Depolarization coefficients of light in multiply scattering media

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The depolarization coefficients are calculated for multiply scattered linearly and circularly polarized light. For a number of media (aqueous suspension of polystyrene particles, water droplets in air), the calculations are carried out both numerically, with solving the vector radiative transfer equation and analytically, within the polarization mode approximation. In the latter case the depolarization coefficients are expressed explicitly in terms of the scattering and absorption coefficients, and the scattering matrix elements of the medium. The range of applicability of the polarization mode approximation is established. For most practically important cases, this method is shown to provide a satisfactory degree of accuracy. We also find the fundamental values of the depolarization coefficients for a Rayleigh medium.

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I. INTRODUCTION

Over the last 20 years, great interest has been taken in polarized light propagation through various turbid media [1–26]. Polarization phenomena play a key role in multiple scattering by birefringent media (e.g., magneto-optic disordered systems [3], liquid crystals [6], etc.) and also accompany phase coherent transport of electromagnetic waves [15]. Many studies were motivated by applications to atmospheric and oceanic optics and diagnostics of biological tissues [5,7,8,12,14,22,24–26]. The discovery of the difference in depolarization rate between linearly and circularly polarized light [1,2,4,5,11,16,21] was one of the striking results of these studies.

From published experimental data it follows that polarization-sensitive methods provide a way of improving the image quality of objects hidden inside turbid media. Among the most promising approaches being currently studied for transillumination of highly scattering media, worthy of mention are the polarization-difference techniques (see, e.g., [7,12,14,22,24–26]) where the "image-bearing" component of light is extracted by subtracting the detected cross-polarized signal from the co-polarized one. As the diffusive photons with completely randomized polarization contribute equally to both polarized components of scattered radiation, subtraction of the cross-polarized component from the co-polarized one filters out the diffusive photons from the "image-bearing" photons. The feasibility of these techniques is contingent on the value of depolarization coefficients that govern the depth dependence of the difference between the polarized components of intensity. These coefficients, in their turn, are influenced by concentration, size of scattering particles, and their refractive index (the effect of these parameters on depolarization is experimentally studied with the use of the well-characterized media, e.g., aqueous suspensions of polystyrene microspheres [1,2,24–26]).

Previous theoretical studies of depolarization of light in multiply scattering media were mainly based on Monte Carlo simulations and various methods of numerical integration of the vector radiative transfer equation (see, e.g., [2,10–12,23]).

The most frequently used analytical results for the depolarization coefficients [2,17–19] (see also review [26]) rely on the assumption of the diffusive propagation of light in a scattering medium. Within the random matrix approach [2,17–19] the coefficients of polarization decay per unit path were calculated. Then, under the assumption that the distribution over paths at a given depth is described by the diffusion law, the depolarization coefficients themselves were found [2,17,19]. Within such an approach, the coefficients of polarization decay per unit path are supposed to be much less than the transport scattering coefficient, and depolarization of light follows the onset of the diffusion regime of wave propagation. This can be valid for circularly polarized light propagating through a medium with large particles (see, e.g., [4]). However, for linearly polarized light, decay of polarization occurs at scales of the order of the transport mean-free path up to the onset of the diffusion regime, and the condition mentioned above is never fulfilled. Approach [2,17–19], as shown below, overestimates noticeably the values of the coefficients of depolarization in the case of Rayleigh scattering. According to [2,17–19,26] the depolarization coefficients for linearly and circularly polarized light turn out to be greater than the scattering coefficient of the medium.

In this paper, we present the results of rigorous calculations of the depolarization coefficients beyond the assumption of the diffusive propagation of light. The numerical calculations are carried out with solving the characteristic equations of the vector radiative transfer equation. The analytical results are based on decoupling the vector radiative transfer equation. Our approach relies on the special features of the phase matrix in the circular representation. Using numerical calculations with the Mie theory, we show that the off-diagonal elements of the phase matrix prove to be small within a wide range of sizes and refractive index of scattering particles. In the first approximation we can neglect the off-diagonal elements. Then the vector radiative transfer equation decouples into independent equations. This corresponds to the basic mode approximation [27–29]. Taking into account the off-diagonal elements, we develop a perturbative procedure for calculating the depolarization coefficients of linearly and circularly polarized light. The obtained analytical results enable us to find explicit interrelation between the depolarization coefficients and the single-scattering characteristics of the medium. The range of applicability of the polarization mode approximation is established and it is shown that our results provide a good accuracy for most practically important cases. We also calculate the fundamental values of the depolarization

coefficients for the medium composed of Rayleigh particles. The obtained values correct result [2,17,19,26], which overestimates noticeably these coefficients in magnitude.

Our results are in good agreement with numerical computations [10,11] and experiments [2,5,8,16,21] and can be of interest for studying strongly scattering media by polarization optical methods.

II. GENERAL RELATIONS

Consider a beam of polarized light incident on a medium normally to its surface. The medium is assumed to be a statistically isotropic disordered ensemble of scatterers. The polarization state of scattered light can be described by the Stokes column vector [23,30–33]

$$\hat{S} = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}, \tag{1}$$

where four Stokes parameters are defined by relations

$$\begin{aligned} I &= \langle E_\parallel E_\parallel^* + E_\perp E_\perp^* \rangle, \\ Q &= \langle E_\parallel E_\parallel^* - E_\perp E_\perp^* \rangle, \\ U &= \langle E_\parallel E_\perp^* + E_\parallel^* E_\perp \rangle, \\ V &= i\langle E_\parallel E_\perp^* - E_\parallel^* E_\perp \rangle. \end{aligned} \tag{2}$$

The Stokes parameters and the components $E_\parallel$ and $E_\perp$ of the electric field appearing in Eq. (2) are defined in the system of unit vectors $\mathbf{e}_\parallel = \partial\mathbf{n}/\partial\theta$, $\mathbf{e}_\perp = [\mathbf{n},\mathbf{e}_\parallel]$, and $\mathbf{n}$. The unit vector $\mathbf{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ is the direction of propagation of the transverse electromagnetic wave, the vector $\mathbf{e}_\parallel$ lies in the plane formed by the vectors $\mathbf{n}_0$ and $\mathbf{n}$ (where $\mathbf{n}_0$ is the internal normal to the surface), and the vector $\mathbf{e}_\perp$ is perpendicular to this plane. The brackets $\langle \ldots \rangle$ denote statistical averaging.

The Stokes parameters obey the vector radiative transfer equation [30–32],

$$\left\{ \mathbf{n}\frac{\partial}{\partial \mathbf{r}} + \sigma_{\rm tot} \right\} \hat{S}(z,\mathbf{n}) = \sigma \int d\mathbf{n}' \hat{Z}(\mathbf{n},\mathbf{n}')\hat{S}(z,\mathbf{n}'), \tag{3}$$

where $\sigma_{\rm tot} = \sigma + \sigma_a$ is the coefficient of total extinction; σ and σ_a are the coefficients of scattering and absorption,

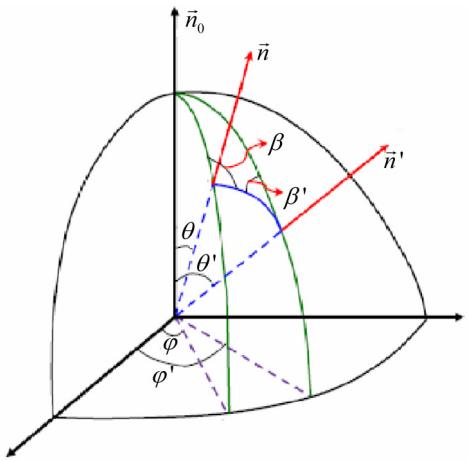

FIG. 1. (Color online) Geometry of scattering.

respectively. The phase matrix $\hat{Z}(\mathbf{n},\mathbf{n}')$ entering into Eq. (3) can be expressed in terms of the scattering matrix as follows (see [23]):

$$\hat{Z}(\mathbf{n},\mathbf{n}') = \hat{L}(\pi - \beta)\hat{F}(\cos\gamma)\hat{L}(-\beta'), \tag{4}$$

where $\cos\gamma = \mathbf{n}\mathbf{n}'$. The scattering matrix $\hat{F}(\cos\gamma)$ describes the intrinsic properties of the medium. The matrix $\hat{L}(-\beta')$ transforms the Stokes parameters of the incident light in going from the system of unit vectors $\{\mathbf{e}'_\parallel, \mathbf{e}'_\perp, \mathbf{n}'\}$ to the scattering plane (i.e., the plane formed by vectors $\mathbf{n}$ and $\mathbf{n}'$; see Fig. 1). The matrix $\hat{L}(\pi - \beta)$ corresponds to the inverse transformation from the scattering plane to the system of unit vectors $\{\mathbf{e}_\parallel, \mathbf{e}_\perp, \mathbf{n}\}$ related to the direction of propagation of the scattered light. The $\hat{L}$ matrix is given in Appendix A. The angles entering into Eq. (4) are defined by formulas

$$\begin{aligned} \cos 2\beta &= 1 - \frac{2(1-\mu'^2)(1-\cos^2\psi)}{1-(\mathbf{n}\mathbf{n}')^2}, \\ \sin 2\beta &= \frac{2\sqrt{1-\mu'^2}(\mu'\sqrt{1-\mu^2} - \mu\sqrt{1-\mu'^2}\cos\psi)\sin\psi}{1-(\mathbf{n}\mathbf{n}')^2}, \end{aligned} \tag{5}$$

$$\begin{aligned} \mathbf{n}\mathbf{n}' &= \mu\mu' + \sqrt{(1-\mu^2)(1-\mu'^2)}\cos\psi, \quad \psi = \varphi - \varphi', \\ \mu &= \mathbf{n}\mathbf{n_0} = \cos\theta, \quad \mu' = \mathbf{n}'\mathbf{n_0} = \cos\theta'. \end{aligned} \tag{6}$$

The functions $\cos 2\beta'$ and $\sin 2\beta'$ are obtained from Eq. (5) by interchanging μ and μ'.

For a macroscopically isotropic and symmetric medium, the scattering matrix $\hat{F}(\cos\gamma)$ appearing in Eq. (4) has the block-diagonal structure (see, e.g., [23,31]):

$$\hat{F}(\cos\gamma) = \begin{pmatrix} a_1(\cos\gamma) & b_1(\cos\gamma) & 0 & 0 \\ b_1(\cos\gamma) & a_2(\cos\gamma) & 0 & 0 \\ 0 & 0 & a_3(\cos\gamma) & b_2(\cos\gamma) \\ 0 & 0 & -b_2(\cos\gamma) & a_4(\cos\gamma) \end{pmatrix}. \tag{7}$$

For the forward scattering ($\mathbf{n} = \mathbf{n}'$, $\cos\gamma = 1$), the $\hat{F}$ matrix is diagonal: $\hat{F}(1) = \mathrm{diag}(a_1,a_2,a_3,a_4)$ and $a_2(1) = a_3(1)$ [23]. The matrix element a_1 appearing in matrix (7) is the scattering phase function. This element is normalized by the relation

$$\int d\mathbf{n}' a_1(\mathbf{n}\mathbf{n}') = 1. \tag{8}$$

Calculation of the Stokes parameters with the vector radiative transfer equation is valid for all directions excluding a narrow vicinity near the backward direction where the coherent backscattering due to wave interference should be taken into account (regarding interrelation between polarization of the coherent backscattering and the solution of Eq. (3), see Refs. [23,34]).

As was shown in [35], to describe multiple scattering of light in turbid media, it is more convenient to go from the linear basis [see Eq. (2)] to the circular basis, where the electric field is defined in the system of unit vectors

$$\mathbf{e}_{\pm} = \frac{1}{\sqrt{2}}(\mathbf{e}_{\parallel} \pm \mathrm{i}\mathbf{e}_{\perp})$$

as superposition of waves with right (+) and left (−) circular polarizations.

For the circular representation, column vector [35,36]

$$\hat{I} = \begin{pmatrix} \langle E_- E_+^* \rangle \\ \langle |E_+|^2 \rangle \\ \langle |E_-|^2 \rangle \\ \langle E_-^* E_+ \rangle \end{pmatrix} = \frac{1}{2}\begin{pmatrix} Q - \mathrm{i}U \\ I - V \\ I + V \\ Q + \mathrm{i}U \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} I_2 \\ I_0 \\ I_{-0} \\ I_{-2} \end{pmatrix} \tag{9}$$

is an analog of Stokes vector (1).

The vector $\hat{I}$ obeys the following vector radiative transfer equation [23,35,36]:

$$\left\{ \mathbf{n}\frac{\partial}{\partial \mathbf{r}} + \sigma_{\mathrm{tot}} \right\} \hat{I}(\mathbf{r},\mathbf{n}) = \sigma \int d\mathbf{n}' \hat{d}(\mathbf{n},\mathbf{n}') \hat{I}(\mathbf{r},\mathbf{n}'). \tag{10}$$

The phase matrix $\hat{d}(\mathbf{n},\mathbf{n}')$ appearing in Eq. (10) is given by [23]

$$\hat{d}(\mathbf{n},\mathbf{n}') = \begin{pmatrix} a_+^{(2,3)} \exp(2i\chi_+) & b_+ \exp(-2i\beta) & b_- \exp(-2i\beta) & a_-^{(2,3)} \exp(2i\chi_-) \\ b_+ \exp(-2i\beta') & a_+^{(1,4)} & a_-^{(1,4)} & b_- \exp(2i\beta') \\ b_- \exp(-2i\beta') & a_-^{(1,4)} & a_+^{(1,4)} & b_+ \exp(2i\beta') \\ a_-^{(2,3)}(-2i\chi_-) & b_- \exp(2i\beta) & b_+ \exp(2i\beta) & a_+^{(2,3)} \exp(-2i\chi_+) \end{pmatrix}, \tag{11}$$

where angles $\chi_{\pm}$ and β, β' are related by

$$\chi_{\pm} = \pi - (\beta \pm \beta'). \tag{12}$$

Functions $a_{\pm}^{(i,j)}$ and $b_{\pm}$ entering into Eq. (11) are expressed in terms of the elements of scattering matrix (7) and have the form

$$a_{\pm}^{(i,j)} = \frac{a_i \pm a_j}{2}, \quad b_{\pm} = \frac{b_1 \pm i b_2}{2}. \tag{13}$$

For spherical particles of given radius and refractive index, matrix elements (13) can be expressed in terms of the scattering amplitudes [23,33]. Quantities a_i (i = 1–4), b_1, and b_2 are equal to

$$a_1(\cos\gamma) = a_2(\cos\gamma) = \frac{n_0}{2\sigma}(|A_{\parallel}(\cos\gamma)|^2 + |A_{\perp}(\cos\gamma)|^2), \tag{14}$$

$$a_3(\cos\gamma) = a_4(\cos\gamma) = \frac{n_0}{\sigma}\mathrm{Re}A_{\parallel}(\cos\gamma)A_{\perp}^*(\cos\gamma), \tag{15}$$

$$b_1(\cos\gamma) = \frac{n_0}{2\sigma}(|A_{\parallel}(\cos\gamma)|^2 - |A_{\perp}(\cos\gamma)|^2), \tag{16}$$

$$b_2(\cos\gamma) = \frac{n_0}{\sigma}\mathrm{Im}A_{\parallel}(\cos\gamma)A_{\perp}^*(\cos\gamma), \tag{17}$$

where $A_{\parallel}$ and $A_{\perp}$ are the amplitudes of the components polarized parallel and perpendicularly to the scattering plane, and n_0 is the number of scattering particles per unit volume. The values of $A_{\parallel}$ and $A_{\perp}$ can be calculated with the Mie theory [23,32,33]. In the limiting case of the Born particles ($ka|n-1| \ll 1$, where $k = 2\pi/\lambda$, and a and n are the radius and the relative refractive index of the particles; λ is a wavelength), amplitude $A_{\parallel}$ is related to amplitude $A_{\perp}$ by equation $A_{\parallel}(\cos\gamma) = A_{\perp}(\cos\gamma)\cos\gamma$ [23,32,33].

In the case of the normal incidence, quantities I_m ($m = \pm 0, \pm 2$) in Eq. (9) can be expanded in azimuthal harmonics [27–29,37,38]:

$$I_{\pm 2}(z,\mathbf{n}) = \frac{1}{\sqrt{2}}[W(z,\mu)\exp(\pm 2i\varphi) + Q_I(z,\mu) \mp i\, U_V(z,\mu) + w(z,\mu)\exp(\mp 2i\varphi)],$$

$$I_{\pm 0}(z,\mathbf{n}) = \frac{1}{\sqrt{2}}[I(z,\mu) \mp V(z,\mu) + I_W(z,\mu)\cos 2\varphi \mp V_W(z,\mu)\sin 2\varphi]. \tag{18}$$

The functions entering into Eq. (18) obey three independent systems of transfer equations:

$$\left\{\mu\frac{\partial}{\partial z}+\sigma_{\rm tot}\right\}\begin{pmatrix} I(z,\mu)\\ Q_I(z,\mu)\end{pmatrix}=\sigma\int d\mathbf{n}'\begin{pmatrix} a_1 & b_1e^{2i\beta'}\\ b_1e^{-2i\beta} & a_+^{(2,3)}e^{2i\chi_+}+a_-^{(2,3)}e^{2i\chi_-}\end{pmatrix}\begin{pmatrix} I(z,\mu')\\ Q_I(z,\mu')\end{pmatrix}, \tag{19}$$

$$\left\{\mu\frac{\partial}{\partial z}+\sigma_{\rm tot}\right\}\begin{pmatrix} W(z,\mu)\\ I_W(z,\mu)\\ V_W(z,\mu)\\ w(z,\mu)\end{pmatrix}=\sigma\int d\mathbf{n}'\begin{pmatrix} a_+^{(2,3)}e^{2i(\chi_+-\psi)} & \frac{b_1}{2}e^{-2i(\beta+\psi)} & -\frac{b_2}{2}e^{-2i(\beta+\psi)} & a_-^{(2,3)}e^{2i(\chi_--\psi)}\\ b_1e^{-2i(\beta'+\psi)} & a_1e^{-2i\psi} & 0 & b_1e^{2i(\beta'-\psi)}\\ b_2e^{-2i(\beta'+\psi)} & 0 & a_4e^{-2i\psi} & -b_2e^{2i(\beta'-\psi)}\\ a_-^{(2,3)}e^{-2i(\chi_-+\psi)} & \frac{b_1}{2}e^{2i(\beta-\psi)} & \frac{b_2}{2}e^{2i(\beta-\psi)} & a_+^{(2,3)}e^{-2i(\chi_++\psi)}\end{pmatrix}\begin{pmatrix} W(z,\mu')\\ I_W(z,\mu')\\ V_W(z,\mu')\\ w(z,\mu')\end{pmatrix}, \tag{20}$$

$$\left\{\mu\frac{\partial}{\partial z}+\sigma_{\rm tot}\right\}\begin{pmatrix} V(z,\mu)\\ U_V(z,\mu)\end{pmatrix}=\sigma\int d\mathbf{n}'\begin{pmatrix} a_4 & -b_2e^{2i\beta'}\\ b_2e^{-2i\beta} & a_+^{(2,3)}e^{2i\chi_+}-a_-^{(2,3)}e^{2i\chi_-}\end{pmatrix}\begin{pmatrix} V(z,\mu')\\ U_V(z,\mu')\end{pmatrix}. \tag{21}$$

The first pair of equations [see Eq. (19)] enables us to calculate the intensity of radiation and the azimuth-independent contribution to the second Stokes parameter. Quantity Q_I coincides with the second Stokes parameter of an initially unpolarized beam [39]. If the incident light is not polarized, the state of scattered light is characterized only by quantities I and Q_I. The state of a linearly polarized beam multiply scattered in the medium is governed by Eqs. (19) and (20). For a circularly polarized incident beam, the third set of equations [see Eq. (21)] comes into play instead of Eq. (20).

Expression (18) and Eqs. (19)–(21) are valid for arbitrary initial polarization which can be taken into account in the boundary condition at $z=0$. In particular, for the elliptically polarized incident light, the initial values of the functions W and V entering into Eq. (18) are equal to

$$W=I\sqrt{1-P_C^2},\quad V=I\,P_C,$$

where I and P_C are the intensity at $z=0$ and the degree of circular polarization of the incident light. The other functions, Q_I, I_W, V_W, w, and U_V, should be put to be equal to zero at $z=0$.

III. BASIC MODE APPROXIMATION

In the case of scattering by large inhomogeneities (size a is larger than wavelength λ) the off-diagonal elements of the phase matrix appearing in Eqs. (19)–(21) turn out to be small as compared to the diagonal ones (see Fig. 2). In the first approximation we can neglect the off-diagonal elements. Then Eqs. (19)–(21) are reduced to independent transfer equations [27–29,37,38]. These equations describe propagation of the individual modes appearing in Eq. (18).

For a linearly polarized incident beam, the boundary condition at $z=0$ takes the form $I=W$, and $Q_I=I_W=V_W=w=0$. Therefore we can consider only the transfer equations for modes I and W; no other modes are excited.

Quantity I is the specific intensity and subject to the ordinary scalar transfer equation (see, e.g., [31,32])

$$\left\{\mu\frac{\partial}{\partial z}+\sigma_{\rm tot}\right\}I(z,\mu)=\sigma\int d\mathbf{n}'a_1(\mathbf{nn}')I(z,\mu'). \tag{22}$$

Quantity W was named [27–29,37,38] by the basic mode of linear polarization. The transfer equation for W has the form

$$\left\{\mu\frac{\partial}{\partial z}+\sigma_{\rm tot}\right\}W(z,\mu)=\sigma\int d\mathbf{n}'a_+^{(2,3)}(\mathbf{nn}')\exp\{2i[\chi_+(\mathbf{n},\mathbf{n}')-\psi]\}W(z,\mu'). \tag{23}$$

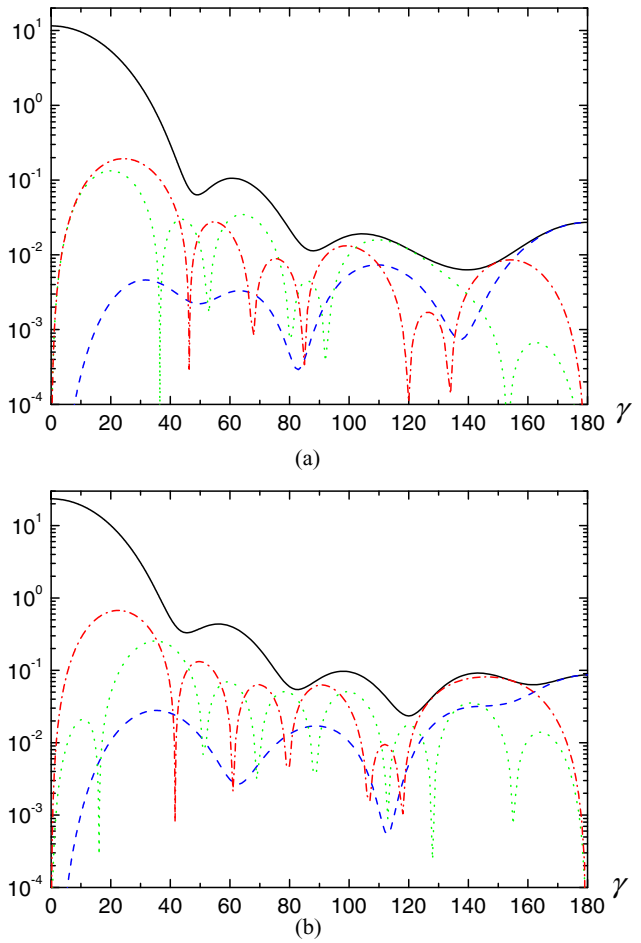

FIG. 2. (Color online) Angular dependence of scattering matrix elements for aqueous suspension of polystyrene microspheres (a) and water droplets in air (b) ($ka=5$): solid black, dashed blue, dotted green, and dash-dotted red lines represent a_1, $a_-^{(2,3)}$, $|b_1|$, and $|b_2|$, respectively. The numerical calculations were carried out with the Mie theory [23,32,33].

The modes Q_I, I_W, V_W, and w come into play only if the off-diagonal elements are taken into account. In the latter case the basic modes I and W generate the "sources" in the corresponding equations for modes Q_I, I_W, V_W, and w.

For a circularly polarized light ($I = V$ and $Q_I = U_V = 0$ at $z = 0$), the basic mode approximation is similar to that discussed above. When neglecting the off-diagonal elements of the phase matrix appearing in Eq. (21), we arrive at the individual transfer equation for the basic mode V of circular polarization,

$$\left\{\mu\frac{\partial}{\partial z} + \sigma_{\rm tot}\right\}V(z,\mu) = \sigma\int d\mathbf{n}' a_4(\mathbf{n}\mathbf{n}')V(z,\mu'). \qquad (24)$$

Within the basic mode approximation, a beam of light propagating through a turbid medium is described by three quantities, I, W, and V; each of them is subject to its own transfer equation. The separate equations for W and V were also derived in a different way, without resorting to the circular representation, in [40].

In the polarization-difference techniques (see, e.g., [22,24–26]) which are based on subtracting the detected cross-polarized signal from the co-polarized one, only two modes W and V are responsible for data of measurements. Intensity I is not required to process polarization-difference images.

The equations for W and V [see Eqs. (23) and (24)] differ from the scalar transfer equation (22) by the form of the effective phase functions. The effective phase functions appearing in Eqs. (23) and (24) are $a_+^{(2,3)}\exp[2i(\chi_+ - \psi)]$ and a_4, respectively. The difference between these phase functions and phase function a_1 entering into Eq. (22) results in the fact that the integral of either of the effective phase functions over directions is less than unity [as opposed to Eq. (8)]. This gives rise to nonzero effective "absorption" in Eqs. (23) and (24) (even in the absence of true absorption). The effective "absorption" in Eqs. (23) and (24) is responsible for the additional attenuation of W and V as compared to intensity I and describes the effect of depolarization of linearly and circularly polarized light [27–29,37,38].

There are two different reasons [4] (see also [27–29,37,38]) for wave depolarization in a scattering medium. These reasons were first pointed out in the context of the study of wave propagation through a turbulent atmosphere [41–43].

The "geometrical" depolarization is due to the Rytov rotation [44] of the polarization plane. According to [44], the plane of polarization turns, as the ray of light propagates along a nonplanar curve. The depolarization observed in multiple scattering of linearly polarized light results from superposition of randomly oriented polarizations of the waves propagating along different random paths. Therefore, the "geometrical" depolarization occurs simultaneously with isotropization of light over the directions of propagation at depths of the order of the transport mean-free path l_{tr} [$l_{tr} = l/(1 - \langle\cos\gamma\rangle)$, where $l = \sigma^{-1}$ is the mean-free path, and $\langle\cos\gamma\rangle$ is the mean cosine of single scattering] [4]. The situation is different for circularly polarized light. Circularly polarized light can be presented as a superposition of two linearly cross-polarized waves shifted in phase by $\pi/2$. In multiple scattering, the Rytov effect results in the turn of the polarization plane of each linearly polarized component, but has no effect on the phase shift between them. Therefore, a circularly polarized wave propagating along any random trajectory is unaffected by the Rytov rotation. The wave remains circularly polarized.

A similar picture of depolarization was also discussed in [45,46] in close analogy with the torsion statistics of semiflexible polymers of the type of DNA considered as fluctuating stiff threads.

The pure "geometrical" depolarization can be realized in the limit $A_\parallel = A_\perp$ (or $a_1 = a_2 = a_3 = a_4$) [37]. In this case, phase matrix (11) takes the diagonal form and is expressed in terms of phase function a_1 and angle χ_+ [see Eq. (12)]. Depolarization results from multiple turns of the polarization plane as the direction of wave propagation changes randomly [4,37].

The difference between the amplitudes $A_\parallel$ and $A_\perp$ (or, the difference between diagonal elements a_i, $i = 1$–4) are responsible for the "dynamical" mechanism of depolarization. The "dynamical" depolarization occurs independently of the initial polarization of light. In particular, circularly polarized light depolarizes only due to the "dynamical" reason (the difference between a_1 and a_4 is responsible for depolarization of circularly polarized waves) [4,27–29].

In the case of the linearly polarized incident beam, the role of one or the other reason for depolarization depends on the optical properties of the scattering particles. The "geometrical" depolarization can be either dominant or as important as the "dynamical" depolarization [27–29].

IV. DEPOLARIZATION COEFFICIENTS IN THE ASYMPTOTIC STATE

We can take advantage of Eqs. (19)–(21) to calculate the coefficients of depolarization in the asymptotic state of propagation ($z > l_{tr}$). Analysis of the asymptotic limit is important for various practical applications (see, e.g., [1,2,5,22,26]).

The elements of phase matrix (11) are expanded in the generalized spherical functions [35] (see also [23]). Therefore, solutions to Eqs. (19)–(21) can be sought in the form of series in these functions as well [35]. In the scalar transfer theory, this approach is known as the P_l approximation [47]. As applied to the vector radiative transfer equation, expansion in the generalized spherical functions was originally proposed in [35], and used then in a number of studies as a basis for numerical and analytical calculations (e.g., see [36,48]). In particular, this approach was applied to calculating spatial moments of the photon distribution generated by a pulsed source of light [9,20].

Solutions of Eqs. (19)–(21) in the asymptotic state can be represented as [27–29]

$$\begin{pmatrix} I(z,\mu) \\ Q_I(z,\mu)\end{pmatrix} = \exp(-\varepsilon_I z)\sum_{l=0}\frac{2l+1}{4\pi}\begin{pmatrix} I(l)P_l(\mu) \\ Q_I(l)P_{20}^l(\mu)\end{pmatrix}, \qquad (25)$$

$$\begin{pmatrix} W(z,\mu) \\ I_W(z,\mu) \\ V_W(z,\mu) \\ w(z,\mu)\end{pmatrix} = \exp(-\varepsilon_W z)\sum_{l=2}\frac{2l+1}{4\pi}\begin{pmatrix} W(l)P_{22}^l(\mu) \\ I_W(l)P_{20}^l(\mu) \\ V_W(l)P_{20}^l(\mu) \\ w(l)P_{2-2}^l(\mu)\end{pmatrix}, \qquad (26)$$

$$\begin{pmatrix} V(z,\mu) \\ U_V(z,\mu) \end{pmatrix} = \exp(-\varepsilon_V z) \sum_{l=0} \frac{2l+1}{4\pi} \begin{pmatrix} V(l)P_l(\mu) \\ U_V(l)P^l_{20}(\mu) \end{pmatrix}, \quad (27)$$

where $P_l(\mu)$ [$P_l(\mu) \equiv P^l_{00}(\mu)$] and $P^l_{mn}(\mu)$ are the Legendre polynomials and the generalized spherical functions, respectively. Detailed definitions and properties of $P^l_{mn}(\mu)$ can be found in [49].

Substituting expansions (25)–(27) into Eqs. (19)–(21) we arrive at an eigenvalue problem for each system of the coupling equations. The corresponding systems of equations are given in Appendix B [see Eqs. (B1)–(B3)]. Coefficients ε_I, ε_W, and ε_V are minimum eigenvalues. The quantities ε_W and ε_V are the depolarization coefficients that govern decay of polarization with depth z. In particular, decrease in the polarization difference (for linearly and circularly polarized light $I_\parallel - I_\perp$ and $I_+ - I_-$, respectively) with depth z is just characterized by the coefficients ε_W and ε_V.

When the optical properties of the medium are known, the solutions to Eqs. (B1)–(B3) can be found numerically, by truncating the dimension of the corresponding system at some $l_{\max}$. Eigenvalues ε_I, ε_W, and ε_V are determined as the roots of the corresponding characteristic equations. For the intensity, the asymptotic state has been studied in detail (see, e.g., [35]). In what follows we pay our attention to analysis of depolarization of linearly and circularly polarized beams in a nonabsorbing medium ($\sigma_a = 0$).

A. Linearly polarized light

Within the basic mode approximation where the off-diagonal elements are neglected, the value of coefficient ε_W is determined from the characteristic equation for mode W. Explicit analytical expressions for ε_W [27–29] can be easily derived within the one-polynomial

$$\varepsilon^{(1)}_W = \frac{3}{2}\sigma^{(2,3)}_+(2) \quad (28)$$

and two-polynomial

$$\varepsilon^{(2)}_W = \frac{7}{6}\left[\sigma^{(2,3)}_+(2) + 2\sigma^{(2,3)}_+(3)\right] \times \left\{1 - \sqrt{1 - \frac{36}{7}\frac{\sigma^{(2,3)}_+(2)\sigma^{(2,3)}_+(3)}{\left[\sigma^{(2,3)}_+(2) + 2\sigma^{(2,3)}_+(3)\right]^2}}\right\} \quad (29)$$

approximations, where quantities $\sigma^{(2,3)}_+(l)$ ($l = 2,3$) are expressed in terms of the expansion coefficients of matrix element $a^{(2,3)}_+$ in the generalized spherical functions [see Appendix B, Eq. (B4)]. Comparison with numerical solution of the characteristic equation for W showed [29] that Eq. (29) gives virtually the exact result.

To analyze the effect of the off-diagonal elements, first we consider the result of calculations of the depolarization coefficient within the one-polynomial approximation. Within this approximation the characteristic equation looks like as follows:

$$\det\begin{pmatrix} \frac{2\varepsilon_W}{3-\sigma^{(2,3)}_+(2)} & \frac{\sigma}{2}b_1(2) & -\frac{\sigma}{2}b_2(2) & \sigma a^{(2,3)}_-(2) \\ \sigma b_1(2) & -\sigma_1(2) & 0 & \sigma b_1(2) \\ \sigma b_2(2) & 0 & -\sigma_4(2) & -\sigma b_2(2) \\ \sigma a^{(2,3)}_-(2) & \frac{\sigma}{2}b_1(2) & \frac{\sigma}{2}b_2(2) & -\frac{2\varepsilon_W}{3} - \sigma^{(2,3)}_+(2) \end{pmatrix} = 0, \quad (30)$$

where quantities $\sigma_1(l)$, $\sigma_4(l)$, $b_1(l)$, and $b_2(l)$ ($l = 2$) are defined in Appendix B by Eq. (B4). Then the ε_W coefficient is equal to

$$\varepsilon_W = \frac{3}{2}\sigma^{(2,3)}_+(2)\left(1 - \frac{\sigma a^{(2,3)}_-(2)}{\sigma^{(2,3)}_+(2)} - \frac{[\sigma b_1(2)]^2}{\sigma^{(2,3)}_+(2)\sigma_1(2)}\right)^{1/2} \times \left(1 + \frac{\sigma a^{(2,3)}_-(2)}{\sigma^{(2,3)}_+(2)} + \frac{[\sigma b_2(2)]^2}{\sigma^{(2,3)}_+(2)\sigma_4(2)}\right)^{1/2}. \quad (31)$$

Comparison of Eqs. (28) and (31) shows that the effect of the off-diagonal elements of the scattering matrix on the value ε_W is characterized by quantities

$$\left(\frac{\sigma a^{(2,3)}_-(2)}{\sigma^{(2,3)}_+(2)}\right)^2, \quad \frac{[\sigma b_1(2)]^2}{\sigma^{(2,3)}_+(2)\sigma_1(2)}, \quad \frac{[\sigma b_2(2)]^2}{\sigma^{(2,3)}_+(2)\sigma_4(2)}. \quad (32)$$

The dependence of quantities (32) on the radius of scattering particles is illustrated in Fig. 3 for two important cases. Polystyrene microspheres in water are frequently used for modeling optical properties of biological tissues (see, e.g., [2,5,16,26]). Water droplets in air are the model of atmospheric aerosols. From Fig. 3 it follows that the effect of the off-diagonal elements can be observable only for relatively small particles $ka(n-1) \leqslant 1$. In the limiting case of Rayleigh scattering particles [23,32,33] quantities (32) are equal to $9/49$, $2/21$, and zero, respectively. As a result, the contribution of the off-diagonal elements to the value of the depolarization coefficient ε_W can be supposed to be small and taken into account within a perturbative method.

With allowance for the off-diagonal elements, generalization of the two-polynomial formula (29) gives the following result:

$$\varepsilon_W = \varepsilon^{(2)}_W + \delta\varepsilon_W, \quad (33)$$

$$\delta\varepsilon_W = 3\sigma^{(2,3)}_+(2)\Psi\left(\frac{\sigma a^{(2,3)}_-(2)}{\sigma^{(2,3)}_+(2)} + \frac{[\sigma b_1(2)]^2}{\sigma^{(2,3)}_+(2)\sigma_1(2)}, \frac{\sigma a^{(2,3)}_-(2)}{\sigma^{(2,3)}_+(2)} + \frac{[\sigma b_2(2)]^2}{\sigma^{(2,3)}_+(2)\sigma_4(2)}, \frac{\sigma^{(2,3)}_+(3)}{\sigma^{(2,3)}_+(2)}\right), \quad (34)$$

where function $\Psi(a,b,c)$ is presented in Appendix C.

Figures 4 and 5 illustrate accuracy of different approximations in calculating the depolarization coefficient ε_W. From Fig. 4 it follows that Eq. (29) provides a good accuracy at $ka(n-1) \geqslant 0.5$. The more general result (33), (34) is applicable all over the range of variations of size parameter ka and refractive index n.

As our calculations show (see Fig. 6) ratio $\varepsilon_W/\sigma_{tr}$ is of the order of unity and depends only slightly on the particle radius at $ka(n-1) > 1$–2. For such values of $ka(n-1)$, coefficient ε_W can be calculated with Eq. (29) (i.e., without allowance

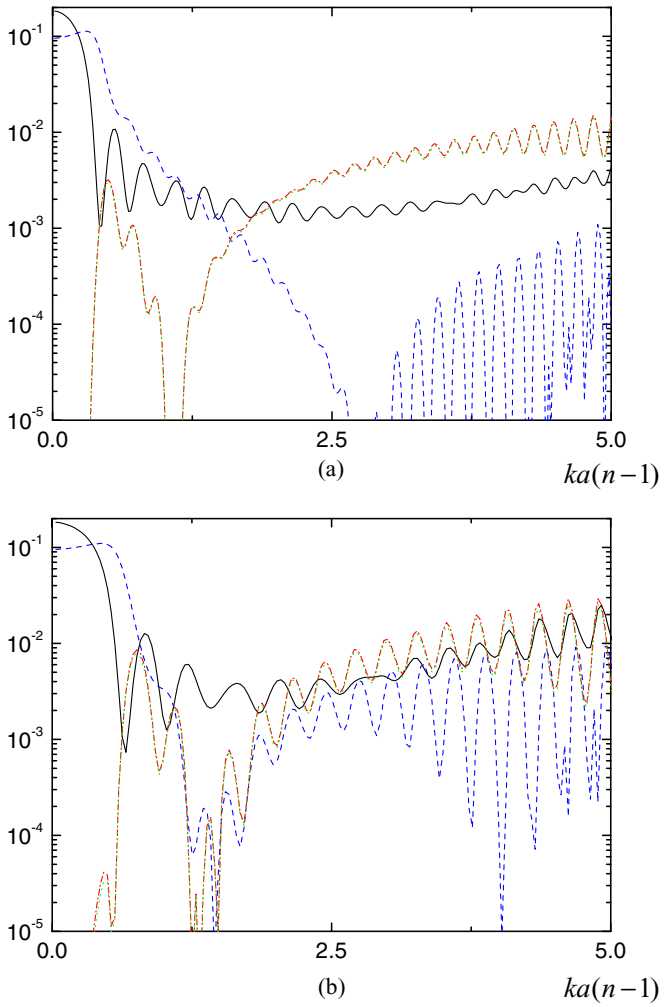

FIG. 3. (Color online) Quantities $(\frac{\sigma a_-^{(2,3)}(2)}{\sigma_+^{(2,3)}(2)})^2$ (solid black), $\frac{[\sigma b_1(2)]^2}{\sigma_+^{(2,3)}(2)\sigma_1(2)}$ (dashed blue), $\frac{[\sigma b_2(2)]^2}{\sigma_+^{(2,3)}(2)\sigma_4(2)}$ (dotted green), and $\frac{[\sigma b_2(2)]^2}{\sigma_3(2)\sigma_4(2)}$ (dash-dotted red) appearing in Eq. (31) as a function of radius a of scattering particles for aqueous suspension of the polystyrene microspheres (a) and water droplets in air (b). The numerical calculations were carried out with the Mie theory [23,32,33].

for the off-diagonal elements of the phase matrix). Moreover, the main contribution to ε_W results from the "geometrical" depolarization [27–29]. The value of $\varepsilon_W^{\text{geom}}$ calculated within the assumption that the depolarization is only due to the Rytov rotation (i.e., $a_1 = a_2 = a_3 = a_4$) differs little from the exact value of ε_W (see the inset in Fig. 6).

B. Circularly polarized light

Under certain conditions, propagation of circularly polarized light through a scattering medium can be considered as the spatial diffusion [4]. The diffusion is described by the two-polynomial approximation in expansion (25) for V. In this approximation the depolarization coefficient ε_V is given by

$$\varepsilon_V = \sqrt{3\sigma_4(0)\sigma_4(1)}, \tag{35}$$

where quantities $\sigma_4(0)$ and $\sigma_4(1)$ are defined in Appendix B by Eq. (B4).

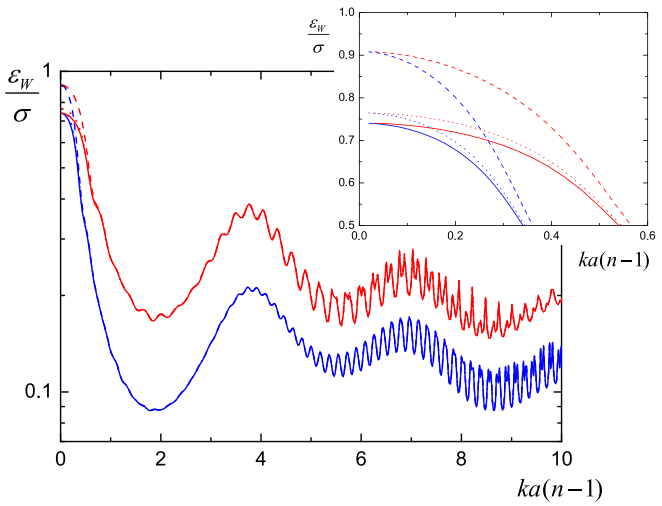

FIG. 4. (Color online) Depolarization coefficient ε_W as a function of radius a of scattering particles for aqueous suspension of polystyrene microspheres [blue (bottom) lines] and water droplets in the air [red (upper) lines]. Calculations of ε_W are carried out with numerical solution of the characteristic equation ($l_{\max} = 10$, solid line) and with Eqs. (29) (dashed lines), (33) (dotted lines).

In a medium with no true absorption, quantity $\sigma_4(0)$ is determined by the difference between the phase function a_1 and the "effective" phase function a_4 entering into the transfer equation (24) for V,

$$\sigma_4(0) = \sigma \int d\mathbf{n}'[a_1(\mathbf{n}\mathbf{n}') - a_4(\mathbf{n}\mathbf{n}')], \tag{36}$$

and is responsible for the appearance of the effective "absorption" in Eq. (24). The effective "absorption" coefficient $\sigma_4(0)$ describes additional attenuation of V as compared to intensity I. According to the notations of [4,27–29] $\sigma_4(0) = \sigma_{\text{dep}}$ is the coefficient of depolarization of circularly polarized light per

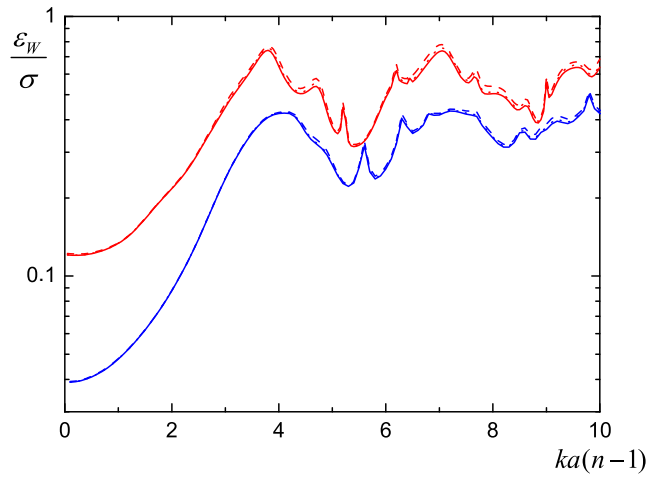

FIG. 5. (Color online) Depolarization coefficient ε_W as a function of relative refractive index for $ka = 5$ [red (upper) lines] and $ka = 10$ [blue (bottom) lines]. Calculations of ε_W are carried out with numerical solution of the characteristic equation ($l_{\max} = 10$, solid lines) and with Eqs. (29) (dashed lines), (33) (dotted lines).

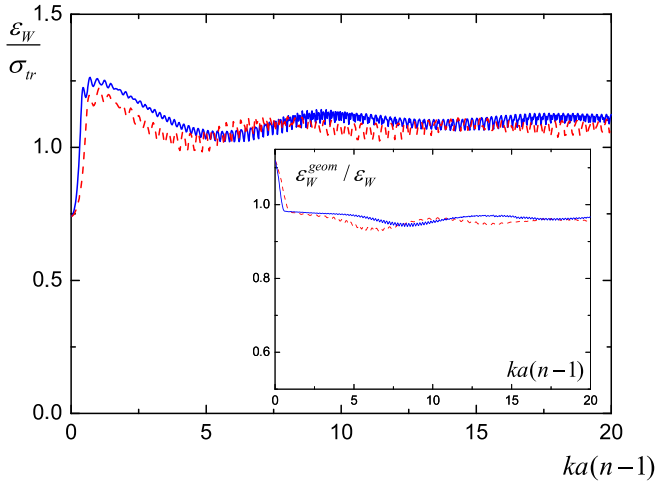

FIG. 6. (Color online) Ratio $\varepsilon_W/\sigma_{tr}$ as a function of radius a of scattering particles for aqueous suspension of polystyrene microspheres (solid blue line), water droplets in air (dashed red line). Calculations are carried out with numerical solution of the characteristic equation of Eq. (B2) ($l_{\max} = 10$). The inset shows the radius dependence of ratio $\varepsilon_W^{\text{geom}}/\varepsilon_W$.

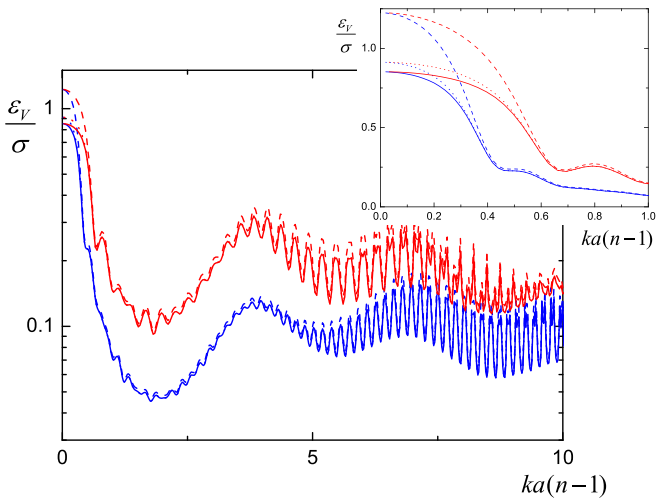

FIG. 7. (Color online) Depolarization coefficient ε_V as a function of radius a for scattering particle aqueous suspension of the polystyrene microspheres [blue (bottom) lines] and water droplets in the air [red (upper) lines]. Calculations are carried out numerically with the characteristic equation of Eq. (B3) ($l_{\max} = 10$, solid lines) and with approximate formulas (35) (dashed lines), (40) (dotted lines).

unit path. Quantity $\sigma_4(1)$ can be written as

$$\sigma_4(1) = \sigma_4(0) + \sigma \int d\mathbf{n}'(1 - \mathbf{n}\mathbf{n}')a_4(\mathbf{n}\mathbf{n}'), \qquad (37)$$

where the second term plays the role of the transport scattering coefficient in the equation for V [27–29].

Formula (35) is valid under conditions of weak effective "absorption" [4,27–29]

$$\sigma_4(0) \ll \sigma_4(1). \qquad (38)$$

Inequality (38) is fulfilled if the difference between a_1 and a_4 is rather small. In this case quantity $\sigma_4(1)$ can be estimated as σ_{tr}, and Eq. (38) can be rewritten in the form [4]

$$\sigma_{\text{dep}} \ll \sigma_{tr}. \qquad (39)$$

More polynomials should be taken into account in expansion (27), as the condition of weak effective "absorption" is violated.

Within the three-polynomial approximation, the contribution from the off-diagonal element b_2 is also allowed for in ε_V, to give

$$\varepsilon_V = \sqrt{3\sigma_4(0)\sigma_4(1)}\left(1 + \frac{[\sigma b_2(2)]^2}{5\sigma_3(2)\sigma_4(2)}\right)^{1/2} \times \left(1 + \frac{4\sigma_4(0)}{5\sigma_4(2)} + \frac{[\sigma b_2(2)]^2}{5\sigma_3(2)\sigma_4(2)}\right)^{-1/2}. \qquad (40)$$

The contribution from the off-diagonal element b_2 to the ε_V is always a small quantity (see Fig. 3),

$$\frac{[\sigma b_2(2)]^2}{5\sigma_3(2)\sigma_4(2)} \ll 1.$$

So, allowance for the additional number of polynomials in expansion of V proves to be of more importance due to a noticeable value of effective "absorption" $\sigma_4(0)$ rather than the off-diagonal element b_2.

Figures 7 and 8 illustrate the depolarization coefficient ε_V as a function of, respectively, radius a and relative refractive index n of scattering particles. From the figures it follows that the diffusion formula for the ε_V coefficient [see Eq. (35)] is applicable only to large ($ka \gg 1$) and slightly refractive ($|n-1| \ll 1$) scattering inhomogeneities. The more advanced result (40) improves considerably the accuracy of calculating ε_V all over the range of variations of size parameter ka and refractive index n.

Ratios ε_W/σ and ε_V/σ reach their maximum values in the limiting case of Rayleigh scattering particles ($ka \ll 1$, $|n-1| \ll 1$; see Figs. 4 and 7). In this case accurate numerical

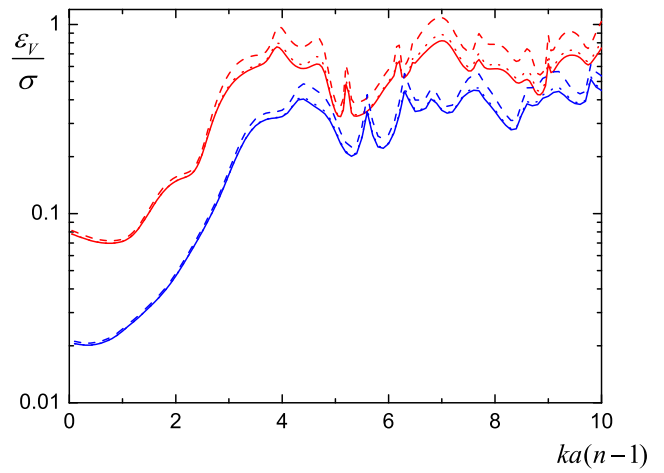

FIG. 8. (Color online) Depolarization coefficient ε_V as a function of relative refractive index for $ka = 5$ [red (upper) lines] and $ka = 10$ [blue (bottom) lines]. Calculations are carried out numerically with the characteristic equation of Eq. (B3) ($l_{\max} = 10$, solid lines) and with approximate formulas (35) (dashed lines), (40) (dotted lines).

calculations with the characteristic equations give

$$\varepsilon_W = 0.74\,\sigma, \quad \varepsilon_V = 0.85\,\sigma. \tag{41}$$

Approximate formulas (33) and (40) give slightly exceeded values

$$\varepsilon_W = 0.76\,\sigma, \quad \varepsilon_V = 0.91\,\sigma. \tag{42}$$

According to the analytical results presented above [see Eqs. (29), (33) and (35), (40)], the depolarization coefficients can be expressed in terms of a few first coefficients of expansion of the scattering matrix elements in the generalized spherical functions. These results establish explicit interrelation between the depolarization properties of the multiply scattering medium and optical characteristics of individual scatterers.

V. ANALYTICAL SOLUTION FOR A RAYLEIGH MEDIUM

In the case of a Rayleigh medium the depolarization coefficients can be found exactly. For Rayleigh particles ($ka \ll 1$, $|n-1| \ll 1$) the scattering matrix (7) is given by [30]

$$\hat{F}(\cos\gamma) = \frac{3}{16\pi}\begin{pmatrix} 1+\cos^2\gamma & \cos^2\gamma - 1 & 0 & 0 \\ \cos^2\gamma - 1 & 1+\cos^2\gamma & 0 & 0 \\ 0 & 0 & 2\cos\gamma & 0 \\ 0 & 0 & 0 & 2\cos\gamma \end{pmatrix}. \tag{43}$$

Element $b_2 \equiv 0$ and we can put $V_W = 0$.

Let us introduce a new function

$$F_W(z,\mu) = (1+\mu)^2 W(z,\mu) - (1-\mu^2) I_W(z,\mu) + (1-\mu)^2 w(z,\mu). \tag{44}$$

Then, combining the equations entering into system (20), we can easily show that function F_W obeys the following equation:

$$\left\{\mu\frac{\partial}{\partial z} + \sigma_{\rm tot}\right\} F_W(z,\mu) = \frac{3\sigma}{16}(1+\mu^2)^2 \int_{-1}^{1} d\mu' F_W(z,\mu'). \tag{45}$$

The characteristic equation corresponding to the asymptotic solution of Eq. (45) has the form

$$1 = \frac{3}{16}\frac{\sigma}{\sigma_{\rm tot}} \int_{-1}^{1} d\mu \frac{(1+\mu^2)^2}{1 - \mu\varepsilon_W/\sigma_{\rm tot}}. \tag{46}$$

Evaluating the integral in Eq. (46) we arrive at the algebraic equation

$$\frac{\varepsilon_W}{\sigma_{\rm tot}} = \frac{3}{16}\frac{\sigma}{\sigma_{\rm tot}}\left\{\left[1 + \left(\frac{\sigma_{\rm tot}}{\varepsilon_W}\right)^2\right]^2 \ln\frac{1+\frac{\varepsilon_W}{\sigma_{\rm tot}}}{1-\frac{\varepsilon_W}{\sigma_{\rm tot}}} - \frac{14}{3}\frac{\sigma_{\rm tot}}{\varepsilon_W} - 2\left(\frac{\sigma_{\rm tot}}{\varepsilon_W}\right)^3\right\}. \tag{47}$$

Equation (46) is among the set of the characteristic equations that were derived in a different way by Chandrasekhar within the problem of diffuse reflection and transmission of light by a slab of Rayleigh particles (see Chap. X, Sec. 69, Eqs. (120) and (124) in [30]). However, as we have learned, nobody has applied the corresponding characteristic equation yet to calculate the coefficient that is responsible for depolarization of initially linearly polarized light.

For Rayleigh scattering matrix (43), the fourth Stokes parameter V is described by the independent equation

$$\left\{\mu\frac{\partial}{\partial z} + \sigma_{\rm tot}\right\} V(z,\mu) = \frac{3\sigma}{4}\mu \int_{-1}^{1} d\mu'\,\mu' V(z,\mu'). \tag{48}$$

The corresponding characteristic equation is given by

$$1 = \frac{3}{4}\frac{\sigma}{\sigma_{\rm tot}} \int_{-1}^{1} d\mu \frac{\mu^2}{1 - \mu\varepsilon_V/\sigma_{\rm tot}} \tag{49}$$

or, in the algebraic form, by

$$\left(\frac{\varepsilon_V}{\sigma_{\rm tot}}\right)^2 = \frac{3}{4}\frac{\sigma}{\sigma_{\rm tot}}\left\{\frac{\sigma_{\rm tot}}{\varepsilon_V}\ln\frac{1+\frac{\varepsilon_V}{\sigma_{\rm tot}}}{1-\frac{\varepsilon_V}{\sigma_{\rm tot}}} - 2\right\}. \tag{50}$$

For a nonabsorbing medium, the roots of Eqs. (47) and (50) are coincident with numerical values given by Eq. (41).

In the Rayleigh case, as opposed to the case of large inhomogeneities, circular polarization decays faster than linear polarization, $\varepsilon_W < \varepsilon_V$. The difference between the values of ε_W and ε_V decreases with increasing absorption. In the limit $\sigma \ll \sigma_a$ ratios ε_W/σ and ε_V/σ tend to unity.

VI. DISCUSSION

There are rather many studies devoted to calculations and measurements of depolarization of light in disordered media (see, e.g., [2,4,5,8–11,16–19,21]). Within the random matrix approach, the coefficients of depolarization per unit path were found for both linearly and circularly polarized light [17–19]. For circular polarization, result [17,19] is coincident with our quantity $\sigma_{\rm dep} = \sigma_4(0)$ (see Eq. (36) and also [4,27–29]). Result [17,18] related to linear polarization is coincident with the formula derived in [9] by solving the nonstationary vector radiative transfer equation integrated over space. In [9] the coefficient of depolarization per unit path is the coefficient of polarization decay with time t as path $s = ct$, where c is the speed of light. Coefficients ε_W and ε_V that are responsible for depolarization of light with depth z were calculated in [17–19] under the assumption that the distribution over paths at given depth z is subject to the diffusion law. This assumption is valid only if the coefficient of depolarization per unit path is much less than transport scattering coefficient σ_{tr}, and decay of polarization follows the isotropization of light over directions. Such conditions are fulfilled for circularly polarized light in media composed of large weakly refractive particles [4,27–29]. In this case result [17,19] coincides with Eq. (35) (see also [4,27–29]). For linearly polarized light, the coefficient of depolarization per unit path is of the order of the transport scattering coefficient and the above-mentioned assumption is never valid. Therefore, approach [17,18] fails for calculations of the ε_W coefficient.

The drawback of approach [17–19] reveals itself clearly for Rayleigh particles. Within approach [17–19] (see also [2,26]) the values of the depolarization coefficients are equal to $\varepsilon_W = 1.03\,\sigma$ and $\varepsilon_V = 1.46\,\sigma$, exceeding noticeably the scattering

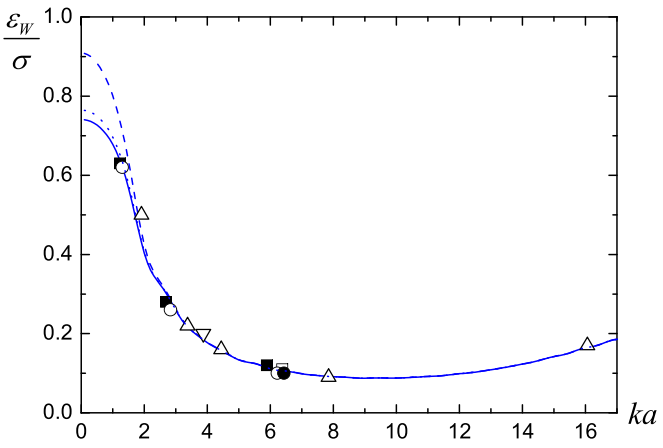

FIG. 9. (Color online) Depolarization coefficient ε_W as a function of radius a of scattering particles for aqueous suspension of polystyrene microspheres. The symbols are the results of experiments (■, [2]; □, [5]; △, [8]; •, [16]; ▽, [21]) and numerical calculation (○, [11]). The curves are the results of our calculations (regarding details, see Fig. 5).

coefficient of the medium (i.e., light is depolarized faster than scattered).

The fundamental values of ε_W and ε_V for a Rayleigh medium [see Eq. (41)] differ essentially from results [2,17,19,26] both quantitatively and qualitatively. According to Eq. (41) these values are less than the scattering coefficient of the medium and the difference between ε_W and ε_V is less noticeably than that predicted in [2,17,19,26].

It is of interest to compare the results obtained in the present work with data of experiments [2,5,8,16,21] and numerical integration [11] of the vector radiative transfer equation. In Figs. 9 and 10 the depolarization coefficients ε_W and ε_V as functions of size parameter ka are shown for aqueous suspension of polystyrene microspheres. Experimental and numerical data on ε_W and ε_V were extracted from the depth

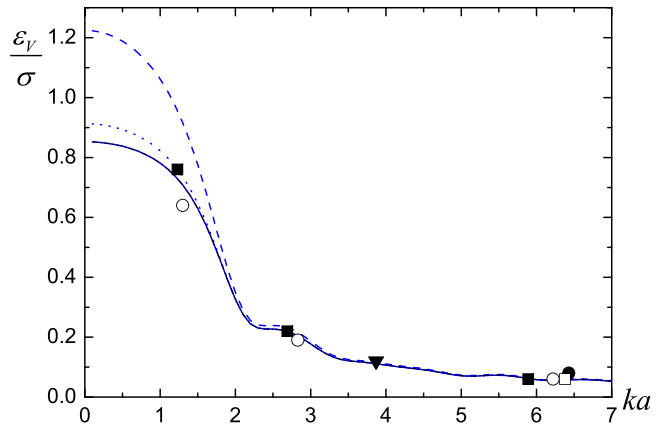

FIG. 10. (Color online) Depolarization coefficient ε_V as a function of radius a of scattering particles for aqueous suspension of polystyrene microspheres. The symbols are the results of experiments (■, [2]; □, [5]; •, [16]; ▽, [21]) and numerical calculation (○, [11]). The curves are the results of our calculations (regarding details, see Fig. 8).

dependence of the degree of polarization. In accordance with [27–29] the degree of linear ($P_L = W/I$) and circular ($P_C = V/I$) polarization falls off by law

$$P \sim z \exp(-\varepsilon z), \tag{51}$$

as $W \sim \exp(-\varepsilon_W z)$, $V \sim \exp(-\varepsilon_V z)$, and the diffusive intensity I decreases in a nonabsorbing medium as $I \sim 1/z$ with increasing z. Fitting of Eq. (51) to data [2,5,8,11,16,21] enables us to extract the corresponding values of the depolarization coefficients.

Our results are also in agreement with the numerical integration of the vector radiative transfer equation carried out in [10]. The values of ε_W and ε_V obtained above and in

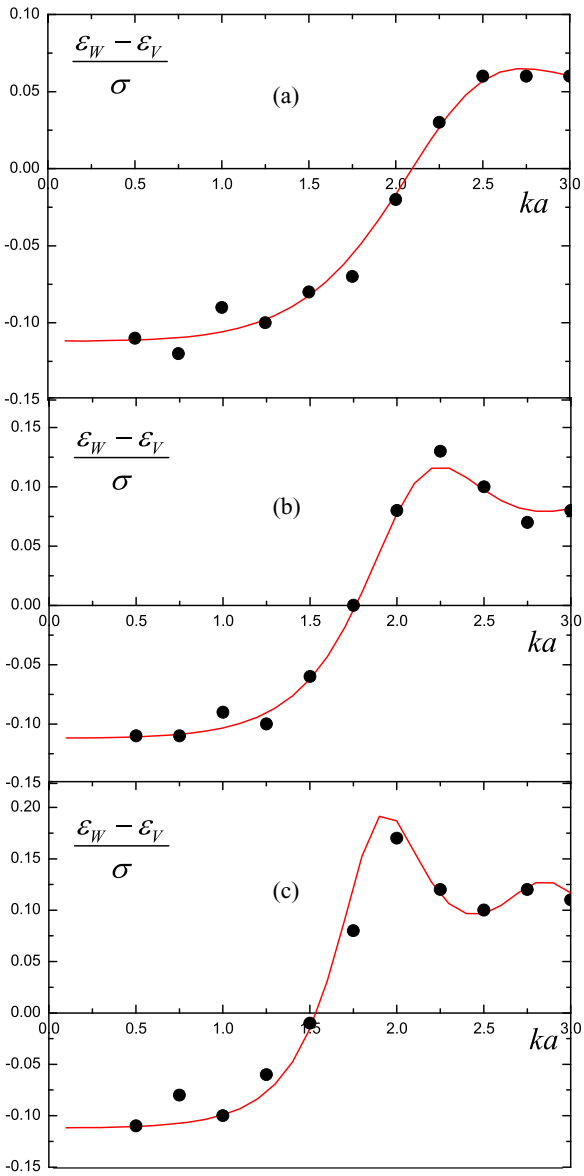

FIG. 11. (Color online) Difference between depolarization coefficients $\varepsilon_W - \varepsilon_V$ as a function of radius a for various relative refractive index of particles $n = 1.01(a)$, $n = 1.20(b)$, and $n = 1.40(c)$. Solid curves are the results of numerical calculations of $\varepsilon_V - \varepsilon_W$ with the characteristic equations of systems (B2) and (B3) ($l_{\max} = 10$). The results of numerical integration of the vector radiative transfer equation [10] are shown by symbols.

[10] are slightly different due to a systematic error which is inherent in data [10]. When extracting ε_W and ε_V from the data for the degree of polarization, the authors of [10] omit the linear prefactor in Eq. (51). The systematic error resulting from this procedure can be excluded in the difference $\varepsilon_W - \varepsilon_V$. Comparison of our results for $\varepsilon_W - \varepsilon_V$ with the corresponding data taken from [10] is illustrated in Fig. 11.

From Fig. 11 it follows that the circular polarization decays slower than the linear one for relatively large particles. There is a critical value of ka where the difference between the depolarization coefficients ε_W and ε_V vanishes. This critical value can be approximated by

$$\ln ka \approx 0.74 - 0.77(n-1) + 0.15(n-1)^2, \qquad (52)$$

where relative refractive index n ranges from 1 to 2.5.

Although the depolarization coefficients are related to the scattering matrix elements of the medium, a qualitative picture of depolarization can be characterized only by two quantities, σ_{tr} and $\sigma_{\rm dep}$, respectively. From the results obtained above it follows that the value of ε_W is of the order of σ_{tr} [for $ka(n-1) \sim 1$, coefficient $\varepsilon_W \sim (1.2\text{–}1.25)\sigma_{tr}$; see Fig. 6], while the value of ε_V, as a rule, is less than ε_W and is approximated by $\varepsilon_V = \sqrt{3\sigma_{\rm dep}\sigma_{tr}}$ [see Eq. (35)]. According to these estimates the depolarization coefficients are equal to each other at $\sigma_{\rm dep} \sim 0.5\,\sigma_{tr}$. Such a qualitative assessment is confirmed by accurate numerical calculations. Within a wide range of ka and $n = 1\text{–}2.5$ equality $\varepsilon_W = \varepsilon_V$ is achieved at ratio $\sigma_{\rm dep}/\sigma_{tr} = 0.58$.

It should be noted that there is correlation between the difference $\varepsilon_W - \varepsilon_V$ and a prevailing mechanism of depolarization. In the case of slow decay of circular polarization, $\sigma_{\rm dep} \ll \sigma_{tr}$, the "geometrical" mechanism (i.e., depolarization due to the Rytov rotation effect) is mainly responsible for depolarization of linearly polarized light. Otherwise $\sigma_{\rm dep} \geqslant 0.5\,\sigma_{tr}$, the "dynamical" and "geometrical" mechanisms contribute equally to ϵ_W.

VII. CONCLUSIONS

In conclusion, the coefficients of depolarization of multiply scattered linearly and circularly polarized beams have been calculated both numerically and analytically. The coefficients of depolarization govern the depth dependence of the difference between the intensities of the polarized components of light and are of great interest for various applications of polarization techniques to remote sensing of highly scattering media (aerosols, colloidal solutions, biological tissues, etc.). Based on the polarization mode approximation, we have expressed the depolarization coefficients in terms of a few first coefficients of expansion of the scattering matrix elements in the generalized spherical functions. This method has been shown to provide a satisfactory degree of accuracy for most practically important cases. The fundamental values of the depolarization coefficients for a Rayleigh medium have been calculated both numerically and analytically. The obtained values correct the frequently used result (see, e.g., [2,17–19,26]), which overestimates noticeably the magnitude of the depolarization coefficients.

APPENDIX A

The Stokes parameters are always defined with respect to a reference frame, in our case the system of unit vectors $\{\mathbf{e}_\parallel, \mathbf{e}_\perp, \mathbf{n}\}$. If the reference frame is rotated through angle α around the direction $\mathbf{n}$, transformation of the Stokes vector is described by

$$\hat{S}' = \hat{L}(\alpha)\hat{S}, \qquad (A1)$$

where rotation matrix $\hat{L}$ is equal to

$$\hat{L}(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\alpha & \sin 2\alpha & 0 \\ 0 & -\sin 2\alpha & \cos 2\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (A2)$$

According to Eq. (A2) the Stokes parameters Q and U are transformed via each other under spatial rotation

$$\begin{aligned} Q' &= Q\cos 2\alpha + U \sin 2\alpha, \\ U' &= -Q\sin 2\alpha + U\cos 2\alpha. \end{aligned} \qquad (A3)$$

In going from the linear to circular representation (i.e., from the Stokes vector $\hat{S}$ to the vector $\hat{I}$) relation (A1) is replaced by

$$\hat{I}' = \hat{\mathcal{L}}(\alpha)\hat{I}, \qquad (A4)$$

where

$$\hat{\mathcal{L}}(\alpha) = \begin{pmatrix} \exp(2i\alpha) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \exp(-2i\alpha) \end{pmatrix}. \qquad (A5)$$

Contrary to the Stokes parameters Q and U, either component of $\hat{I}$ is transformed via itself [36],

$$I'_m = \exp(2im\alpha) I_m. \qquad (A6)$$

So within the circular representation we exclude coupling between the different Stocks parameters Q and U that results from spatial rotations of the reference frame.

APPENDIX B

Taking into account expansions of the scattering matrix elements in the generalized spherical functions [35] (see also [9,23]) we arrive at the following equations for the coefficients entering into Eqs. (25) and (26):

$$\begin{aligned} &\varepsilon_I\left(\frac{l}{(2l+1)}I(l-1) + \frac{(l+1)}{(2l+1)}I(l+1)\right) - \sigma_1(l)I(l) = \sigma b_1(l)Q_I(l), \\ &\varepsilon_I\left(\frac{\sqrt{(l+2)(l-2)}}{(2l+1)}Q_I(l-1) + \frac{\sqrt{(l+3)(l-1)}}{(2l+1)}Q_I(l+1)\right) - \sigma_2(l)Q_I(l) = -\sigma b_1(l)I(l), \end{aligned} \qquad (B1)$$

$$
\begin{aligned}
&\varepsilon_W\left(\frac{l^2-4}{l(2l+1)}W(l-1)+\frac{4}{l(l+1)}W(l)+\frac{(l+3)(l-1)}{(l+1)(2l+1)}W(l+1)\right)-\sigma_+^{(2,3)}(l)W(l)\\
&\quad=-\frac{\sigma}{2}b_1(l)I_W(l)+\frac{\sigma}{2}b_2(l)V_W(l)-\sigma a_-^{(2,3)}(l)w(l),\\
&\varepsilon_W\left(\frac{\sqrt{(l+2)(l-2)}}{(2l+1)}I_W(l-1)+\frac{\sqrt{(l+3)(l-1)}}{(2l+1)}I_W(l+1)\right)-\sigma_1(l)I_W(l)=-\sigma b_1(l)W(l)-\sigma b_1(l)w(l),\\
&\varepsilon_W\left(\frac{\sqrt{(l+2)(l-2)}}{(2l+1)}V_W(l-1)+\frac{\sqrt{(l+3)(l-1)}}{(2l+1)}V_W(l+1)\right)-\sigma_4(l)V_W(l)=-\sigma b_2(l)W(l)+\sigma b_2(l)w(l),\\
&\varepsilon_W\left(\frac{(l+2)(l-2)}{l(2l+1)}w(l-1)-\frac{4}{l(l+1)}w(l)+\frac{(l+3)(l-1)}{(l+1)(2l+1)}w(l+1)\right)-\sigma_+^{(2,3)}(l)w(l)\\
&\quad=-\sigma a_-^{(2,3)}(l)W(l)-\frac{\sigma}{2}b_1(l)I_W(l)-\frac{\sigma}{2}b_2(l)V_W(l),
\end{aligned}
\tag{B2}
$$

$$
\begin{aligned}
&\varepsilon_V\left(\frac{l}{(2l+1)}V(l-1)+\frac{(l+1)}{(2l+1)}V(l+1)\right)-\sigma_4(l)V(l)=\sigma b_2(l)U_V(l),\\
&\varepsilon_V\left(\frac{\sqrt{(l+2)(l-2)}}{(2l+1)}U_V(l-1)+\frac{\sqrt{(l+3)(l-1)}}{(2l+1)}U_V(l+1)\right)-\sigma_3(l)U_V(l)=-\sigma b_2(l)V(l),
\end{aligned}
\tag{B3}
$$

where

$$
\sigma_{1,4}(l)=\sigma(1-a_{1,4}(l))+\sigma_a,\quad a_{1,4}(l)=2\pi\int_{-1}^{1}d\mu\, a_{1,4}(\mu)P_l(\mu),\quad \sigma_\pm^{(2,3)}(l)=\sigma\left(1-a_\pm^{(2,3)}(l)\right)+\sigma_a,
$$

$$
a_\pm^{(2,3)}(l)=2\pi\int_{-1}^{1}d\mu\, a_\pm^{(2,3)}(\mu)P_{2\pm2}^{l}(\mu),\quad b_{1,2}(l)=2\pi\int_{-1}^{1}d\mu\, b_{1,2}(\mu)P_{20}^{l}(\mu).
\tag{B4}
$$

Coefficients $\sigma_2(l)$ and $\sigma_3(l)$ appearing in Eqs. (B1), (B3) can be expressed in terms of $a_\pm^{(2,3)}(l)$ as follows:

$$
\sigma_{2,3}(l)=\sigma_+^{(2,3)}(l)\pm\sigma_-^{(2,3)}(l)\mp(\sigma+\sigma_a).
\tag{B5}
$$

Solutions to Eqs. (B1)–(B3) can be found numerically with the use of truncating at some $l_{\max}$.

Within the basic mode approximation, coefficients $b_1(l)$, $b_2(l)$, and $a_-^{(2,3)}(l)$ should be put equal to zero. Then expansion coefficients $I(l)$, $W(l)$, and $V(l)$ are subject to the separate equations that coincide with the first equations of systems (B1)–(B3) with nil on the right-hand side.

APPENDIX C

The simplest generalization of the two-polynomial formula derived within the basic mode approximation [see Eq. (29)] can be obtained with allowance for one-polynomial expansion of the off-diagonal elements. In this case the characteristic equation takes the form

$$
\det\begin{pmatrix}
\frac{\varepsilon_W}{3-\sigma_+^{(2,3)}(3)} & \frac{5}{21}\varepsilon_W & 0 & 0 & 0\\
\frac{1}{3}\varepsilon_W & \frac{2\varepsilon_W}{3}-\sigma_+^{(2,3)}(2) & \frac{\sigma}{2}b_1(2) & -\frac{\sigma}{2}b_2(2) & \sigma a_-^{(2,3)}(2)\\
0 & \sigma b_1(2) & -\sigma_1(2) & 0 & \sigma b_1(2)\\
0 & \sigma b_2(2) & 0 & -\sigma_4(2) & -\sigma b_2(2)\\
0 & \sigma a_-^{(2,3)}(2) & \frac{\sigma}{2}b_1(2) & \frac{\sigma}{2}b_2(2) & -\frac{2\varepsilon_W}{3}-\sigma_+^{(2,3)}(2)
\end{pmatrix}=0.
\tag{C1}
$$

Equation (C1) can be rewritten as

$$
x^3-x^2\left(\frac{5}{18}-\frac{5}{36}(a-b)+\frac{14}{9}c\right)-\frac{7}{18}(1-a)(1+b)x+\frac{7}{18}(1-a)(1+b)c=0,
\tag{C2}
$$

where $x=\varepsilon_W/3\sigma_+^{(2,3)}(2)$,

$$
a=\frac{\sigma a_-^{(2,3)}(2)}{\sigma_+^{(2,3)}(2)}+\frac{[\sigma b_1(2)]^2}{\sigma_+^{(2,3)}(2)\sigma_1(2)},\quad b=\frac{\sigma a_-^{(2,3)}(2)}{\sigma_+^{(2,3)}(2)}+\frac{[\sigma b_2(2)]^2}{\sigma_+^{(2,3)}(2)\sigma_4(2)},\quad c=\frac{\sigma_+^{(2,3)}(3)}{\sigma_+^{(2,3)}(2)}.
\tag{C3}
$$

From Eq. (C2) it follows that the dependence of ε_W on the parameters of the scattering matrix has the form of Eqs. (33), (34). A solution to Eq. (C2) can be obtained numerically or expressed in terms of the Cardano formula. As one of the roots of Eq. (C2)

is much greater than the others, the sought-for Ψ function appearing in Eq. (34) can be approximated by the analytical relation

$$\Psi(a,b,c) \approx -\frac{B}{A} + \sqrt{\left(\frac{B}{A}\right)^2 - 2\frac{C}{A}}, \tag{C4}$$

where

$$A = -\frac{108}{7}x_0 + 8c + \frac{5}{7}(2 - a + b),$$

$$B = -\left(\frac{5}{7}(a-b) + 4c + \frac{32}{7}\right)x_0 + (1-a)(1+b) + 6c,$$

$$C = \frac{5}{14}(b-a)x_0^2 - (x_0 - c)(a - b + ab),$$

$x_0 = \varepsilon_W^{(2)}/3\sigma_+^{(2,3)}(2)$; the coefficient $\varepsilon_W^{(2)}$ is determined by Eq. (29).

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