mathcal{T}_e - \mathcal{T}_c}{\mathcal{T}_0 - \mathcal{T}_c} = 1 + \frac{\mathcal{T}_0 - \mathcal{T}_e}{\mathcal{T}_c - \mathcal{T}_0}. \quad (29)$$

Thus, the RO is only a function of $\mathcal{T}_e$ and $\mathcal{T}_0$ and, according to Eqs. (26) and (27), it is finally given by the following expression:

$$\mathfrak{R} = 1 + \frac{I - \psi(1/2)}{2\bar{a}(\mathcal{T}_c - \mathcal{T}_0)}. \quad (30)$$

This theoretical result is plotted in Fig. 2 as a function of the external force F_e for representative τ values and compared with numerical simulation results, with an excellent agreement in all depicted instances. In the Markovian case ($\tau = 0$) the detection process is initiated at an approximate critical value of $F_e \approx 1.28 \times 10^{-6}$, less than the considered non-Markovian cases $F_e \approx 1.67 \times 10^{-6}$ and $F_e \approx 1.95 \times 10^{-6}$ corresponding to the values $\tau = 3.0$ and $\tau = 7.0$, respectively. This result indicates that, as the memory time of the non-Markovian dynamics increases, a corresponding growth in the critical value of the parameter F_e is obtained.

IV. STATISTICS OF THE PASSAGE TIME DISTRIBUTION AND QD APPROACH

The other criterion for the detection of weak signals we mentioned in the Introduction of this work is related to the statistics of the first passage time distribution. As shown below, for the application of this criterion it is sufficient to know the time characterization of the unstable state in the linear approximation of the bistable potential profile. However, the

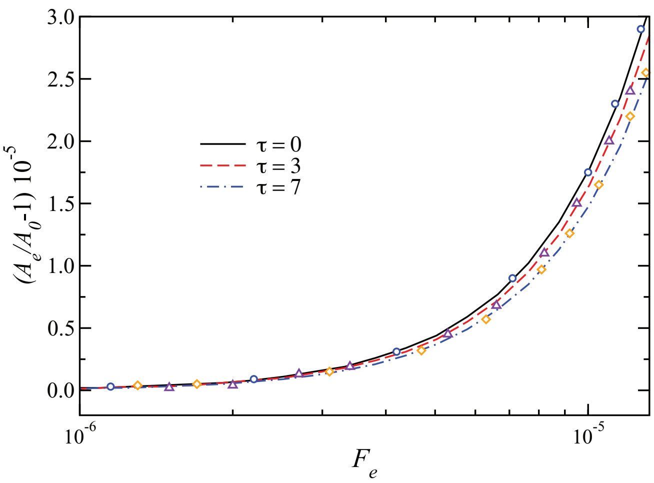

FIG. 2. (Color online) Receiver output given by Eq. (30) as a function of the external force F_e for $\tau = 0$ (solid line), $\tau = 3$ (dashed line), and $\tau = 7$ (dot-dashed line) compared to numerical simulation results (circles, triangles, and diamonds for that same set of τ values). Same a_0, b_0, γ_0, and λ values as in Fig. 1.

criterion also works if the time characterization takes into account nonlinear potential profiles. Herein we study both ensuing time scales. We begin with the linear approximation to Eq. (9) for which the x variable is restricted in the interval $-R \leqslant x \leqslant R$, R being a prescribed reference value chosen proportional to the steady value of the bistable potential in the absence of the external force; hence, $R = C_0 x_{st}$, with $0 < C_0 < 1$ and $x_{st} = \sqrt{a/b} = \sqrt{a_0/b_0}$. In this case, the Langevin dynamics is the same as Eq. (15) and, by applying the QD approach, it is shown that the random passage time required by the Brownian particle to reach the reference value R^2 is then

$$t = \frac{1}{2\bar{a}} \ln\left(\frac{R^2}{h^2}\right). \tag{31}$$

The statistics of this passage time distribution is given by both the MFPT, denoted by $\langle t \rangle$, and its variance, defined by $\langle (\Delta t)^2 \rangle \equiv \langle t^2 \rangle - \langle t \rangle^2$. Both quantities can be calculated from the moment generating function (MGF) $G(2\bar{a}\nu) = \langle e^{-2\bar{a}\nu t} \rangle = \langle (R^2/h^2)^{-\nu} \rangle$. This function and the statistics of the first passage time distribution are explicitly calculated in Appendix B, showing that the MFPT is given by

$$\langle t \rangle_{_L} = \langle t \rangle_{_L}^0 - \frac{e^{-\beta^2}}{\bar{a}} \sum_{n=1}^{\infty} \frac{(\beta)^{2n}}{n!} \sum_{k=1}^{n} \frac{1}{2k-1}, \tag{32}$$

where the parameter β is the same as defined before and

$$\langle t \rangle_{_L}^0 = \frac{1}{2\bar{a}} \left\{ \ln\left(\frac{aR^2}{2D_e}\right) - \psi(1/2) \right\} \tag{33}$$

is the MFPT in the absence of the force field ($\beta = 0$). The variance is then

$$\langle (\Delta t)^2 \rangle = \frac{1}{4\bar{a}^2}\psi'(1/2) + \frac{1}{\bar{a}^2} e^{-\beta^2} \sum_{n=1}^{\infty} \frac{\beta^{2n}}{n!} \left(\sum_{k=1}^{n} \frac{1}{2k-1} \right)^2$$
$$- \frac{1}{\bar{a}^2} e^{-\beta^2} \sum_{n=1}^{\infty} \frac{\beta^{2n}}{n!} \sum_{k=1}^{n} \frac{1}{(2k-1)^2}$$
$$- \frac{1}{\bar{a}^2} \left[e^{-\beta^2} \sum_{n=1}^{\infty} \frac{\beta^{2n}}{n!} \sum_{k=1}^{n} \frac{1}{2k-1} \right]^2, \tag{34}$$

where $\psi'(1/2) = \pi^2/2 = 4.934$. To deal with nonlinear contributions, we use again Eq. (11) and its connection with the QD approach to show that the nonlinear passage time reads as

$$t = \int_{h^2}^{R^2} \frac{d\mathfrak{r}}{\mathfrak{f}(\mathfrak{r})} = \int_{h^2}^{R^2} \frac{C_0 + \mathfrak{r} g(\mathfrak{r})}{\mathfrak{r}(\mathfrak{r}_{st} - \mathfrak{r})}. \tag{35}$$

Its mean value can be written as $\langle t \rangle_{_{NL}} = (1/2\bar{a})\langle \ln(R^2/h^2) \rangle + K_{_{NL}}$, where $K_{_{NL}}$ takes into account the nonlinear contribution of the potential profile such that

$$K_{_{NL}} = \lim_{h \to 0} \left[\frac{1}{2\bar{a}} \int_{h^2}^{R^2} \frac{d\mathfrak{r}}{\mathfrak{r}_{st} - \mathfrak{r}} + \int_{h^2}^{R^2} \frac{g(\mathfrak{r})}{\mathfrak{r}_{st} - \mathfrak{r}} \right]. \tag{36}$$

For the bistable potential $g(\mathfrak{r}) = 0$ and $K_{_{NL}} = (1/2\bar{a}) \ln[1/(1-M^2)]$, with $M^2 = R^2/x_{st}^2$. The MFPT including the nonlinear contributions can be shown to be

$$\langle t \rangle_{_{NL}} = \langle t \rangle_{_{NL}}^0 - \frac{e^{-\beta^2}}{\bar{a}} \sum_{n=1}^{\infty} \frac{\beta^{2n}}{n!} \sum_{k=1}^{n} \frac{1}{2k-1}, \tag{37}$$

where now

$$\langle t \rangle_{_{NL}}^0 = \frac{1}{2\bar{a}} \left\{ \ln\left[\frac{a x_{st}^2 M^2}{2(1-M^2)D_e}\right] - \psi(1/2) \right\} \tag{38}$$

is the nonlinear MFPT in the absence of the external force. Notice that M measures how close the particle is from its the stationary-state value, notwithstanding that it never reaches that value, contrary to what happens with the NLRT given by Eq. (26). The variance remains the same as Eq. (34). The result given in Eq. (38) without the external field can be compared with those obtained in other works in the Markovian case. This case means that $D_e = D$ in Eq. (38) and then it has exactly the same algebraic structure than that calculated by Haake *et al.* [26] and also the one given by Eq. (16) in [29]. On the other hand, in way similar to that studied in [29], when the absorbing barrier is removed, the point x can cross the point $x = \pm R$ any number of times and in any direction; in this case a study of the terms in the inverse probability current [29,30] must be performed. This study leads to an unexpected effect called *noise delayed decay*, wherewith the stochastic fluctuations can considerably increase the decay time of unstable and metastable states. A fact which can be considered elsewhere in the context of the GLE. The method in [29] has been developed to calculate the NLRT for any fluctuation intensity and arbitrary potential profile, but without the external force field. In the particular case of small fluctuations, the NLRT coincides with the MFPT. The NLRT defined in [29] has been calculated for the symmetric bistable potential and other potential profiles. For the symmetric bistable potential it has been shown that, for small noise intensity such that $q \ll \Phi(x_m)$, with q being the noise intensity, $x_m = \pm\sqrt{a/b}$, and $\Phi(x_m) = -a^2/4b$ as the depth of the potential profile, the NLRT given by Eq. (15) in [29] coincides with the MFPT calculated by Haake *et al.* [26] and given in Eq. (16) of the same Ref. [29], as expected. In conclusion, when we use the QD approach, which is valid for small noise intensity, in the time characterization of the decay of the unstable state, this relaxation process is bounded by fixed and absorbing barriers, so that the inverse probability current becomes negligible.

Weak signal detection

The statistics of the first passage time given by Eqs. (34) and (37) can now be used to establish the other criterion for the detection process of weak signals. It makes it possible to find a critical value of β parameter (recall that the β parameter is proportional to the external force to be measured) for which the reduction of the MFPT is greater than or of the same order as the maximum variance, that is [20,21],

$$[\langle t\rangle_{\beta_c} - \langle t\rangle_{\beta=0}]^2 \geqslant \langle(\Delta t)^2_{\beta=0}\rangle. \tag{39}$$

The value for β_c found through this criterion will tell us the weak external forces which can be measured, and, in fact, they must satisfy the condition $F_{ec} < F_e$. It should be noticed that the difference on the left hand side leads to the same result for linear (32) or nonlinear (37) MFPT. The variance for $\beta = 0$ is simply $\langle(\Delta t)^2_{\beta=0}\rangle = \psi'(1/2)/4\bar{a}^2$ and therefore the critical value β_c can be calculated from the condition

$$\left(e^{-\beta_c^2}\sum_{n=1}^{\infty}\frac{\beta_c^{2n}}{n!}\sum_{k=1}^{n}\frac{1}{2k-1}\right)^2 \geqslant \frac{\psi'(1/2)}{4}. \tag{40}$$

It should be said that Eq. (40) can also be written in terms of the first derivative of the hypergeometric Kummer function; then the condition to determine the critical value for β reads

$$\left\{{}_1F_1^{(1,0,0)}[0,1/2,-\beta_c^2]\right\}^2 = \frac{\psi'(1/2)}{4}. \tag{41}$$

Its numerical solution, up to the *Mathematica* tool precision, gives $\beta_c = 1.36\,543$ and the external force to be measured can be obtained as $F_{ec}^2 = \frac{2a\lambda\beta_c^2}{(1-a\tau)}$. Now it is clear that the external signal, which can be measured by means of this criterion, depends on the noise correlation time: The bigger time τ drives to a bigger external critical force F_{ec}. It should be recalled that the factor $(1 - a\tau) > 0$ must be maintained; otherwise, the treatment is not valid.

V. CONCLUDING REMARKS

By assuming an exponential decay in the friction memory kernel and extending the number of stochastic variables, the GLE (1) and (2) are transformed into an equivalent Markovian process given by Eqs. (5)–(7). By assuming also the restriction $1 - a\tau \gg 3b\tau x^2$ for the dynamics, these equations reduce in the overdamped approximation to the nonlinear Langevin Eq. (9), wherein all the Markovian parameters a, b, F_e/γ_0, and λ/γ_0^2 have been rescaled by the non-Markovian factor $1/(1 - a\tau)$. The decay of the unstable state of a Brownian particle in this case has been achieved through two time scales, namely, the NLRT and MFPT, along with the QD approach. Both have been expressed for arbitrary nonlinear unstable potential, for which a bistable one is just a particular case. A criterion for each time scale is used when the decay process is taken place in the bistable potential profile. The RO as a function of the non-Markovian NLRT is given in Eq. (30) and it has been compared with the numerical simulation results for different τ values. As shown in Fig. 2, the bigger the correlation time τ drives to an increase in the critical value for the parameter β, and an easier way for the external force F_e to be measured. Similar behavior is obtained when we applied the statistics of the MFPT distribution. We emphasize that the criterion established in Eq. (40) gives the same result for linear (32) or nonlinear (37) MFPT, whereas the RO is strictly given as a function of the NLRT. An interesting point we would like to make in these Concluding Remarks regards the study of the detection of weak oscillatory signals in the decay process of the unstable states of Brownian particles. This problem has already been considered in [24] to detect weak oscillatory optical signals in the transient dynamics of a class-A laser, in which the resonance effect is known to exist. The study for a Brownian particle in the presence of crossed electric and magnetic fields will be considered in future works.

Finally, we note that the study of the decay process in terms of the inverse probability current [29,30], the time evolution of averages in dynamical systems driven by noise [31,32], can, in principle, be extended to the case of Brownian particles embedded in memory heat baths using the GLE with an exponential decay in the friction memory kernel.

APPENDIX A: NLRT IN THE LINEAR APPROXIMATION

The NLRT given by Eq. (19) of Sec. III, along with $C_L = 1/2\bar{a}$, can be written as

$$\mathcal{T}_L = \frac{1}{2\bar{a}}[\ln(\alpha^2 \mathfrak{r}_{st}) - \langle\ln(\alpha^2 h^2)\rangle - 1], \tag{A1}$$

where $\alpha^2 = 1/2\sigma^2$. To evaluate the mean value $I \equiv \langle\ln(\alpha^2 h^2)\rangle$, we use Eq. (20) and define the variable $z = \alpha h$ as well as the parameter $\beta = \alpha\langle h\rangle$ in such a way that the mean value can be written as $I = I_1 + I_2$, where

$$I_1 = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{0}\ln(\alpha^2 h^2)e^{-\alpha^2(h-\langle h\rangle)^2}dh = \frac{e^{-\beta^2}}{\sqrt{\pi}}\int_0^{+\infty}\ln z^2 e^{-z^2-2\beta z}dz, \tag{A2}$$

$$I_2 = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{0}^{+\infty}\ln(\alpha^2 h^2)e^{-\alpha^2(h-\langle h\rangle)^2}dh = \frac{e^{-\beta^2}}{\sqrt{\pi}}\int_0^{+\infty}\ln z^2 e^{-z^2+2\beta z}dz. \tag{A3}$$

For each integral it can be shown that [49]

$$I_1 = \frac{e^{-\beta^2}}{2\sqrt{\pi}}\sum_{k=0}^{\infty}\Gamma\left(\frac{k+1}{2}\right)\psi\left(\frac{k+1}{2}\right)\frac{(-2\beta)^k}{k!}, \tag{A4}$$

$$I_2 = \frac{e^{-\beta^2}}{2\sqrt{\pi}}\sum_{k=0}^{\infty}\Gamma\left(\frac{k+1}{2}\right)\psi\left(\frac{k+1}{2}\right)\frac{(2\beta)^k}{k!}. \tag{A5}$$

After some algebra it can be shown that

$$I = e^{-\beta^2}\psi(1/2) + \frac{e^{-\beta^2}}{\sqrt{\pi}}\sum_{n=1}^{\infty}\psi\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)\frac{(2\beta)^{2n}}{(2n)!}, \quad (A6)$$

and therefore Eq. (A1) can be written as

$$\mathcal{T}_{_L} = \frac{1}{2\bar{a}}\left\{\ln\left(\frac{x_{st}^2}{2\sigma^2}\right) - \psi(1/2) - 1\right\} + \frac{1}{2\bar{a}}\psi(1/2) - \frac{1}{2\bar{a}}I, \quad (A7)$$

where $\psi(1/2) = -\gamma - 2\ln 2 = -1.936$, $\gamma = 0.577$ being the Euler constant.

APPENDIX B: STATISTICS OF THE FIRST PASSAGE TIME DISTRIBUTION

The statistics of the passage time distribution can be calculated from its first and second moments, defined by $\langle 2\bar{a}t\rangle = -dG(2\bar{a}\nu)/d\nu|_{\nu=0}$ and $\langle 2\bar{a}t^2\rangle = d^2G(2\bar{a}\nu)/d\nu^2|_{\nu=0}$, respectively, which are, in turn, defined in terms of the MGF $G(2\bar{a}\nu) = \langle (R^2/h^2)^{-\nu}\rangle$, which, in this particular case, is expressed as

$$G(2\bar{a}\nu) = \frac{R^{-2\nu}}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} h^{2\nu}e^{-(h-\langle h\rangle)^2/2\sigma^2}dh. \quad (B1)$$

To evaluate this integral we use the same change of variable as above, that is, $z = \alpha h$, and define $G(2\tilde{a}\nu) = [(\alpha^2R^2)^{-\nu}/\sqrt{\pi}][J_1 + J_2]$, where

$$J_1 = e^{-\beta^2}\int_0^{\infty} z^{2\nu}e^{-z^2-2\beta z}dz = \frac{e^{-\beta^2}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(2\beta)^n}{n!}\Gamma\left(\nu+n+\frac{1}{2}\right), \quad (B2)$$

$$J_2 = e^{-\beta^2}\int_0^{\infty} z^{2\nu}e^{-z^2+2\beta z}dz = \frac{e^{-\beta^2}}{2}\sum_{n=0}^{\infty}\frac{(2\beta)^n}{n!}\Gamma\left(\nu+n+\frac{1}{2}\right). \quad (B3)$$

After some algebra it can be shown that

$$G(2\bar{a}\nu) = G_0(2\bar{a}\nu)e^{-\beta^2}\sum_{n=0}^{\infty}\frac{\Gamma(\nu+n+1/2)}{\Gamma(\nu+1/2)}\frac{(2\beta)^{2n}}{(2n)!}, \quad (B4)$$

$G_0(2\bar{a}\nu) = (\alpha^2R^2)^{-\nu}\Gamma(\nu+1/2)/\sqrt{\pi}$ being the MGF in the absence of the external force and $\beta^2 = \alpha^2\langle h\rangle^2 = (F_e/\gamma_0)^2/2aD_e$. After some straightforward algebra it is not difficult to show that the MFPT reads as

$$\langle 2\bar{a}t\rangle = \ln\left(\frac{aR^2}{2D_e}\right) - \psi(1/2) - 2e^{-\beta^2}\sum_{n=1}^{\infty}\frac{(\beta)^{2n}}{n!}\sum_{k=1}^{n}\frac{1}{2k-1}. \quad (B5)$$

The variance of the passage time is $\langle(2\bar{a}\Delta t)^2\rangle = \langle(2\bar{a}t)^2\rangle - \langle 2\bar{a}t\rangle^2$. Again, after some algebra and using another identity for all natural numbers $(n = 1,2,3,\ldots)$ $\psi'(n+1/2) = \psi'(1/2) - 4\sum_{k=1}^{n}1/(2k-1)^2$, with $\psi'(1/2) = \pi^2/2 = 4.934$, it can be shown that

$$\langle(2\bar{a}\Delta t)^2\rangle = \psi'(1/2) + 4e^{-\beta^2}\sum_{n=1}^{\infty}\frac{\beta^{2n}}{n!}\left(\sum_{k=1}^{n}\frac{1}{2k-1}\right)^2 - 4e^{-\beta^2}\sum_{n=1}^{\infty}\frac{\beta^{2n}}{n!}\sum_{k=1}^{n}\frac{1}{(2k-1)^2} - 4\left[e^{-\beta^2}\sum_{n=1}^{\infty}\frac{\beta^{2n}}{n!}\sum_{k=1}^{n}\frac{1}{2k-1}\right]^2. \quad (B6)$$

[1] R. Benzi, A. Sutera, and A. Vulpiani, J. Phys. A: Math. Gen. **14**, L453 (1981); **18**, 2239 (1985).
[2] B. McNamara and K. Wiesenfeld, Phys. Rev. A **39**, 4854 (1989).
[3] I. Klik and L. Gunther, J. Stat. Phys. **60**, 473 (1990).
[4] P. Jung and P. Hänggi, Phys. Rev. A **44**, 8032 (1991).
[5] J. K. Douglass, L. Wilkens, E. Pantazelou, and F. Moss, Nature (London) **365**, 337 (1993).
[6] K. Weisenfeld and F. Moss, Nature (London) **373**, 33 (1995).
[7] A. Pérez-Madrid and J. M. Rubí, Phys. Rev. E **51**, 4159 (1995).
[8] S. M. Bezrukov and I. Vodyanoy, Nature (London) **378**, 362 (1995).
[9] X. Pei, L. Wilkens, and F. Moss, J. Neourophysiol. **76**, 3002 (1996).
[10] J. J. Collins, T. T. Imhoff, and P. Grigg, Nature (London) **383**, 770 (1996).
[11] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998).
[12] L. F. Yang, Z. H. Hou, and H. W. Xin, J. Chem. Phys. **110**, 3591 (1999).
[13] H. Li, Z. Hou, and H. Xin, Chem. Phys. Lett. **402**, 444 (2005).
[14] T. A. Engel, B. Helbig, D. F. Russell, L. Schimansky-Geier, and A. B. Neiman, Phys. Rev. E **80**, 021919 (2009).
[15] B. Kosko and S. Mitaim, Phys. Rev. E **64**, 051110 (2001).
[16] B. Dybiec and E. Gudowska-Nowak, J. Stat. Mech. (2009) P05004.
[17] T. Srokowski, Eur. Phys. J. B. **86**, 239 (2013).
[18] G. Vemuri and R. Roy, Phys. Rev. A **39**, 2539 (1989).
[19] I. Littler, S. Balle, K. Bergmann, G. Vemuri, and R. Roy, Phys. Rev. A **41**, 4131 (1990).
[20] S. Balle, F. De Pasquale, and M. San Miguel, Phys. Rev. A **41**, 5012 (1990).
[21] J. I. Jiménez Aquino and J. M. Sancho, Phys. Rev. A **43**, 589 (1991).
[22] J. I. Jiménez-Aquino and M. Romero-Bastida, Phys. Rev. E **84**, 011137 (2011).
[23] J. M. Sancho and M. San Miguel, Phys. Rev. A **39**, 2722 (1989).
[24] J. Dellunde, M. C. Torren, J. M. Sancho, and M. San Miguel, Opt. Commun. **109**, 435 (1994).
[25] F. de Pasquale and P. Tombesi, Phys. Lett. A **72**, 7 (1979).

[26] F. Haake, J. W. Haus, and R. Glauber, Phys. Rev. A **23**, 3255 (1981).
[27] W. Nadler and K. Schulten, J. Chem. Phys. **82**, 151 (1985).
[28] J. Casademunt, J. I. Jiménez-Aquino, and J. M. Sancho, Phys. Rev. A **40**, 5905 (1989).
[29] N. V. Agudov and A. N. Malakhov, Phys. Rev. E **60**, 6333 (1999).
[30] N. V. Agudov and B. Spagnolo, Phys. Rev. E **64**, 035102(R) (2001).
[31] A. L. Pankratov, Phys. Lett. A **234**, 329 (1997); **255**, 17 (1999).
[32] A. L. Pankratov and B. Spagnolo, Phys. Rev. Lett. **93**, 177001 (2004).
[33] A. N. Malakhov and A. L. Pankratov, Adv. Chem. Phys. **121**, 357 (2002).
[34] A. N. Malakhov and A. L. Pankratov, Physica C **269**, 46 (1996).
[35] J. Wu, R. J. Silbey, and J. Cao, Phys. Rev. Lett. **110**, 200402 (2013).
[36] M. Torchala, P. Chelminiak, and P. A. Bates, Eur. Phys. J. B **85**, 116 (2012).
[37] R. Yvinec, M. R. D'Orsogna, and T. Chou, J. Chem. Phys. **137**, 244107 (2012).
[38] J. A. Morrone, T. E. Markland, M. Ceriotti, and B. J. Berne, J. Chem. Phys. **134**, 014103 (2011).
[39] A. P. Roberts and C. P. Haynes, Phys. Rev. E **83**, 031113 (2011).
[40] B. H. Shargel, M. R. D' Orsogna, and T. Chou, J. Phys. A: Math. Theor. **43**, 305003 (2010).
[41] A. Dienst and R. Friedrich, Chaos **17**, 033104 (2007).
[42] A. Baura, M. Kumar Sen, and B. C. Bag, Chem. Phys. **417**, 30 (2013).
[43] P. Hänggi and P. Jung, in *Colored Noise in Dynamical Systems*, edited by I. Prigogine and Stuart A. Rice, Advances in Chemical Physics Vol. LXXXIX (Wiley & Sons, New York, 1995).
[44] M. James, F. Moss, P. Hänggi, and C. Van den Broeck, Phys. Rev. A **38**, 4690 (1988).
[45] J. I. Jiménez-Aquino and M. Romero-Bastida, Phys. Rev. E **86**, 031110 (2012).
[46] J. Masoliver, B. J. West, and K. Lindenberg, Phys. Rev. A **35**, 3086 (1987).
[47] H. Mori, Progr. Theor. Phys. **33**, 423 (1965).
[48] R. Zwanzing, J. Stat. Phys. **9**, 215 (1973).
[49] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).