Controlling symmetry-breaking states by a hidden quantity in multiplicative noise

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(Received 3 June 2014; revised manuscript received 19 August 2014; published 14 November 2014)

The inhomogeneity of multiplicative white noise leads to various coupling modes between deterministic and stochastic forces. We investigate the phase transition induced by the variation of the coupling mode through manipulating its characteristic parameter continuously. Even when the noise strength is fixed, an increase of this parameter can enhance or inhibit the symmetry-breaking state. We also propose a scheme to implement these phase transitions experimentally. Our result demonstrates that the coupling mode previously considered to be a mathematical convention serves as an additional quantity leading to physically observable phase transitions. This observation provides a mechanism to control the effect of noise without regulating the noise strength.

DOI: 10.1103/PhysRevE.90.052121 PACS number(s): 05.40.−a, 05.10.Gg, 05.70.Ln

I. INTRODUCTION

Many complex phenomena in physics and biology exhibit a stochastic nature. Adding noise to deterministic dynamical equations has been shown to be effective in describing such processes [1]. The separated time scales between deterministic and stochastic forces lead to possible variation of their coupling mode. In particular, the solution for the stochastic differential equation with multiplicative white noise depends on the sampling points of the integral sum. A general notation for this coupling mode is given by introducing a parameter $\alpha \in [0,1]$, and the prepoint ($\alpha = 0$), midpoint ($\alpha = 0.5$), and postpoint ($\alpha = 1$) inside each bin of the integral sum correspond to Ito's [2], Stratonovich's [3], and anti-Ito's [4] integral, respectively. In previous studies, the coupling mode was usually considered as these isolated cases, and different fields have their preferred choice, such as Ito's mode in mathematical economics [5], and Stratonovich's [6,7] or anti-Ito's [8–11] mode in physics. For a given stochastic system, the use of α is not *a priori* clear, and it was still under debate recently. For example, the transition implemented with an analog simulator agrees with that of Stratonovich [12], while the force measurements [13–16] and the drift measurements [17] on a Brownian particle near a wall favor anti-Ito's mode.

Rich phenomena induced by multiplicative noise have been discovered, including noise-induced fronts [18], pattern formation [19], and phase transitions [20,21]. For the phase transitions that are the main topic here, previous studies have mainly investigated several isolated coupling modes. For example, the qualitative independence of Ito's and Stratonovich's modes was found in the disorder-order phase transition [22,23], while the disorder-order-disorder phase transition was supposed to appear only with Stratonovich's mode [24,25]. As an intrinsic mathematical property, the parameter α can vary continuously [11]. From a physical point of view, one may naturally ask the following question: can the continuous variation of α lead to physically observable phase transitions? In this paper, we study the symmetry-breaking phase transition induced by the continuous variation of α. In contrast with previous conclusions [21–23], we find that the order-disorder phase transition does depend on the coupling mode. Surprisingly, an increase of α can either enhance or inhibit the symmetry-breaking state even when the noise strength is fixed.

We show the effect of the α mode on the order parameter by studying two representative cases of the following lattice model Eq. (1). The two models serve as the simplest possible examples to demonstrate the present phase transition. The variation of the α mode has opposite effects on the two models, which can help us to demonstrate the regulatory mechanism of the α mode by comparison. In addition, the continuous variation of the coupling mode recently became partially controllable ($\alpha \in [0,0.5]$) in noisy electric circuits [26], where α is manipulated by the ratio between the driving noise correlation time and the feedback delay time. With this mechanism of controlling α, we further provide an experimental scheme to implement the present phase transition.

This paper is organized as follows. In Sec. II, we introduce the lattice model and the method to track the phase transition. We then show the phase transition induced by the coupling mode. In Sec. III, we present a detailed discussion on the difference between our result and the previous phase transition induced by the noise strength. We also demonstrate the physical origin and experimental realization of the present phase transition. In Sec. IV, we summarize our work.

II. PHASE TRANSITION INDUCED BY THE COUPLING MODE

To state our investigation, we consider the following type of d-dimensional lattice model given by the Langevin

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equation:

$$\dot{q}_i = f(q_i) + \sqrt{\varepsilon} g(q_i)\xi_i(t) - \frac{J}{2d} h(q_i) \sum_{j\in n(i)} (q_i - q_j), \quad (1)$$

where the scalar variable $q_i (i = 1, \ldots, L^d)$ denotes the lattice point, and the symbol $n(i)$ is the set of the $2d$ nearest neighbors of the site i. The noise intensity ε plays the role of the temperature ($k_B T$), and J represents the coupling strength. Here $\{\xi_i(t)\}$ are Gaussian white noises with $\langle \xi_i(t)\rangle = 0$ and $\langle \xi_i(t_1)\xi_j(t_2)\rangle = \delta_{ij}\delta(t_1 - t_2)$, and the average is taken with respect to the noise distribution. The drift term is $f(q_i)$, the coupling function is $h(q_i)$, and the diffusion matrix $D_{ij}(q) \doteq \delta_{ij} g(q_i) g(q_j)/2$. For Eq. (1), there is a freedom in choosing α: $\int_{t_0}^{t_N} g[q(t)]dW(t) \doteq \lim_{N\to\infty} \sum_{n=1}^{N} g[q(t_n^*)][W(t_n) - W(t_{n-1})]$ with $t_n^* = \alpha t_n + (1-\alpha)t_{n-1}$, where the Wiener process $W(t)$ is defined as $\Delta W(t)|_{t_0}^{t_N} \doteq \int_{t_0}^{t_N} \xi(t)dt$ [1]. As $W(t)$ is not a bounded variation function, various α values lead to different results of integration, corresponding to different coupling modes between stochastic and deterministic forces.

A. Method

We next introduce a method to track the phase transition. We analyze the equilibrium distribution $\rho_{\rm eq}(q_i)$ solved through the Fokker-Planck equation for Eq. (1) with the α mode [10]:

$$\partial_t \rho(q,t) = -\partial_q \bigg[f(q) + \alpha g^{'} g(q) - \frac{J}{2d} h(q) \sum_{j\in n} (q - q_j) - \frac{\varepsilon}{2}\partial_q g^2(q) \bigg] \rho(q,t), \quad (2)$$

where we drop the subscript i as each site can be analyzed similarly by neglecting the boundary effect, and the superscript prime denotes the derivation with respect to q. The maximum point of the equilibrium distribution corresponds to the symmetry-breaking state. More specifically, by setting the left side of Eq. (2) to be zero, we first get the equilibrium distribution $\rho_{\rm eq}(q)$. The mean value of each site $\langle q\rangle$ then satisfies the condition

$$\langle q\rangle = \int_{-\infty}^{+\infty} dq[q\rho_{\rm eq}(q)]. \quad (3)$$

To solve this equation, we adopt the classical Weiss mean-field method [21–25,27], which has been applied successfully to study phase transitions in a wide class of stochastic dynamics, and its validity has been demonstrated by the simulations. From this method, the expected value for neighbor sites is assumed to be a constant $\langle q\rangle$. Then, the order parameter $m = |\langle q\rangle|$ can be calculated by the self-consistency Eq. (3). Two solutions for the symmetry-breaking state are obtained from the maximum point of the equilibrium distribution, which leads to the phase diagram in terms of parameters J, ε, and α.

B. Phase transition in two models

For the first model, when the noise intensity is fixed, the symmetry-breaking state can be generated by increasing α. For some fixed α values, the reentrant phase transition (disorder-order-disorder) can be observed when the noise strength increases. Specifically, the symmetry-breaking state does not appear for Ito's mode ($\alpha = 0$) with arbitrary noise strength. In contrast with the first model, the symmetry-breaking state can be inhibited through increasing α for the second model. For some fixed α values, the symmetry-breaking phase transition happens when the noise strength increases. The symmetry-breaking state does not appear for anti-Ito's mode ($\alpha = 1$) when the noise strength varies. We provide a detailed analysis in the following.

The first model is given by Eq. (1) with

$$f(q) = -q(1+q^2)^2, \quad g(q) = 1 + q^2, \quad h(q) = 1. \quad (4)$$

The equilibrium distribution is obtained by solving Eq. (2):

$$\rho_{\rm eq}(q) = \frac{1}{Z(1+q^2)^{2(1-\alpha)}} \exp\bigg\{ -\frac{q^2}{\varepsilon} - \frac{(J/\varepsilon)q^2}{1+q^2} + \frac{J}{\varepsilon}\bigg[\frac{q}{1+q^2} + \arctan(q)\bigg]\langle q\rangle \bigg\}, \quad (5)$$

where $Z = \int_{-\infty}^{+\infty} dq[1/(1+q^2)^{2(1-\alpha)}]\exp\{-q^2/\varepsilon - (J/\varepsilon)q^2/(1+q^2) + (J/\varepsilon)[q/(1+q^2) + \arctan(q)]\langle q\rangle\}$ is the normalization constant. From Eq. (3), the order parameter as a function of J, ε, and α can be solved. With $F(\langle q\rangle) \doteq Z\langle q\rangle$, the phase diagram for the symmetry-breaking state is obtained by $[(\partial F/\partial\langle q\rangle)/Z]|_{\langle q\rangle = 0} = 1$, which leads to the relation of J, ε, and α:

$$\frac{J}{N\varepsilon}\int_{-\infty}^{+\infty} dq \bigg\{ \frac{1}{(1+q^2)^{2(1-\alpha)}} \exp\bigg[-\frac{q^2}{\varepsilon} - \frac{(J/\varepsilon)q^2}{1+q^2} \bigg] \times \bigg[\frac{q^2}{1+q^2} + q\arctan(q) \bigg] \bigg\} = 1, \quad (6)$$

with the normalization constant $N = \int_{-\infty}^{+\infty} dq[1/(1+q^2)^{2(1-\alpha)}] \exp[-q^2/\varepsilon - (J/\varepsilon)q^2/(1+q^2)]$.

The J-ε relations as phase diagrams for different α values from Eq. (6) are plotted in the upper panel of Fig. 1. Each curve separates the phase region $\langle q\rangle = 0$ (below the curve) from the region $\langle q\rangle \neq 0$ (above the curve). For a fixed ε, the coupling strength J for the appearance of the symmetry-breaking state increases when α decreases. The order parameter m as a function of ε and α when $J = 20$ is plotted in the upper panel of Fig. 2. For a different α mode, the interval of ε corresponding to the symmetry-breaking state with $J = 20$ varies, including (i) no symmetry-breaking state for $\alpha = 0$, (ii) approximately $1 < \varepsilon < 20$ for $\alpha = 1/2$, and (iii) approximately $1 < \varepsilon < 310$ for $\alpha = 1$. In addition, the reentrant phase transition happens once the symmetry-breaking state appears for various α modes. Our result is consistent with the previous result, where only Stratonovich's mode was considered [24].

The second model has

$$f(q) = -\frac{aq}{1+cq^2}, \quad g(q) = \sqrt{\frac{2}{1+cq^2}}, \quad h(q) = \frac{1}{1+cq^2}, \quad (7)$$

where a and c are positive coefficients. The equilibrium distribution is

$$\rho_{\rm eq}(q) = \frac{1}{Z(1+cq^2)^{\alpha-1}} \exp\bigg\{ -\frac{(a+J)q^2}{2\varepsilon} + \frac{J}{\varepsilon} q\langle q\rangle \bigg\}, \quad (8)$$

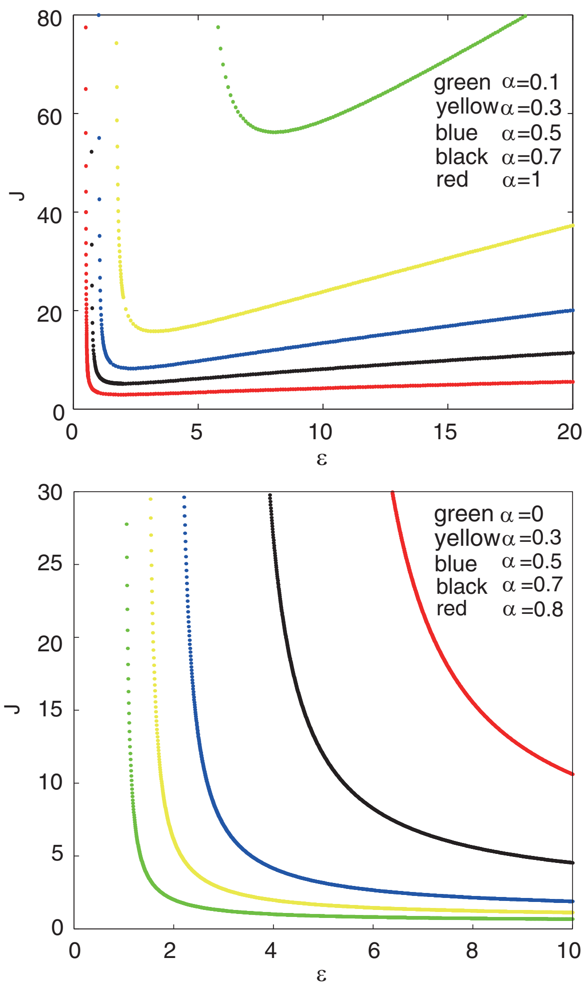

FIG. 1. (Color online) The phase diagram is plotted in terms of coupling strength J and noise intensity ε. Each curve separates the two phase regions: $\langle q\rangle = 0$ and $\langle q\rangle \neq 0$. The upper panel is for model 1: $\langle q\rangle = 0$ is below the curve and $\langle q\rangle \neq 0$ is above; no symmetry-breaking state appears for Ito's noise with any fixed coupling strength J and noise intensity ε. The lower panel is for model 2: $\langle q\rangle = 0$ is on the left of the curve and $\langle q\rangle \neq 0$ is on the right. No symmetry-breaking state appears for anti-Ito's noise with any fixed coupling strength J and the noise intensity ε. The colors denote different α values: for the upper panel, the colors in the legend correspond to the colors of lines from right to left; for the lower panel, the colors in the legend correspond to the colors of lines from left to right.

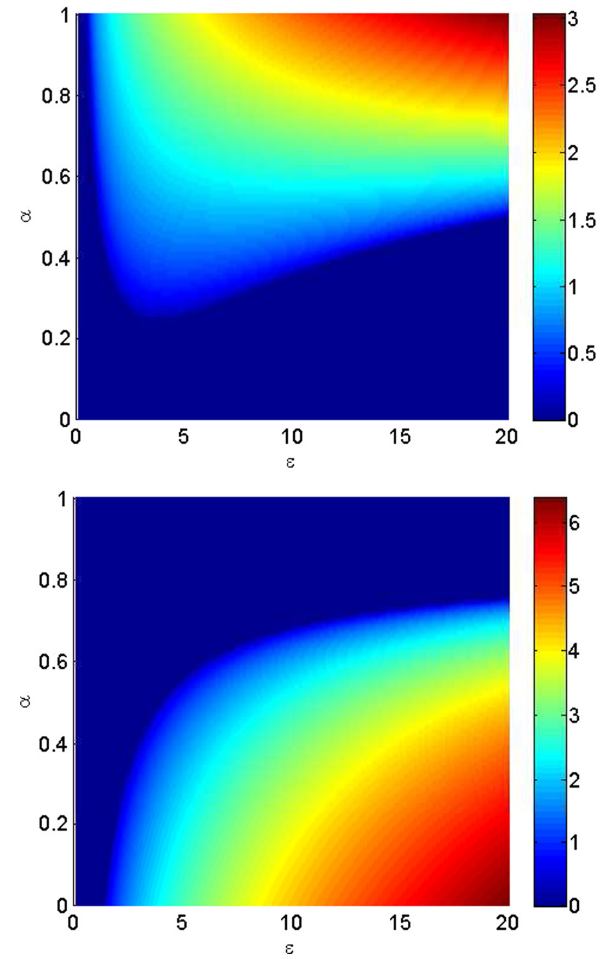

FIG. 2. (Color online) The order parameter m is plotted as a function of noise intensity ε and coupling mode α. The upper panel is for model 1 when the coupling strength $J = 20$: the symmetry-breaking state does not appear for Ito's noise; it appears approximately when $1 < \varepsilon < 20$ for Stratonovich's noise and approximately when $1 < \varepsilon < 310$ for anti-Ito's noise. The lower panel is for model 2 when $a = 1$, $c = 0.5$, and the coupling strength $J = 4$: the symmetry-breaking state appears approximately when $1 < \varepsilon$ for Ito's noise, when $4 < \varepsilon$ for Stratonovich's noise, and it does not appear for anti-Ito's noise. The increase of α enhances (inhibits) the ordered state for model 1 (2).

where $Z = \int_{-\infty}^{+\infty} dq[1/(1+cq^2)^{\alpha-1}]\exp[-(a+J)q^2/(2\varepsilon) + (J/\varepsilon)q\langle q\rangle]$. We obtain the order parameter from Eq. (3). The relation of J, ε, and α is

$$\frac{J}{N\varepsilon}\int_{-\infty}^{+\infty} dq \frac{q^2}{(1+cq^2)^{\alpha-1}} \exp\left[-\frac{(a+J)q^2}{2\varepsilon}\right] = 1, \quad (9)$$

where $N = \int_{-\infty}^{+\infty} dq[1/(1+cq^2)^{\alpha-1}]\exp[-(a+J)q^2/(2\varepsilon)]$.

The J-ε relations with $a = 1$ and $c = 0.5$ from Eq. (9) are plotted in the lower panel of Fig. 1. Different α values have corresponding different phase diagrams. Each curve separates the phase region $\langle q\rangle = 0$ (on the left of the curve) from the region $\langle q\rangle \neq 0$ (on the right of the curve). The order parameter m as a function of ε and α when $a = 1$, $c = 0.5$, and $J = 4$ is plotted in the lower panel of Fig. 2. For a different α mode, the noise intensity ε corresponding to the symmetry-breaking state varies, including (i) approximately $1 < \varepsilon$ for $\alpha = 0$, (ii) approximately $4 < \varepsilon$ for $\alpha = 1/2$, and (iii) no symmetry-breaking state for $\alpha = 1$.

The fact that the second model stays in the symmetry-breaking state once it is reached and no reentrant transition occurs can also be shown by the analysis of the maximum of the effective potential [21]: the symmetry-breaking state arises near $\varepsilon \approx a/[2c(1-\alpha)]$. From this method, we notice that the noise intensity $\varepsilon \to \infty$ when $\alpha \to 1$. Therefore, no symmetry-breaking state appears for $\alpha = 1$, and we reach the same conclusion as the above result. As a result, the phase transition for Eq. (7) does depend on α. In addition, the dependence of the phase transition on the coupling mode is in contrast with the previous conclusion [22,23], where the phase transition was regarded as qualitatively independent of α.

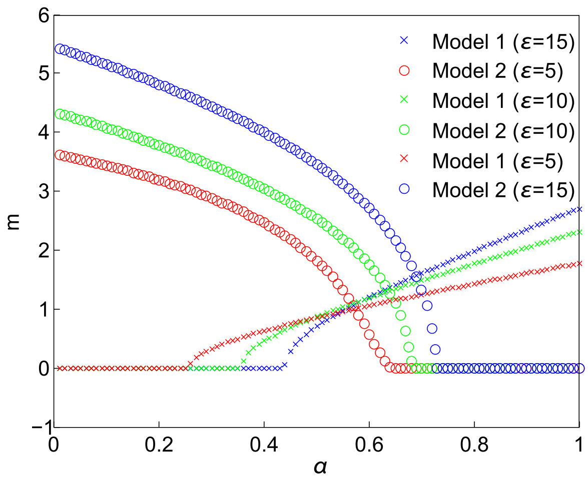

FIG. 3. (Color online) The order parameter m is plotted as a function of the coupling mode α: it is positively correlated with α for the first model (lines of cross) as the relation between the diffusion coefficient and the spatial coordinate $\partial D/\partial q|_{q>0} > 0$, while it is negatively correlated for the second one (lines of cycle) as $\partial D/\partial q|_{q>0} < 0$. The parameter setting is as follows: the noise intensity $\varepsilon = 15$ (blue upper line), $\varepsilon = 10$ (green middle line), and $\varepsilon = 5$ (red lower line); the coupling strength $J = 20$ for the first model; $a = 1$, $c = 0.5$, and $J = 4$ for the second model.

An increase of α has opposite effects on the order parameter of the above two models, as illustrated in Fig. 3. The detailed analysis is as follows. In our models, when the spatial coordinate of state variable gets further from the origin, the position of sampling points $\alpha q_n + (1-\alpha)q_{n-1}$ (n is positive integer) inside each bin of the integral sum has greater value for larger α. Therefore, the system takes stronger diffusibility with increasing α when the correlation of the diffusion coefficient $D(q)$ and the spatial coordinate q is positive, i.e., $\partial D/\partial q|_{q>0} > 0$, and vice versa. As the order parameter is defined as $\langle q\rangle$, the system becomes more ordered when the diffusibility is stronger. The first (second) model has $\partial D/\partial q|_{q>0} > 0\ (< 0)$, and thus an increase of α can enhance (inhibit) the ordered state.

III. DISCUSSION

The phase transition discussed above is different from those induced by the noise strength [20,21], where α was considered to be isolated points. Physically, the noise strength induces the reentrant phase transition in the first model, while the coupling mode does not lead to reentrant behavior. Mathematically, different stochastic integration for a Langevin equation can be transformed from one to another by adding a corresponding drift term in the equation [1,28,29]. The variation of the coupling mode is then equivalent to adding a noise-induced drift, while the change of the noise strength mainly influence the diffusion term. Thus, the coupling mode and noise strength are two different sources with cooperative effects for the phase transition.

The physical origin of the phase transition induced by α is the variation on the coupling mode between the deterministic and the stochastic forces [30]. Different α modes come from the uncertainty between forces with different time scales [1]: the mesoscopic time scale for the focused level and the microscopic time scale with fine structures for the noise. Intuitively, the fast variable noise term can be considered as a series of random pulses. A different kicking position for the pulses on the system's state leads to distinct effects on the order. The coupling mode determines at which position the random pulses drive the state variable of the system, and larger α represents later pulses. When the amplitude of a pulse is positively (negatively) correlated with the state variable, or equivalently the diffusion coefficient $\partial D/\partial q|_{q>0} > 0$ (<0), an increase of α enhances (inhibits) the ordered state by providing stronger (weaker) pulses. As a result, with a fixed noise intensity, the variation of α can lead to the appearance of multistable equilibrium probability and a symmetry-breaking phase transition.

A recent experiment sheds light on a possible way to control the α mode. A transition among α modes is implemented (Stratonovich-to-Ito transition with $\alpha \in [0,0.5]$) in an electric circuit [26]. This method can be applied to experimentally test our result on the phase transition induced by α. More specifically, the equilibrium distribution can be measured by the analog simulator for a given stochastic differential equation with $J = 0$ in Eq. (1) [12]. When $J \neq 0$, the effect of neighbors $\sum_{j\in n(i)} q_j/2d = \langle q_j\rangle$ is constant according to the Weiss mean-field method, and thus it can also be implemented by the analog simulator. As a result, given an *a priori* value for the order parameter $m = \langle q_j\rangle$ in Eq. (1), the equilibrium distribution can be measured. Then, Eq. (3) generates a value for the order parameter. When the generated value agrees with the *a priori* one, we get an actual value of the order parameter. We can repeat this process of measurements for various coupling modes α and noise intensity ε, and we plot $m - (\varepsilon,\alpha)$ diagrams comparable with Fig. 2.

IV. CONCLUSION

We have demonstrated that taking the coupling mode between deterministic and stochastic forces as a continuous parameter induces physically observable phase transitions. When the diffusion coefficient has a positive or negative correlation with the spatial coordinate q, an increase of α can enhance or inhibit the symmetry-breaking state, respectively. This mechanism is useful to control the effect of noise through manipulating the α mode without varying the noise strength. The origin of different α modes is the uncertainty in the mutual interaction between the focused level of description (mesoscopic) to its lower (microscopic) level. Our result provides an impetus for the search of physical consequences induced by the continuous variation of the coupling mode between different hierarchical levels both theoretically and experimentally.

ACKNOWLEDGMENTS

We thank Richard Blythe, Xiaomei Zhu, Min Tang, and Yuke Huang for critical comments. This work is supported in part by the National 973 Project No. 2010CB529200 and by the Natural Science Foundation of China Projects No. NSFC91029738.

[1] C. W. Gardiner, *Handbook of Stochastic Methods: For Physics, Chemistry, and the Natural Sciences*, 3rd ed. (Springer, Berlin, 2004).
[2] K. Itô, Proc. Imp. Acad. **20**, 519 (1944).
[3] R. L. Stratonovich, J. SIAM Control **4**, 362 (1966).
[4] P. Hänggi, Z. Phys. B **30**, 85 (1978).
[5] B. Øksendal, *Stochastic Differential Equations: An Introduction with Applications*, 3rd ed. (Springer, Berlin, 1992).
[6] N. Van Kampen, J. Stat. Phys. **24**, 175 (1981).
[7] U. Seifert, Rep. Prog. Phys. **75**, 126001 (2012).
[8] P. Hänggi, Helv. Phys. Acta **51**, 183 (1978).
[9] Y. Tang, R. Yuan, and P. Ao, Phys. Rev. E **89**, 062112 (2014).
[10] R. Yuan and P. Ao, J. Stat. Mech. (2012) P07010.
[11] P. Ao, C. Kwon, and H. Qian, Complexity **12**, 19 (2007).
[12] J. Smythe, F. Moss, and P. V. E. McClintock, Phys. Rev. Lett. **51**, 1062 (1983).
[13] G. Volpe, L. Helden, T. Brettschneider, J. Wehr, and C. Bechinger, Phys. Rev. Lett. **104**, 170602 (2010).
[14] R. Mannella and P. V. E. McClintock, Phys. Rev. Lett. **107**, 078901 (2011).
[15] G. Volpe, L. Helden, T. Brettschneider, J. Wehr, and C. Bechinger, Phys. Rev. Lett. **107**, 078902 (2011).
[16] T. Brettschneider, G. Volpe, L. Helden, J. Wehr, and C. Bechinger, Phys. Rev. E **83**, 041113 (2011).
[17] P. Lançon, G. Batrouni, L. Lobry, and N. Ostrowsky, Europhys. Lett. **54**, 28 (2001).
[18] J. Armero, J. M. Sancho, J. Casademunt, A. M. Lacasta, L. Ramirez-Piscina, and F. Sagues, Phys. Rev. Lett. **76**, 3045 (1996).
[19] J. Buceta, M. Ibañes, J. M. Sancho, and K. Lindenberg, Phys. Rev. E **67**, 021113 (2003).
[20] W. Horsthemke and R. Lefever, *Noise-Induced Transitions: Theory and Applications in Physics, Chemistry, and Biology*, 2nd ed. (Springer, Berlin, 2006), Vol. 15 .
[21] F. Sagués, J. M. Sancho, and J. García-Ojalvo, Rev. Mod. Phys. **79**, 829 (2007).
[22] M. Ibañes, J. García-Ojalvo, R. Toral, and J. M. Sancho, Phys. Rev. Lett. **87**, 020601 (2001).
[23] O. Carrillo, M. Ibañes, J. García-Ojalvo, J. Casademunt, and J. M. Sancho, Phys. Rev. E **67**, 046110 (2003).
[24] C. Van den Broeck, J. M. R. Parrondo, and R. Toral, Phys. Rev. Lett. **73**, 3395 (1994).
[25] C. Van den Broeck, J. M. R. Parrondo, R. Toral, and R. Kawai, Phys. Rev. E **55**, 4084 (1997).
[26] G. Pesce, A. McDaniel, S. Hottovy, J. Wehr, and G. Volpe, Nat. Commun. **4**, 2733 (2013).
[27] J.-h. Li and Z.-q. Huang, Phys. Rev. E **53**, 3315 (1996).
[28] P. Tupper and X. Yang, Proc. R. Soc. London, Ser. A **468**, 3864 (2012).
[29] F. Marchesoni, Materials **6**, 3598 (2013).
[30] P. Ao, Phys. Life Rev. **2**, 117 (2005).