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# Vibrational higher-order resonances in an overdamped bistable system with biharmonic excitation

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Experimental evidence of vibrational higher-order resonances in a bistable vertical-cavity surface-emitting laser driven by two harmonic signals with very different frequencies is reported. The phenomenon shows up in a parameter space (the dc current, the amplitude of the high-frequency signal) as well-defined structures with multiple local maxima at higher harmonics of the low-frequency signal. Such structures appear due to a strong suppression of higher harmonics for certain values of the high-frequency amplitude and the dc current. Complexity of the structures and the total number of the local maxima depend on the harmonic order $k$. The behavior of nonlinear distortion factor is also studied. The experimental results are in a good agreement with the numerical results which were obtained in the model of the bistable overdamped oscillator with biharmonic excitation.

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## I. INTRODUCTION

Driven bistable systems display a variety of nonlinear phenomena, among which one can note, for example, stochastic resonance (SR) [1–3] and vibrational resonance (VR) [4–8], which have a similarity in the response to the effect of the low-frequency (LF) excitation. The latter is a deterministic analog of the phenomenon of stochastic resonance where noise is replaced by a high-frequency (HF) signal. The phenomenon of VR shows up as a resonance behavior on the frequency of the LF input signal depending on the amplitude or frequency of the additional HF excitation. Vibrational resonance has been evidenced experimentally in analog electric circuits [5–7] and in a bistable vertical-cavity surface-emitting laser (VCSEL) [8]. Improvement of a detection of weak periodic and aperiodic binary signals by VR has been experimentally demonstrated as well [9,10]. Until now almost all theoretical, numerical, and experimental investigations of VR in bistable systems were focused on the response on the first harmonic of the LF signal except recent theoretical studies of nonlinear VR [11] and ghost-vibrational resonance [12]. In the first case the phenomenon of VR on the second harmonic of the LF signal in bistable systems with the symmetric double-well potential was theoretically investigated [11]. The response on subharmonic frequencies in an underdamped bistable oscillator with a multifrequency excitation was theoretically studied in the second case [12].

The generation of higher-order harmonics in dynamical systems is an indication of a strong nonlinear interaction of signals with the nonlinear systems and may characterize the system itself. For instance, the absence or presence of even harmonics in the spectra of the system's response indicates the symmetry of nonlinear systems. Another important point is that a study of the responses at higher harmonics may result in finding conditions for a passage of the signal through nonlinear systems with minimal distortions. The latter is of special interest for vibrational resonance, since in this case a strong amplification of periodic and aperiodic signals can be achieved in bistable systems [8,10]. It should be noted that the generation of higher harmonics in driven stochastic nonlinear systems was a subject of the theoretical and experimental investigations in the context of studying SR [13–23]. Some nonlinear phenomena were found in these investigations such as noise-induced resonances or suppression of higher harmonics of the LF signal for a certain value of the noise intensity [14–17], the generation of multiple maxima on higher harmonics in the bistable system due to a broken symmetry by dc signal [18,19] or noise-induced linearization [20].

Here we present experimental evidence of vibrational higher-order resonances (VHORs) in a VCSEL in the regime of bistability between two polarization states. The laser is driven by two harmonic signals with very different frequencies. Specifically, the response amplitude of a bistable VCSEL was studied in the broad range of the dc injection current depending on the amplitude of the HF signal at the frequencies $f_L^{(k)} = kf_L$, where $f_L$ is the frequency of the LF driving signal, $k = 1,2,3,\ldots,9$. In these conditions the response amplitude on higher-order harmonics of the LF signal displays multiple local maxima which form in a parameter space (the dc current, the amplitude of the HF signal) clear-cut structures. The total number of maxima depending on the harmonic order $k$ obey a simple empirical expression. The experimental regularities are in excellent agreement with results of the numerical simulation performed in the model of the overdamped bistable oscillator with biharmonic excitation.

## II. EXPERIMENTAL SETUP

Experiments were performed on the experimental setup recently used for studying VR in a multistable VCSEL [24]. A 850 nm VCSEL (Honeywell HFE4080-321) with a threshold current $J_{\mathrm{th}} \approx 5.6$ mA was used in the experiments. Depending on the value of the dc injection current $j_{\mathrm{dc}}$ two, three, or four polarization states can coexist in the laser. The dc current $j_{\mathrm{dc}}$ was chosen in such a way that only two coexisting polarization states could appear. The temperature of the laser diode was controlled within 0.01 °C. All measurements were performed at temperature of the laser diode equal to 15 °C. The collimated VCSEL emission was split into two polarization components by a half-wave plate and a Glan prism. The temporal behavior on one selected polarization was recorded by a fast photodiode

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and USB oscilloscope with sampling frequency up to 100 MHz and the input bandwidth of 50 MHz. Two sine-wave LF and HF periodic signals from function generators with frequencies $f_L = 0.5$ kHz and $f_H = 50$ kHz and different amplitudes $A_L$ and $A_H$ were added to the dc current $j_{dc}$. Both driving frequencies are less than the cutoff frequency of the amplitude frequency characteristic of the polarization-resolved laser response, which is of the order of 200 kHz. The amplitude of the LF signal $A_L$ was set to 12.5 mV. The amplitude of the HF signal $A_H$ is a control parameter here. The dc current and the amplitudes of both periodic signals were controlled from a computer.

## III. MODEL

In parallel with the experimental study, for a generalization purpose, the numerical simulation in the framework of the model of an overdamped oscillator with a double-well potential function was performed. It was shown previously that many features of the polarization dynamics of a bistable VCSEL can be well described by this model (see, e.g., Refs. [25,26]). The following equation is used in the simulation:

$$dx/dt = 4(x - x^3) - \Delta + A_L \sin 2\pi f_L t + A_H \sin 2\pi f_H t, \tag{1}$$

where $\Delta$ is a level of the asymmetry, and $f_L$ and $f_H$ are low and high frequencies, respectively. In this case a bistable potential function $V(x)$ has the following form: $V(x) = x^4 - 2x^2 + \Delta x$. In the simulation we used $f_L = 0.001$ and $f_H = 0.1$, which are less than the intrawell relaxation frequency $f_r = 4/\pi$ in the symmetrical configuration ($\Delta = 0$) of a bistable potential.

## IV. VR ON THE HIGHER-ORDER HARMONICS OF THE LF SIGNAL

Resonances at higher harmonics were experimentally investigated from the spectra of the Fourier transformed time series of the laser intensity on the selected polarization at the frequencies $kf_L$ ($k = 1,2,\dots,9$) depending on the amplitude $A_H$ in a wide range of the dc current $j_{dc}$. Figure 1 demonstrates typical amplitude spectra shown for different values of $A_H$. First nine harmonics of a LF signal are shown here. One can see that some higher harmonics may appear and disappear with a change of the $A_H$. Therefore, we studied in a systematic way the response amplitude $R_L^{(k)}$ on the grid of 101 × 41 values of $j_{dc}$ and $A_H$, which were changed with steps ∼0.006 mA and 1 mV, respectively. The response amplitude $R_L^{(k)}$ was estimated as the height of peaks in spectra of the laser responses. In a similar way, the response amplitude $R_L^{(k)}$ depending on the level of asymmetry $\Delta$ and the HF amplitude $A_H$ was estimated in the numerical simulation.

Contour plots in Figs. 2 and 3 demonstrate normalized experimental and numerical response amplitudes $R_{L,\mathrm{norm}}^{(k)}$ as a function of the HF amplitude and the level of asymmetry ($\Delta j_{dc}$ and $\Delta$, respectively). $R_{L,\mathrm{norm}}^{(k)}$ is defined here as follows: $R_{L,\mathrm{norm}}^{(k)} = R_L^{(k)}/R_{L,\max}^{(k)}$, where $R_{L,\max}^{(k)}$ is a maximal value for each higher harmonic. The level of asymmetry $\Delta j_{dc}$ is defined as $\Delta j_{dc} = j_{dc} - j_{sym}$, where $j_{dc}$ is a current value, and $j_{sym}$ is a value of the $j_{dc}$ corresponding to the symmetrical configuration of a bistable potential. In the experiment, the value of $j_{sym}$ is determined as a minimal value of the switching amplitude of the HF periodical signals depending on the $j_{dc}$. Comparing Figs. 2 and 3, one can see a substantial similarity between them. Therefore, in what follows we analyze the experimental results with the numerical ones.

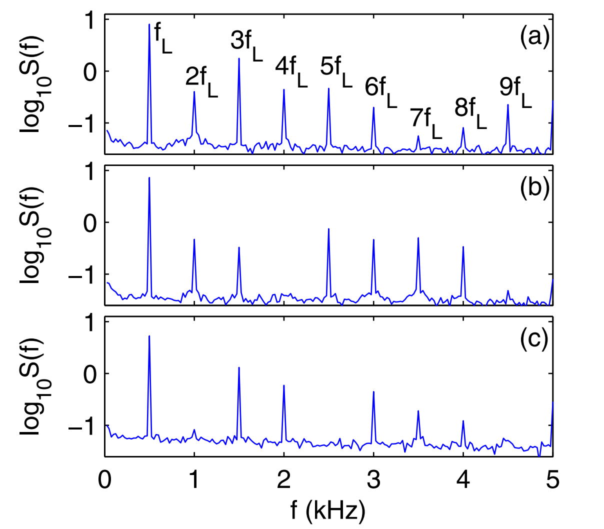


FIG. 1. (Color online) Experiment: Amplitude spectrum $S(f)$ of the the laser response on the selected polarization shown for three different values of $A_H$: $j_{dc} = 17.44$ mA; $A_H = 78$ (a), 87 (b), 98 (c) mV.

First, one can note an appearance of well-defined structures with multiple local maxima in the parameter space ($j_{dc}$, $A_H$). Complexity of the structures increases with increasing the ultraharmonic number $k$. These patterns are structured so that multiple local maxima are symmetrically located with respect to $\Delta = 0$. Based on the data processing for nine patterns we found that the total number of the local maxima $N_{tot}^{(k)}$ depending on the ultraharmonic number $k$ can be evaluated from the following empirical expression:

$$N_{tot}^{(k)} = [k(k+2) + 0.5(1 - (-1)^k)]/4. \tag{2}$$

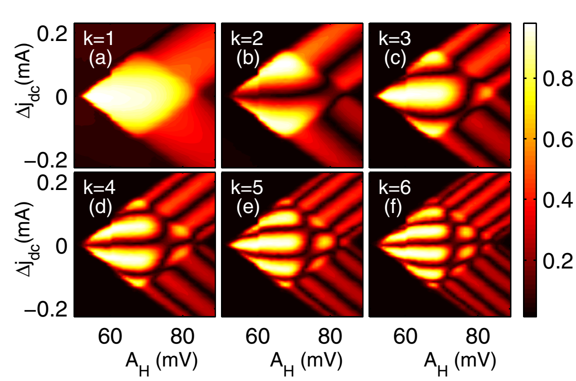


FIG. 2. (Color online) Experiment: The normalized laser response amplitude $R_{L,\mathrm{norm}}^{(k)}$ on the frequency $f_k = kf_L$ ($k = 1, 2, \dots, 6$) versus the level of asymmetry $\Delta j_{dc}$ and the HF amplitude $A_H$.

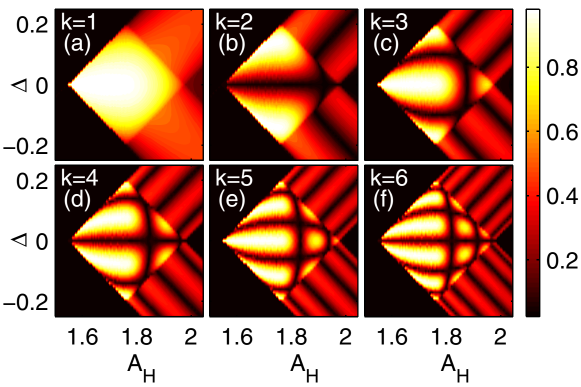


FIG. 3. (Color online) Numerical simulation: The normalized response amplitude $R_{L,\mathrm{norm}}^{(k)}$ of a bistable system on the frequency $f_k = kf_L$ versus the level of asymmetry $\Delta$ and the HF amplitude $A_H$ ($f_L = 0.001$, $f_H = 0.1$).

From contour plots in in Figs. 2 and 3 it is seen that for the symmetrical configuration of a bistable potential ($\Delta j_{\mathrm{dc}} = 0$) all even ultraharmonics vanish so that only odd higher harmonics remain in the spectra. The generation of odd harmonics of the driving frequency is a typical feature of symmetric nonlinear systems [13]. Fitting of the experimental and numerical data gives that the amplitude of odd higher harmonics $A_k, k = 1,3,5,\ldots$ decreases as $A_k \sim k^{-\alpha}$, where the numerical $\alpha_{\mathrm{num}} \approx 1.028$ and the experimental $\alpha_{\mathrm{exp}} \approx 1.045$. Both values are close to the value $\alpha = 1$, which corresponds to a harmonic series with odd numbers only.

In Fig. 4 the experimental and numerical results are directly compared for fixed values of the HF amplitude showing a good agreement between them. For this comparison the amplitude of the HF signal in the experiment $A_H^{\mathrm{exp}} \approx 69$ mV and that in the simulation $A_H^{\mathrm{sim}} = 1.77$ were chosen. A good agreement also can be seen from Fig. 5 where experimental and numerical results are compared for the case of the symmetrical configuration of a bistable potential. From Figs. 2–5 it follows directly that there are certain values of the HF amplitude and the level of asymmetry for which a strong suppression of ultraharmonics of the LF signal occurs. In fact, such a suppression is very similar to the phenomenon of the noise-induced resonance previously observed in SR [15,16] where a strong suppression of the higher harmonics of the harmonic signal for a certain level of a noise intensity was theoretically demonstrated. From presented results here it is seen that such a suppression in VR is observed for all higher-order harmonics of the LF signal. As noted in Ref. [14,15] noise-induced resonance (or suppression of higher harmonics) is accompanied by phase jumps by $\pi$. Similar phase jumps were observed in the Fourier spectra of numerical time series when the amplitude of the HF signal is close to the suppression point.

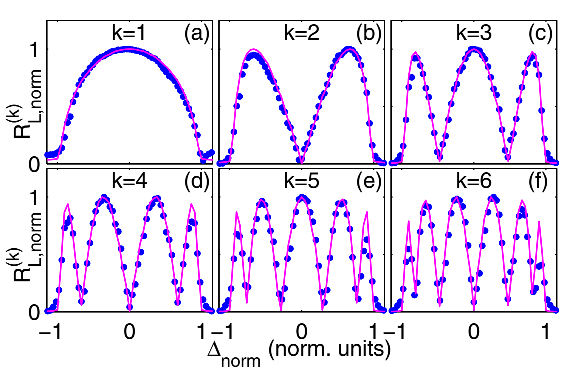


FIG. 4. (Color online) The normalized response amplitude $R_{L,\mathrm{norm}}^{(k)}$ versus normalized level of asymmetry $\Delta_{\mathrm{norm}}$. The amplitude of the HF signal $A_H = 68.6$ mV (experiment) and 1.77 (numerical simulation). Points, the experiment; solid line, the numerical simulation.

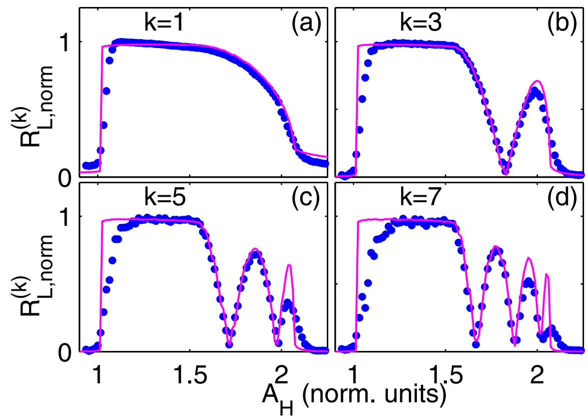


FIG. 5. (Color online) The normalized response amplitude $R_{L,\mathrm{norm}}^{(k)}$ versus normalized amplitude of HF signal $A_H$ for the symmetrical configuration of a bistable potential. Points, the experiment; solid line, the numerical simulation.

## V. NONLINEAR DISTORTION FACTOR IN VR

From the above consideration it is seen that a passage of the harmonic signal through a bistable system in the regime of VR is followed by a generation of a large number of ultraharmonics of the LF signal. Therefore it is naturally to characterize such a behavior by a coefficient of distortion which widely used in electronic engineering. We here define the nonlinear distortion factor (NDF) as follows:

$$\chi = \left[\sum_{k=2}^{m} A_k^2\right]^{1/2} \Big/ A_1, \tag{3}$$

where $A_k$ is an amplitude of $k$th harmonics. In the experiment and in the numerical simulation we have measured amplitudes of the first nine harmonics; therefore in our case $m = 9$. The experimental and numerical results for NDF are shown in Figs. 6(a) and 6(b), respectively, which demonstrate a good agreement between them. One can see from both figures that there are ranges of the level asymmetry of a bistable potential and the control parameter $A_H$ for which NDF attains the minimal value. Comparing these figures with the response curves $R_L^{(1)}$ on Figs. 2(a) and 3(a) one can see that the minimal value in VR is attained in the beginning of a descending part of the response curve $R_L^{(1)}$. In Fig. 7 the experimental and numerical NFSs are directly compared for the case of symmetrical configuration bistable potential ($\Delta j_{\mathrm{dc}} = 0$ and

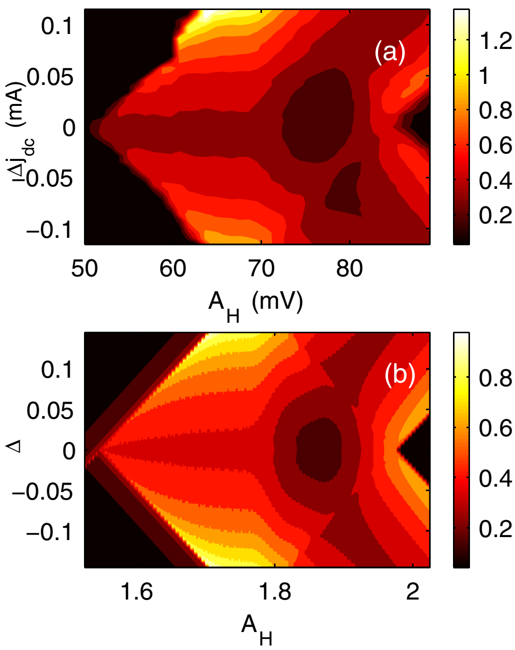


FIG. 6. (Color online) The nonlinear distortion factor $\chi$ versus the HF amplitude $A_H$ and the level of asymmetry: (a) experiment ($\Delta j_{\rm dc}$) and (b) simulation ($\Delta$).

$\Delta = 0$, respectively) showing a good agreement between them. Here only the control parameter $A_H$ was normalized. It is clear that such a reduction of distortion is caused by a suppression of higher-order harmonics. Since the suppression of each harmonics occurs for different values of the control parameter $A_H$, therefore a residual distortion is rather large. In fact, the minimal distortion in VR takes place near the value of the control parameter $A_H$ corresponding to the maximal suppression of third harmonic. It is worthy of note that the NDF is changed insignificantly when we take into account more higher harmonics.

In the regime of strong HF modulation (for $A_H > 2$; Fig. 7) beyond the range of VR, the response of the system is characterized by very low NDF. This regime corresponds to loss of bistability, and in this case the system response can be considered as a linear response of a monostable system. In fact, such a decrease of the NDF seems to be very similar to the noise-induced linearization which was observed in the studying SR for the large enough noise intensity [20].

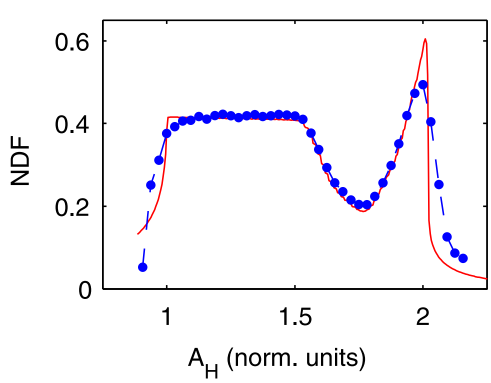


FIG. 7. (Color online) The nonlinear distortion factor $\chi$ versus the normalized HF amplitude $A_H$ for the case of a symmetrical configuration of a double-well potential. Points, experiment; solid line, simulation.

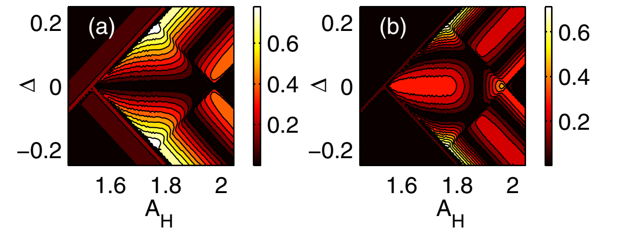


FIG. 8. (Color online) Numerical simulation: The ratio $R_L^{(2)}/R_L^{(1)}$ (a) and $R_L^{(3)}/R_L^{(1)}$ (b) versus the HF amplitude $A_H$ and the level of asymmetry $\Delta$.

In order to give an estimate of the amplitude of higher-order resonances with respect to the response on the fundamental frequency $f_L$, Figs. 8(a) and 8(b) show the ratio for the response amplitude at the second ($R_2 = R_L^{(2)}/R_L^{(1)}$) and the third ($R_3 = R_L^{(3)}/R_L^{(1)}$) harmonics, which mostly contribute to NDF, to the response amplitude $R_L^{(1)}$ as function of the level of asymmetry $\Delta$ and the HF amplitude $A_H$. One can note that maximal relative amplitudes $R_2$ and $R_3$ are observed near the edge of the response amplitude $R_L^{(1)}$. For instance, for HF amplitude $A_H = 1.79$ the quantities $R_2$ and $R_3$ attain the values of $\approx$0.86 and $\approx$0.78, respectively.

## VI. EFFECT OF THE INTRAWELL RELAXATION FREQUENCY ON VHOR

The consideration above was performed in the adiabatic regime. In this regime, both periodic forces were very slow with respect to the intrawell relaxation times. Here we will consider the numerical results which concern the influence of intrawell relaxation frequency on the response of the bistable system on higher-order harmonics. The results presented in

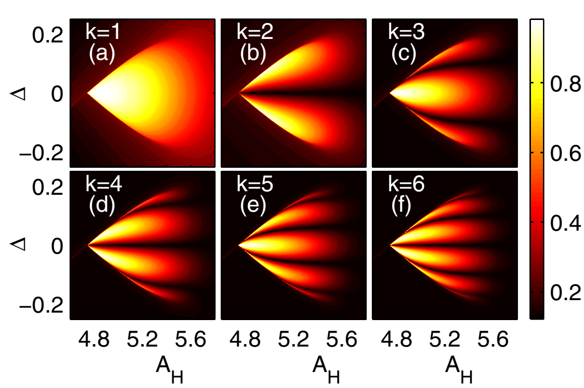


FIG. 9. (Color online) Numerical simulation: The normalized response amplitude $R_{L,\rm norm}^{(k)}$ of a bistable system on the frequency $f_k = k f_L$ versus the level of asymmetry $\Delta$ and the HF amplitude $A_H$ ($f_L = 0.01$, $f_H = 1$).

Fig. 9 correspond to biharmonic excitation with frequencies $f_L = 0.01$ and $f_H = 1$, respectively.

In this case the frequency $f_H$ is close to the relaxation frequency $f_r$. At same time we keep the relationship between $f_L$ and $f_H$ the same as in Fig. 3. One can see that the distribution of higher harmonics in parameter space $(\Delta, A_H)$ is significantly changed. The dependence of the response amplitude $R_L^{(k)}$ for the fixed value of $\Delta$ becomes smooth without multiple maxima. However, the local maxima remain in the $R_L^{(k)}$ for the fixed value of the HF amplitude $A_H$. We did not make such experimental investigations due to very large amplitudes of the HF signal needed in the experiment. However, taking account that all other experimental results are in excellent agreement with numerical ones, one can expect that the same behavior as in Fig. 8 should be observed experimentally.

## VII. CONCLUSIONS

To conclude, we have presented an experimental evidence of the phenomenon of vibrational resonance on higher-order harmonics of the LF harmonic input signal in a VCSEL operating in the regime of polarization bistability. The phenomenon of VR manifests itself in the parameter space $(j_{dc}, A_H)$ as clear-cut patterns with local maxima which number depends on the ultraharmonic number. We have also demonstrated a nonmonotic behavior of the nonlinear distortion factor in VR. All experimental results are in a good agreement with the results of the numerical simulation performed in the framework of a overdamped bistable oscillator. The phenomenon of VR on higher harmonics can be used in dynamical sensors for the detection of dc signals via the broken symmetry of a bistable or multistable potential functions, for instance, in a SQUID [18,19].


[1] R. Benzi, A. Sutera, and A. Vulpiani, J. Phys. A **14**, L453 (1981).
[2] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998).
[3] T. Wellens, V. Shatokhin, and A. Buchleitner, Rep. Prog. Phys. **67**, 45 (2004).
[4] P. S. Landa and P. V. E. McClintock, J. Phys. A: Math. Gen. **33**, L433 (2000).
[5] A. A. Zaikin, L. López, J. P. Baltanás, J. Kurths, and M. A. F. Sanjuán, Phys. Rev. E **66**, 011106 (2002).
[6] E. Ullner, A. Zaikin, J. García-Ojalvo, R. Bóscones, and J. Kurths, Phys. Lett. A **312**, 348 (2003).
[7] J. P. Baltanás, L. López, I. I. Blechman, P. S. Landa, A. Zaikin, J. Kurths, and M. A. F. Sanjuán, Phys. Rev. E **67**, 066119 (2003).
[8] V. N. Chizhevsky, E. Smeu, and G. Giacomelli, Phys. Rev. Lett. **91**, 220602 (2003).
[9] V. N. Chizhevsky and G. Giacomelli, Phys. Rev. A **71**, 011801(R) (2005).
[10] V. N. Chizhevsky and G. Giacomelli, Phys. Rev. E **77**, 051126 (2008).
[11] S. Ghosh and D. S. Ray, Phys. Rev. E **88**, 042904 (2013).
[12] S. Rajamani, S. Rajasekar, and M. A. F. Sanjun, Commun. Nonlinear Sci. Numer. Simulat. **19**, 4003 (2014).
[13] P. Jung and P. Hanggi, Europhys. Lett. **8**, 505 (1989).
[14] R. Bartussek, P. Hanggi, and P. Jung, Phys. Rev. E **49**, 3930 (1994).
[15] P. Jung and P. Talkner, Phys. Rev. E **51**, 2640 (1995).
[16] R. Bartussek, P. Jung, and P. Hanggi, Chaos Solitons Fractals **5**, 1775 (1995).
[17] P. Jung and R. Bartussek, in *Fluctuations and Order*, Institute for Nonlinear Science, edited by M. Millonas (Springer, New York, 1996), pp. 35–52.
[18] A. R. Bulsara, M. E. Inchiosa, and L. Gammaitoni, Phys. Rev. Lett. **77**, 2162 (1996).
[19] M. E. Inchiosa, A. R. Bulsara, and L. Gammaitoni, Phys. Rev. E **55**, 4049 (1997).
[20] D. G. Luchinsky, P. V. E. McClintock, and M. I. Dykman, Rep. Prog. Phys. **61**, 889 (1998).
[21] A. N. Grigorenko, S. I. Nikitin, and G. V. Roschepkin, Phys. Rev. E **56**, R4907(R) (1997).
[22] A. N. Grigorenko, P. I. Nikitin, and G. V. Roschepkin, JETP **85**, 343 (1997).
[23] R. L. Badzey and P. Mohanty, Nature (London) **437**, 995 (2005).
[24] V. N. Chizhevsky, Phys. Rev. E **89**, 062914 (2014).
[25] G. Giacomelli and F. Marin, Quantum Semiclass. Opt. **10**, 469 (1998).
[26] M. B. Willemsen, M. U. F. Khalid, M. P. van Exter, and J. P. Woerdman, Phys. Rev. Lett. **82**, 4815 (1999).