Spectral density of the noncentral correlated Wishart ensembles

Vinayak*
Instituto de Ciencias Físicas, Universidad Nacional Autónoma de México, C.P. 62210 Cuernavaca, Mexico
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Wishart ensembles of random matrix theory have been useful in modeling positive definite matrices encountered in classical and quantum chaotic systems. We consider nonzero means for the entries of the constituting matrix $\mathbb{A}$ which defines the correlated Wishart matrix as $\mathbb{W} = \mathbb{A}\mathbb{A}^{\dagger}$, and refer to the ensemble of such Wishart matrices as the noncentral correlated Wishart ensemble (nc-CWE). We derive the Pastur self-consistent equation which describes the spectral density of nc-CWE at large matrix dimension.

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I. INTRODUCTION

Random matrix theory (RMT) has been applied in a vast domain of science [1–10] and it is now in frontiers of modern research on complex systems. The Wishart model for the correlation matrices [11] is probably the origin of RMT. In recent research this model has received much attention, with various generalizations in trend, to model the symmetric positive definite matrices encountered in classical or quantum chaotic systems. For instance the Wishart model which incorporates actual correlations [12–20] provides an improved paradigm to understand the correlations in the financial market [21–24], and also in practical statistical signal processing applications such as synthetic aperture radar, extra-solar planet detection, and multi-antenna wireless communications [25]. The ensemble of such Wishart matrices is known as the correlated Wishart ensemble (CWE). Also the fixed-trace generalization of Wishart matrices to model density matrices in quantum entanglement problems [26], or power-map-deformed Wishart matrices [27,28] in the context of short time series analysis of multivariate systems, have been useful.

In a general sense the Wishart model may be defined as $\mathbb{W} = \mathbb{A}\mathbb{A}^{\dagger}$ where $\mathbb{A}$ is of dimension $N \times T$. The matrix entries $A_{j\nu}$, for $1 \leqslant j \leqslant N$ and $1 \leqslant \nu \leqslant T$, are Gaussian variables with mean $\mu_{j\nu}$, variance σ^2, and correlations, ξ_{jk}, between the jth and kth rows of $\mathbb{A}$. In a usual setup, where $\mu_{j\nu} = 0$ and ξ is diagonal with 1, this model defines the Wishart or Laguerre ensemble (WE) where a lot is known in terms of Laguerre polynomials for the eigenvalue statistics [29,30]. The WE has been extensively used in diverse fields, particularly in QCD where it is referred to as the chiral ensemble [31].

If the off-diagonal terms of ξ are not 0 then the model defines the CWE. Using Dyson's classification of invariant ensembles [1,32], the three invariant CWE can be defined as the correlated Wishart orthogonal ensemble (CWOE), correlated Wishart unitary ensemble (CWUE), and correlated Wishart symplectic ensemble (CWSE). In this paper we consider rather a simple generalization for all three invariant CWEs, using $\mu_{j\nu} \neq 0$. This generalization defines the noncentral correlated Wishart ensembles (nc-CWEs) [33]. While CWEs are natural for correlation matrices [34], nc-CWEs are important in several other applications. For instance, noncentral Wishart ensembles (nc-WEs) have been used in QCD calculations at finite temperature and/or chemical potential [35–37] where exact results are known in much detail but for the unitary case. Recently, nc-CWEs have been used in the context of density matrices in Ref. [38] with a remark that the zero-mean condition is *a priori* not valid for the density matrices. Besides, nc-WEs have been revisited in the context of signal processing [39] and also in mathematical statistics [40–42]. It is also important to mention that the noncentral generalization of Gaussian and other ensembles [43–48] has also been useful in several other contexts; see Refs. [48,49] and the references therein.

For CWEs, the spectral density is known in terms of a Pastur self-consistent equation [12–14,16,18,50] which is valid for large N and T with finite ratio $N/T = \kappa$. For $\xi = \mathbf{1}_N$, where $\mathbf{1}_N$ is the $N \times N$ identity matrix in our notation, this Pastur equation yields the famous Marčenko-Pastur law for the spectral density of WEs. For finite N and T, the CWE poses a serious difficulty, yet the exact result is known for the spectral density [15,19,20] and the two-point spectral correlation is known asymptotically [18] for large matrices. The Pastur equation, however, has never been investigated for nc-CWE and our focus in this paper is to obtain the Pastur equation for nc-CWEs. We use the binary correlation method [2,18,46,51,52] to obtain our analytic results and investigate some important features such as the effect of nonzero means and cross correlations on the ensemble-averaged bulk density and on the ensemble-averaged mean positions of the eigenvalues separated from the bulk.

The paper is organized as follows. In the next section, Sec. II, we shall describe the model and fix our notations. In Sec. III we shall derive the loop equation. A similar loop equation has also been obtained in Ref. [52] in the context of nonsymmetric correlation matrices. In Sec. IV we shall rederive the Pastur equation by solving the loop equation for the CWE case and discuss analytic results for the separated eigenvalues. In the first part of Sec. V we shall specialize in the Pastur equation for nc-WEs and in the second part we shall derive the result for the ensemble-averaged mean position of the separated eigenvalues and compare this result with that for the CWE. In the final part of Sec. V we shall analyze our result for a nontrivial case. Similarly, in Sec. VI we shall generalize the method of Secs. IV and V and derive the Pastur equation for the nc-CWE in the first part. Next, we shall discuss the separated eigenvalues and the bulk density. Finally, we summarize our work with discussions in Sec. VII.

*vinayaksps2003@gmail.com

II. PRELIMINARIES

The model we are interested in is defined as

$$\mathbb{W} = \mathbb{A}\mathbb{A}^{\dagger}/T, \tag{1}$$

where $\mathbb{A}$ is $N \times T$ and

$$\mathbb{A} = \xi^{1/2}\mathbf{A} + \mathsf{B}, \tag{2}$$

so that, $\overline{\mathbb{A}} = \mathsf{B}$ and $\overline{\mathbb{W}} = v_1^2\xi + \mathsf{B}\mathsf{B}^{\dagger}/T$. Here ξ is the $N \times N$ real symmetric positive definite fixed (nonrandom) matrix which defines the correlation between the rows of $\mathbb{A}$, B is the $N \times T$ fixed matrix which represents the ensemble average matrix $\mathbb{A}$, and the overbar represents the ensemble averaging. Matrix $\mathbf{A}$ is the random matrix where the matrix entries A_{jk} are statistically independent real Gaussian variables with mean 0 and variance v_1^2. These choices define the nc-CWOE where the subscript in v_1^2 represents the value of the Dyson index β. Similarly for nc-CWUE, $\beta = 2$ and we consider $\mathbf{A} = \mathbf{A}^{(1)} + \mathrm{i}\mathbf{A}^{(2)}$ where $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ are statistically equivalent but independent Gaussian matrices described with mean 0 and variance v_2^2. Finally for the nc-CWSE, $\mathbf{A}$ is composed of four statistically equivalent but independent Gaussian matrices described by mean 0 variance v_4^2 and written in terms of a two-dimensional identity matrix $\mathbf{1}_2$ and a two-dimensional matrix representative of quaternion units τ_γ where $\gamma = 1,\ldots,3$. $\mathbb{A}^{\dagger}$ is the transpose, Hermitian conjugate and dual of $\mathbb{A}$, respectively for $\beta = 1$, 2, and 4. The joint probability density (jpd) of the matrix elements of $\mathbf{A}$ is given by the Gaussian probability measure,

$$\mathcal{P}(\mathbf{A}) \propto \exp\left[-\mathrm{Tr}\frac{\mathbf{A}\mathbf{A}^{\dagger}}{2v_\beta^2}\right]. \tag{3}$$

Since variance supplies the scale for the statistics, we fix the scale as $v_\beta^2 = \sigma^2\beta^{-1}$ [53]. Without loss of generality we consider $T \geqslant N$.

We use the binary correlation method to obtain the ensemble-averaged spectral density, $\overline{\rho}_{\mathbb{W}}(\lambda)$. In this method it is convenient to deal with the Stieltjes transform or the resolvent or the Green's function of the density. The resolvent, $\overline{\mathbf{g}}_{\mathbb{W}}(z)$, is defined as

$$\overline{\mathbf{g}}_{\mathbb{W}}(z) = \overline{\langle (z\mathbf{1}_N - \mathbb{W})^{-1}\rangle_N}, \tag{4}$$

where $z = \lambda \pm \mathrm{i}\epsilon$ with $\epsilon > 0$ and the angular brackets represent the spectral averaging. For example, $\langle\mathbf{H}\rangle_K = K^{-1}\,\mathrm{tr}\,\mathbf{H}$ where the matrix $\mathbf{H}$ is $K \times K$. Then $\overline{\rho}_{\mathbb{W}}(\lambda)$ can be determined uniquely via the relation

$$\overline{\rho}_{\mathbb{W}}(\lambda) = \lim_{\epsilon\to 0}\frac{\mp}{\pi}\mathrm{Im}\,\overline{\mathbf{g}}_{\mathbb{W}}(z). \tag{5}$$

In order to calculate $\overline{\mathbf{g}}_{\mathbb{W}}(z)$ we may use the moment expansion, since for large z

$$\overline{\mathbf{g}}_{\mathbb{W}}(z) = \sum_{n=0}^{\infty}\frac{\overline{\mathbf{m}}_n}{z^{n+1}}. \tag{6}$$

Here $\overline{\mathbf{m}}_n$ is the nth moment, of $\overline{\rho}_{\mathbb{W}}(\lambda)$, defined as

$$\overline{\mathbf{m}}_n = \int d\lambda\,\lambda^n\overline{\rho}_{\mathbb{W}}(\lambda) = \overline{\langle\mathbb{W}^n\rangle_N}. \tag{7}$$

In principle, the problem is solved once we obtain a closed form of $\overline{\mathbf{g}}_{\mathbb{W}}(z)$. As in Ref. [18], we could have started the moment expansion to obtain $\overline{\mathbf{g}}_{\mathbb{W}}(z)$. However, due to the additional matrix B the expansion results in nontrivial combinations of $\xi^{1/2}\mathbf{A}$ and B. Further complications will arise in the ensemble averaging of this series with respect to the jpd given in (3). We simplify the problem regarding the ensemble averaging first by using the trick of linearization [54]. Following Ref. [54], we define

$$\mathsf{X} = \frac{1}{\sqrt{T}}\left[\begin{pmatrix}\mathbf{0} & \mathsf{A}\\ \mathsf{A}^{\dagger} & \mathbf{0}\end{pmatrix} + \begin{pmatrix}\mathbf{0} & \mathsf{B}\\ \mathsf{B}^{\dagger} & \mathbf{0}\end{pmatrix}\right], \tag{8}$$

where we used A instead of $\xi^{1/2}\mathbf{A}$ for notational convenience. In the QCD language the above form of the matrix is nothing but the Dirac operator in the chiral basis [31]. In our calculation we also use $(N+T)\times(N+T)$ matrices, defined as

$$\tilde{\mathsf{A}} = \begin{pmatrix}\mathbf{0} & \mathsf{A}\\ \mathsf{A}^{\dagger} & \mathbf{0}\end{pmatrix},\ \tilde{\mathsf{B}} = \begin{pmatrix}\mathbf{0} & \mathsf{B}\\ \mathsf{B}^{\dagger} & \mathbf{0}\end{pmatrix}. \tag{9}$$

Notice that the eigenvalues of X^2 coincide with those of $\mathbb{W}$ with a twofold degeneracy for each. Next we define the resolvent, $\overline{\mathbf{g}}_{\mathsf{X}}(u)$, for the spectral density $\overline{\rho}_{\mathsf{X}}(y)$ of X as

$$\overline{\mathbf{g}}_{\mathsf{X}}(u) = \overline{\langle(\mathsf{U}-\mathsf{X})^{-1}\rangle_{N+T}},\ \text{where } \mathsf{U} = u\mathbf{1}_{N+T}, \tag{10}$$

and $u = y \pm \mathrm{i}\epsilon$. In what follows, we calculate $\overline{\mathbf{g}}_{\mathbb{W}}(z)$ from $\overline{\mathbf{g}}_{\mathsf{X}}(u)$ using the relation between them as given below. For the first, and so on, moments of $\overline{\rho}_{\mathsf{X}}$ are related with the moments of $\overline{\rho}_{\mathbb{W}}$ via

$$z\overline{\mathbf{g}}_{\mathbb{W}}(z) - 1 = \frac{N+T}{2N}[u(z)\overline{\mathbf{g}}_{\mathsf{X}}(u(z)) - 1], \tag{11}$$

where $u^2 = z$.

We will calculate the ensemble average of a matrix-valued Green's function

$$\mathsf{G}_{\mathsf{L}}^{(\mathsf{X})}(u) = \mathsf{L}\,(U-X)^{-1}, \tag{12}$$

where

$$\mathsf{G}^{(\mathsf{X})} = \begin{pmatrix}G_{11} & G_{12}\\ G_{21} & G_{22}\end{pmatrix}. \tag{13}$$

In the right-hand-side (rhs) of the above equation, G_{jj}'s are the square blocks, of dimensions $N \times N$ and $T \times T$ respectively for $j = 1$ and 2, and G_{12} and G_{21} are rectangular blocks, respectively of dimensions $N \times T$ and $T \times N$. We use L as an $(N+T)\times(N+T)$ arbitrary fixed matrix. For $\mathsf{L} = \mathbf{1}_{N+T}$, $\mathsf{G}_{\mathsf{L}}^{(\mathsf{X})}$ gives $\mathsf{G}^{(\mathsf{X})}$ and on the spectral averaging the latter yields $\overline{\mathbf{g}}_{\mathsf{X}}(u)$, as $\overline{\langle\mathsf{G}^{(\mathsf{X})}(u)\rangle_{N+T}} = \overline{\mathbf{g}}_{\mathsf{X}}(u)$. Finally, we define the ratio

$$\kappa = N/T. \tag{14}$$

III. LOOP EQUATION

We notice that the large-u expansion of $\mathsf{G}_{\mathsf{L}}^{(\mathsf{X})}(u)$ has nontrivial combinations of $\tilde{\mathsf{A}}$ and $\tilde{\mathsf{B}}$. Since $\tilde{\mathsf{B}}$ is a fixed matrix, we may use

$$\mathsf{K} = (\mathsf{U}-\tilde{\mathsf{B}})^{-1}, \tag{15}$$

and expand $\mathsf{G}_{\mathsf{L}}^{(\mathsf{X})}(u)$ the for small K (or equivalently for large u). It is worth mentioning that this trick has been used in the context of noncentral Gaussian ensembles in Ref. [46]. Then the large-u expansion of Eq. (12) can be written as

$$\mathsf{G}_{\mathsf{L}}^{(\mathsf{X})}(u) = \mathsf{LK}\sum_{n=0}^{\infty}(\tilde{\mathsf{A}}\mathsf{K})^n. \tag{16}$$

Since the A_{jk}'s are centered, at 0, on the ensemble average, the odd-n terms of the above expansion are identically 0. Thus the ensemble-averaged series reduces to

$$\overline{\mathsf{G}_{\mathsf{L}}^{(\mathsf{X})}}(u) = \mathsf{LK} + \mathsf{LK}\overline{\tilde{\mathsf{A}}\mathsf{G}^{(\mathsf{X})}\tilde{\mathsf{A}}\mathsf{G}^{(\mathsf{X})}}. \tag{17}$$

In order to perform the ensemble averaging for the remaining terms we use the jpd (3) with $\mathsf{A} = \xi^{1/2}\mathbf{A}$ and derive the following exact identities, valid for arbitrary fixed Φ and Ψ:

$$\frac{1}{T}\overline{\mathsf{A}\Phi\mathsf{A}^\dagger\Psi} = \sigma^2\langle\Phi\rangle_T\xi\Psi, \tag{18}$$

$$\frac{1}{T}\overline{\mathsf{A}^\dagger\Phi\mathsf{A}\Psi} = \sigma^2\langle\xi\Phi\rangle_T\Psi, \tag{19}$$

$$\overline{\mathsf{A}\Phi\mathsf{A}\Psi} = \frac{(2-\beta)\sigma^2}{\beta}\Psi\tilde{\Phi}, \tag{20}$$

where $\tilde{\Phi} = \Phi^t$, for $\beta = 1$ where Φ^t is the transpose of Φ, $\tilde{\Phi} = \Phi$ for $\beta = 2$, and $\tilde{\Phi} = -\tau_2\Phi^t\tau_2$ for $\beta = 4$ [18]. As the identities suggest, we consider only the terms resulting from the binary associations of A with $\mathsf{A}^\dagger$ and avoid terms resulting from the binary associations of A with A. With the help of these identities we calculate only the leading order terms of the series in Eq. (17). We find

$$\overline{\mathsf{G}_{\mathsf{L}}^{(\mathsf{X})}} = \mathsf{LK} + \mathsf{LK}\Sigma\overline{\mathsf{G}^{(\mathsf{X})}}, \tag{21}$$

where the equality is valid only in the leading order and

$$\Sigma = \sigma^2\begin{pmatrix}\xi\,\overline{\langle G_{22}\rangle}_T & \mathbf{0}\\ \mathbf{0} & \kappa\,\overline{\langle\xi G_{11}\rangle}_N\mathbf{1}_T\end{pmatrix}. \tag{22}$$

In the derivation of Eqs. (21) and (22) we have avoided binary associations across the traces as those also result terms of $O(N^{-1})$. Substituting now $\mathsf{L}\to\mathsf{L}(\mathbf{1}_N - \mathsf{K}\Sigma)^{-1}$ in Eq. (21), and then using Eq. (15), we finally get

$$\overline{\mathsf{G}_{\mathsf{L}}^{(\mathsf{X})}}(u) = \mathsf{L}(\mathsf{U} - \tilde{\mathsf{B}} - \Sigma)^{-1}. \tag{23}$$

In order to calculate the inverse of the matrix in the rhs of Eq. (23), we use the Schur decomposition. For instance, using $\mathsf{M} = \mathsf{U} - \tilde{\mathsf{B}} - \Sigma$, we may write

$$\mathsf{M}^{-1} = \begin{pmatrix}\mathbf{a} & \mathbf{b}\\ \mathbf{c} & \mathbf{d}\end{pmatrix}^{-1} = \begin{pmatrix}\mathsf{S}^{-1} & -\mathsf{S}\mathbf{b}\mathbf{d}^{-1}\\ -\mathbf{d}^{-1}\mathbf{c}\mathsf{S}^{-1} & (\mathbf{d}-\mathbf{c}\mathbf{a}^{-1}\mathbf{b})^{-1}\end{pmatrix}, \tag{24}$$

where $\mathsf{S} = \mathbf{a} - \mathbf{b}\mathbf{d}^{-1}\mathbf{c}$ and

$$\mathbf{a} = u\mathbf{1}_N - \sigma^2\xi\overline{g}_{22},\quad \mathbf{b} = -\frac{1}{\sqrt{T}}\mathsf{B},$$
$$\mathbf{c} = -\frac{1}{\sqrt{T}}\mathsf{B}^\dagger,\quad \mathbf{d} = (u - \sigma^2\kappa\,\overline{g}_{11;\xi})\mathbf{1}_T. \tag{25}$$

We have used here more general spectral averaged quantities, defined as

$$\overline{g}_{jj;\mathcal{L}} = \langle\mathcal{L}\overline{G}_{jj}\rangle_K, \tag{26}$$

with $\mathcal{L}$ as an arbitrary fixed matrix and K is N and T, respectively, for $j = 1$ and 2. For example, the spectral-averaged quantity $\overline{g}_{22}$ is obtained by using $\mathcal{L} = \mathbf{1}_T$ in definition (26), for the corresponding upper diagonal-block matrix in the rhs of Eq. (22). Similarly, for the lower diagonal-block matrix we have used $\mathcal{L} = \xi$; $\overline{g}_{11;\xi} = \langle\xi\overline{G}_{11}\rangle_N$.

Next, we use $\mathsf{L} = \mathbf{1}_{N+T}$ in Eq. (23) and compute $\overline{\mathbf{g}}_{\mathsf{X}}(u)$ using Eq. (10). We get

$$\overline{\mathbf{g}}_{\mathsf{X}}(u) = \langle(\overline{g}_{11}\mathbf{1}_N \oplus \overline{g}_{22}\mathbf{1}_T)\rangle_{N+T}, \tag{27}$$

where $\oplus$ stands for the direct sum and

$$\overline{g}_{11} = \left\langle\frac{1}{u\mathbf{1}_N - \sigma^2\xi\,\overline{g}_{22} - \frac{\zeta}{(u-\sigma^2\kappa\overline{g}_{11;\xi})}}\right\rangle_N, \tag{28}$$

$$\overline{g}_{22} = \left\langle\frac{1}{(u - \sigma^2\kappa\overline{g}_{11;\xi})\mathbf{1}_T - \frac{1}{T}\mathsf{B}^\dagger(u-\sigma^2\xi\,\overline{g}_{22})^{-1}\mathsf{B}}\right\rangle_T. \tag{29}$$

In the above equation we have introduced a positive definite matrix $\zeta = \mathsf{B}\mathsf{B}^\dagger/T$. As mentioned above, we calculate $\overline{\mathbf{g}}_{\mathbb{W}}(z)$ from $\overline{\mathbf{g}}_{\mathsf{X}}(u)$ using Eq. (27) for the latter and then use relation (11) to obtain the former.

IV. PASTUR EQUATION FOR CWE

For our model, $B_{jk} = 0$ defines the CWE. The Pastur equation of the CWE has been derived by several authors [12–14,16,18] using different techniques. As mentioned before, for large N and T with finite ratio κ, the spectral density is known in terms of a Pastur self-consistent equation. Below we give an alternative method to obtain the Pastur density for the CWE by solving the loop equation (28) and (29).

We first note that in this case Eqs. (28) and (29) reduce to

$$\overline{g}_{11}(u) = \langle(u\mathbf{1}_N - \sigma^2\xi\,\overline{g}_{22})^{-1}\rangle,$$
$$\overline{g}_{22}(u) = (u - \sigma^2\kappa\overline{g}_{11;\xi})^{-1}, \tag{30}$$

where

$$\overline{g}_{11;\xi}(u) = \langle\xi\,(u\mathbf{1}_N - \sigma^2\xi\,\overline{g}_{22})^{-1}\rangle = \frac{u\overline{g}_{11}(u) - 1}{\sigma^2\overline{g}_{22}(u)}. \tag{31}$$

We may also write $\overline{g}_{22}$ as

$$u\overline{g}_{22}(u) = 1 + \sigma^2\kappa\overline{g}_{11;\xi}(u)\overline{g}_{22}(u). \tag{32}$$

Using the second equality of Eq. (31) in the above equation, we get

$$\overline{g}_{22} = \frac{\kappa\,u\overline{g}_{11} + 1 - \kappa}{u}. \tag{33}$$

This is a very useful equation because not only does it establish a linear relation between $\overline{g}_{11}$ and $\overline{g}_{22}$, that we need to solve the loop equation, but also when inserted in Eqs. (11) and (27) it leads to another useful identity, viz.

$$z\overline{\mathbf{g}}_{\mathbb{W}}(z) = u(z)\overline{g}_{11}(u(z)). \tag{34}$$

It will be shown ahead that the above two relations (33) and (34) are also valid for nc-WEs and nc-CWEs. Finally, we use these two relations in Eq. (30), with $u^2 = z$, and obtain the Pastur equation for the CWE:

$$\overline{\mathbf{g}}_{\mathbb{W}}(z) = \langle \{z\mathbf{1}_N - \sigma^2[1-\kappa + z\kappa \overline{\mathbf{g}}_{\mathbb{W}}(z)]\xi\}^{-1}\rangle_N. \quad (35)$$

As noted in Ref. [18], this result is independent of the Dyson index β because of the scaling $v_\beta^2 = \sigma^2/\beta$. The same holds true for the Pastur equations of nc-WEs and nc-CWEs.

Notice that Eq. (35) depends on the spectrum of ξ. For a nontrivial spectrum of ξ, the analytic solution is complicated. Thus, this equation has to be solved numerically. To this end an efficient numerical algorithm has been discussed in Ref. [18] where various cases of ξ have been worked out. However, analytically we can solve the Pastur equation when it is quadratic. For instance, consider $\xi_{jk} = \delta_{jk}$. For this choice the Pastur equation (35) yields the resolvent

$$\overline{\mathbf{g}}_{\mathbb{W}}(z) = \frac{z - \sigma^2(1-\kappa) - \sqrt{[z-\sigma^2(1-\kappa)]^2 - 4z\kappa\sigma^2}}{2\kappa z\sigma^2}, \quad (36)$$

where we have considered the negative sign for the square root, so that $\overline{\mathbf{g}}_{\mathbb{W}}(z)$ behaves as z^{-1} for large z. Next, the inverse transform of this resolvent gives the famous Marčenko-Pastur density [12]:

$$\overline{\rho}_{\mathrm{MP}}(\lambda) = \frac{\sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)}}{2\pi\kappa\sigma^2\lambda}, \quad (37)$$

where $\lambda_\pm = \sigma^2(\sqrt{\kappa} \pm 1)^2$.

It has been shown in Ref. [18] that Eq. (35) can also be solved for the equal-cross-correlation matrix model, viz. $\xi_{jk} = \delta_{jk} + (1-\delta_{jk})\,\mu_0^2$. Notice that in this case ξ is diagonal plus a rank-1 matrix. Thus for $N\mu_0^2 > \sqrt{\kappa}$ the spectral density we find is composed of a bulk and a separated eigenvalue:

$$\overline{\rho}_{\mathbb{W}}(\lambda) = \overline{\rho}_0(\lambda) + N^{-1}\delta(\lambda - \overline{\lambda}_N). \quad (38)$$

The bulk density, $\overline{\rho}_0(\lambda)$, is described by the Marčenko-Pastur law with a rescaled variance $\sigma^2(1-\mu_0^2)$. The ensemble-averaged mean position of the separated eigenvalues, $\overline{\lambda}$, is given by

$$\overline{\lambda}_N = \sigma^2 \frac{\left[(N-1)\mu_0^2 + 1\right]\left[(N-\kappa)\mu_0^2 + \kappa\right]}{N\mu_0^2} \simeq \sigma^2 \frac{\left(N\mu_0^2 + 1\right)\left(N\mu_0^2 + \kappa\right)}{N\mu_0^2}. \quad (39)$$

A simple generalization of the equal-cross-correlation matrix is a block diagonal matrix where each block is an equal-cross-correlation matrix. This matrix can also be seen as a diagonal plus a finite-rank matrix. In this case the above result can be easily generalized for other separated eigenvalues. However, it has been shown in Ref. [18] that even for a more complicated spectrum of ξ, the analytic result for the kth separated eigenvalue $\overline{\lambda}_k$ can be written as

$$\overline{\lambda}_k = \sigma^2\lambda_k^{(\xi)}\left(1 - \kappa + \kappa\,\lambda_k^{(\xi)}\left\langle \mathbb{Q}_k\left(\lambda_k^{(\xi)}\,\mathbf{1}_N - \xi\right)^{-1}\right\rangle_N\right), \quad (40)$$

where $\lambda_k^{(\xi)}$ is the kth eigenvalue of ξ and $\mathbb{Q}_k = \mathbf{1}_N - |k\rangle\,\langle k|$ is the projection operator to the kth eigenstate $|k\rangle$ of ξ.

V. PASTUR EQUATION FOR NC-WE

Analytically, nc-WEs are perhaps the simplest case next to WEs. nc-WEs have already been addressed in Refs. [35–37,42,48] where many important results have been derived using different methods. Since the Pastur equation has never been given explicitly, below we derive the Pastur equation for the nc-WE.

We begin with using $\xi = \mathbf{1}_N$, in Eqs. (28) and (29), which results in

$$\overline{g}_{11} = \left\langle \frac{1}{(u - \sigma^2\overline{g}_{22})\mathbf{1}_N - \frac{\zeta}{u - \sigma^2\kappa\overline{g}_{11}}} \right\rangle_N, \quad (41)$$

$$\overline{g}_{22} = \left\langle \frac{1}{(u - \sigma^2\kappa\overline{g}_{11})\mathbf{1}_T - \frac{\eta}{u - \sigma^2\overline{g}_{22}}} \right\rangle_T, \quad (42)$$

where in the second equality we have used $\mathsf{B}^\dagger\mathsf{B}/T = \eta$. Notice that except for the zeros, ζ and η both have the same spectrum. As mentioned above, Eqs. (33) and (34) also hold here. To show this we first write

$$\overline{g}_{22} = \frac{1}{T}\sum_{j=1}^{N}\left[\frac{u - \sigma^2\overline{g}_{22}}{(u\mathbf{1}_N - \sigma^2\kappa\overline{g}_{11})(u - \sigma^2\overline{g}_{22}) - \lambda_j^{(\zeta)}}\right] + \frac{(1-\kappa)}{(u\mathbf{1}_N - \sigma^2\kappa\overline{g}_{11})}. \quad (43)$$

Next, we use Eq. (41) in the above equality and obtain the relation (33), which consequently implies the relation (34). Finally, we use these relations, (33) and (34), with $u^2 = z$, to simplify the loop equation (41) into a self-consistent equation for $\overline{\mathbf{g}}_{\mathbb{W}}(z)$. This method yields the Pastur equation for the nc-WE:

$$\overline{\mathbf{g}}_{\mathbb{W}}(z) = \left\langle \frac{1}{\{z - \sigma^2[1-\kappa + z\kappa\overline{\mathbf{g}}_{\mathbb{W}}(z)]\}\mathbf{1}_N - \frac{\zeta}{1 - \sigma^2\kappa\,\overline{\mathbf{g}}_{\mathbb{W}}(z)}} \right\rangle_N. \quad (44)$$

If we set now $\zeta = 0$, then we retrieve the resolvent of the Marčenko-Pastur density as given in Eq. (36). Otherwise, if we set $\sigma^2 = 0$ then it will give the resolvent corresponding to the spectrum of ζ. Like the Pastur equation for the CWE, here as well, Eq. (44) depends on the spectrum of ζ and thus has to be solved numerically when ζ has a nontrivial spectrum. Below we consider a rank-1 matrix B which is closely related with the equal-cross-correlation matrix model of the CWE. However, unlike the CWE in this case the bulk density is not rescaled with variance but remains the same as for the WE. Using the techniques of Refs. [18,46] we start with this simple choice to calculate the ensemble-averaged mean position of the separated eigenvalues and generalize this result for the bulk density being different from the Marčenko-Pastur density.

A. Separation of eigenvalues

We begin with a simple choice for B, viz.

$$B_{jk} = \mu. \quad (45)$$

Then the only nonzero eigenvalue of ζ is $\lambda_N^{(\zeta)} = N\mu^2$. In this case, from Eq. (44) we get

$$\overline{\mathbf{g}}_{\mathbb{W}}(z) = \overline{\mathbf{g}^{(0)}}(z) + \frac{N^{-1}}{z - \sigma^2[1-\kappa+z\kappa\overline{\mathbf{g}}_{\mathbb{W}}(z)] - \frac{\lambda_N^{(\zeta)}}{1-\sigma^2\kappa\,\overline{\mathbf{g}}_{\mathbb{W}}(z)}}. \tag{46}$$

Here we have used

$$\overline{\mathbf{g}^{(0)}}(z) = \left\langle \mathbb{Q}_N\left[z\mathbf{1}_N - \sigma^2[1-\kappa+z\kappa\overline{\mathbf{g}}_{\mathbb{W}}(z)]\mathbf{1}_N - \frac{\zeta}{1-\sigma^2\kappa\,\overline{\mathbf{g}}_{\mathbb{W}}(z)}\right]^{-1}\right\rangle_N, \tag{47}$$

where $\mathbb{Q}_k^{(\zeta)} = \mathbf{1}_N - |k\rangle\langle k|$ and $|k\rangle\langle k|$ is the projection operator for the eigenstate $|k\rangle$ corresponding to the eigenvalue $\lambda_k^{(\zeta)}$. Solving Eq. (46), while ignoring the second term, we retrieve the Marčenko-Pastur result (36) for the bulk density; it is understood that the bulk density is normalized to $1 - 1/N$. However, in the the above equation we do not drop the term containing ζ and treat this term as for general ζ. The ensemble-averaged mean position of the separated eigenvalues can be identified from the pole in the second term of Eq. (46) as

$$\overline{\lambda}_N = \sigma^2[1-\kappa+\overline{\lambda}_N\kappa\overline{\mathbf{g}^{(0)}}(\overline{\lambda}_N)] + \frac{\lambda_N^{(\zeta)}}{1-\sigma^2\kappa\,\overline{\mathbf{g}^{(0)}}(\overline{\lambda}_N)}, \tag{48}$$

where we have used $\overline{\mathbf{g}^{(0)}}$ instead of $\overline{\mathbf{g}}$ and ignored the $O(N^{-1})$ term. Using this in Eq. (47) we obtain

$$\overline{\mathbf{g}^{(0)}}(\overline{\lambda}_N) = \frac{\Phi_N}{1+\sigma^2\kappa\Phi_N}, \tag{49}$$

where

$$\Phi_N = \left\langle \mathbb{Q}_N\left(\lambda_N^{(\zeta)}\mathbf{1}_N - \zeta\right)^{-1}\right\rangle_N. \tag{50}$$

Next, using Eq. (49) in Eq. (48), we obtain $\overline{\lambda}_N$. Following Ref. [18] we can also generalize this result for the kth separated eigenvalue, $\overline{\lambda}_k$, as

$$\overline{\lambda}_k = (1+\sigma^2\kappa\Phi_k)[\sigma^2(1-\kappa) + \lambda_k^{(\zeta)}(1+\sigma^2\kappa\Phi_k)]. \tag{51}$$

The above result is of course different from that for the CWE. However, for the rank-1 matrix B this result gives

$$\overline{\lambda}_N = \frac{(N\mu^2+\sigma^2)(N\mu^2+\sigma^2\kappa)}{N\mu^2}, \tag{52}$$

which is valid only if $N\mu^2 > \sqrt{\kappa}\sigma^2$ otherwise the separated eigenvalue will be absorbed in the Marčenko-Pastur bulk density. Interestingly, it also coincides with (39) for $\mu = \mu_0$ and $\sigma^2 = 1$. In Ref. [38], this correspondence has been exploited without any analytical treatment for the nc-WE. There are the parameters chosen as $\mu = \sqrt{r/N}$ and $\sigma = (1-r)/\sqrt{N}$ in the nc-WUE case and $\mu_0 = \sqrt{r}$ and $\sigma = 1/\sqrt{N}$ in the CWUE case. Indeed, for these parameters the two results (39) and (52) coincide in the leading order.

B. Bulk density

It is important to point out that Eq. (44) describes only the bulk density and not the density of the separated eigenvalues. It has been proven for CWUE that the density of the separated eigenvalues is described by a Gaussian distribution [17] and numerically the same has been found for the nc-WUE case [38]. To obtain the bulk density for a nontrivial spectrum of ζ, Eq. (44) has to be solved numerically. We thus use Newton's method, similar to the method described in Ref. [18], to solve Eq. (44). We consider

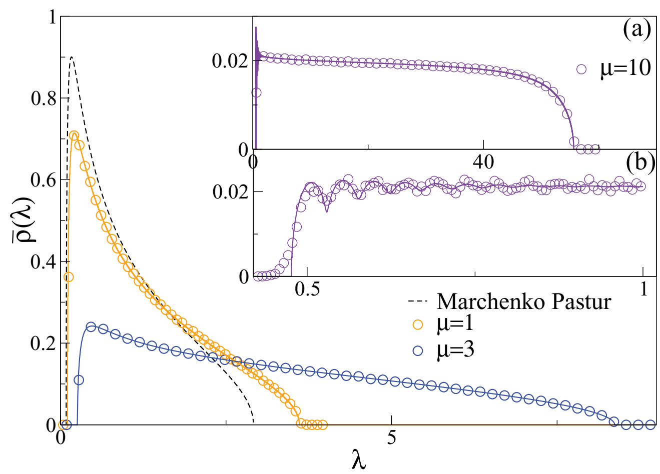

FIG. 1. (Color online) Spectral density of the nc-WOE where $B_{j\nu} = \delta_{j\nu}\mu\sqrt{j}$, $N = 1024$, $T = 2N$, $\sigma^2 = 1$, and $\mu = 0$, 1, 3, and 10. Solid lines in this figure represent the theory obtained from the numerical solution of Eq. (44) and open circles represent the histogram data obtained from the Monte Carlo simulation of C. Dashed line represents the Marčenko-Pastur density, i.e., the $\mu = 0$ case. In the inset (a) we show the density for $\mu = 10$ and in (b) we show the density for $\lambda \leqslant 1$ for the same μ.

$$f\left(\overline{\mathbf{g}_{\mathbb{W}}^{(n)}}(z)\right) - \overline{\mathbf{g}_{\mathbb{W}}^{(n)}}(z) = 0, \tag{53}$$

at given z where $f(\overline{\mathbf{g}_{\mathbb{W}}^{(n)}}(z))$ is the rhs of Eq. (44) for $\overline{\mathbf{g}_{\mathbb{W}}^{(n)}}(z)$, and n represents the iteration number starting from 0 with an initial guess $\overline{\mathbf{g}_{\mathbb{W}}^{(0)}}(z)$.

To illustrate the result (44) we use $B_{j\nu} = \delta_{j\nu}\mu\sqrt{j}$, for $0 \leqslant j \leqslant N$ and $\sigma^2 = 1$. In Fig. 1 we compare the numerical solution of our theory for $N = 1024$ with the Monte Carlo simulations for $N = 1024$ and $T = 2N$. In the main figure, we show results for $\mu = 1$ and 3, and for $\mu = 0$ we plot only the Marčenko-Pastur density. As can be seen from this figure, the density tends to attain a uniform shape as μ is increased. This is closely predicted by the theory. In two insets, (a) and (b), we show the result for $\mu = 10$. As shown in (a), our theory gives a reasonable account for the histogram data throughout the support for the density. In (b) we notice oscillations, for $\lambda < 1$, which is almost consistent with the theory.

VI. PASTUR-EQUATION FOR NC-CWE

Having specialized in CWE and nc-WE cases we now consider $\xi \neq \mathbf{1}_N$ and $\zeta \neq 0$ in Eqs. (28) and (29). We first note that Eq. (29) can be written as

$$\overline{g}_{22} = \frac{1-\kappa}{u-\sigma^2\kappa\overline{g}_{11;\xi}} + \kappa\langle[(u-\sigma^2\kappa\overline{g}_{11;\xi})\mathbf{1}_N - \zeta(u-\sigma^2\xi\overline{g}_{22})^{-1}]^{-1}\rangle_N. \tag{54}$$

Using this and Eq. (28) one finds the relation (33) and consequently the relation (34). Next, exploiting relations (33) and (34) with $u^2 = z$ in Eq. (28) we obtain a coupled-Pastur equation:

$$\overline{\mathbf{g}}_{\mathbb{W};L}(z) = \left\langle L \frac{1}{z\mathbf{1}_N - \alpha_1(z,\overline{\mathbf{g}}_{\mathbb{W}}(z))\,\xi - \alpha_2(\overline{\mathbf{g}}_{\mathbb{W};\xi}(z))\zeta} \right\rangle_N, \tag{55}$$

where L is an arbitrary $N \times N$ matrix and

$$\begin{aligned}\alpha_1(z,\overline{\mathbf{g}}_{\mathbb{W}}(z)) &= \sigma^2[1-\kappa+\kappa z\overline{\mathbf{g}}_{\mathbb{W}}(z)],\\ \alpha_2(\overline{\mathbf{g}}_{\mathbb{W};\xi}(z)) &= [1-\sigma^2\kappa\overline{\mathbf{g}}_{\mathbb{W};\xi}(z)]^{-1}.\end{aligned} \tag{56}$$

Choices $L = \mathbf{1}_N$ and $L = \xi$ yield respectively $\overline{\mathbf{g}}_{\mathbb{W}}(z)$ and $\overline{\mathbf{g}}_{\mathbb{W};\xi}(z)$ and thus complete the result. It is easy to see that results (35) and (44) are immediate from the result (55), respectively, for the choices $L = \mathbf{1}_N$ and $\zeta = 0$, and $L = \mathbf{1}_N$ and $\xi = \mathbf{1}_N$.

A. Separation of eigenvalues

It is also important to note that in general Eq. (55) cannot be simplified to the eigenvalues of ξ and ζ unless they commute with each other. Therefore, unlike the Pastur equation, it is difficult to extend the results (39) and (52) to the nc-CWE case.

We consider $\xi_{jk} = \delta_{jk} + (1-\delta_{jk})\mu_0^2$ and $B_{jk} = \mu\delta_{jk}$. In this case we can write Eq. (55) as

$$\overline{\mathbf{g}}_{\mathbb{W}}(z) = \overline{\mathbf{g}_{\mathbb{W}}^{(0)}}(z) + \frac{1}{N}\frac{1}{z-\overline{\lambda}_N}. \tag{57}$$

Here, since $(N-1)$ eigenvalues of ζ are identically zero, we have

$$\overline{\mathbf{g}_{\mathbb{W}}^{(0)}}(z) = \left\langle \mathbb{Q}_N \frac{1}{z\mathbf{1}_N - \alpha_1(z,\overline{\mathbf{g}}_{\mathbf{W}}(z))\,\xi} \right\rangle_N, \tag{58}$$

where $\mathbb{Q}_k$ corresponds to the kth eigenstates of ξ and

$$\overline{\lambda}_N = \lambda_N^{(\xi)}\alpha_1(\overline{\lambda}_N,\overline{\mathbf{g}}_{\mathbb{W}}(\overline{\lambda}_N)) + \lambda_N^{(\zeta)}\alpha_2(\overline{\mathbf{g}}_{\mathbb{W};\xi}(\overline{\lambda}_N)). \tag{59}$$

Next, we write

$$\begin{aligned}\overline{\mathbf{g}_{\mathbb{W};\xi}}(z) &= \overline{\mathbf{g}_{\mathbb{W};\xi}^{(0)}}(z) + O(N^{-1}),\\ \overline{\mathbf{g}_{\mathbb{W};\xi}^{(0)}}(z) &= \left\langle \mathbb{Q}_N\,\xi \frac{1}{z\mathbf{1}_N - \alpha_1(z,\overline{\mathbf{g}}_{\mathbb{W}}(z))\xi} \right\rangle_N.\end{aligned} \tag{60}$$

We notice a relation between $\alpha_1(z,\overline{\mathbf{g}_{\mathbb{W}}^{(0)}}(z))$ and $\alpha_2(z,\overline{\mathbf{g}_{\mathbb{W};\xi}^{(0)}}(z))$:

$$\alpha_1(z,\overline{\mathbf{g}_{\mathbb{W}}^{(0)}}(z)) = \sigma^2\alpha_2(\overline{\mathbf{g}_{\mathbb{W};\xi}^{(0)}}(z)). \tag{61}$$

To obtain the above result we write $\overline{\mathbf{g}_{\mathbb{W}}^{(0)}}(z)$ in terms of $\overline{\mathbf{g}_{\mathbb{W};\xi}^{(0)}}(z)$ and then use Eq. (56). This relation simplifies Eq. (59) as

$$\overline{\lambda}_N = \alpha_2(\overline{\mathbf{g}_{\mathbb{W};\xi}^{(0)}}(z))\left[\sigma^2\lambda_N^{(\xi)} + \lambda_N^{(\zeta)}\right]. \tag{62}$$

Further, using the above equation in Eq. (60), we find

$$\overline{\mathbf{g}_{\mathbb{W};\xi}^{(0)}}(\overline{\lambda}_N) = [\alpha_2(\overline{\mathbf{g}}_{\mathbb{W};\xi}(\overline{\lambda}_N))]^{-1}\Psi_N, \tag{63}$$

where

$$\Psi_N = \left\langle \mathbb{Q}_N\,\xi \frac{1}{\sigma^2\big(\lambda_N^{(\xi)}\mathbf{1}_N - \xi\big) + \lambda_N^{(\zeta)}\mathbf{1}_N} \right\rangle_N. \tag{64}$$

Substituting Eq. (63) in the definition of α_2, we find

$$\alpha_2(\overline{\mathbf{g}}_{\mathbb{W};\xi}(\overline{\lambda}_N)) = 1 + \sigma^2\kappa\Psi_N. \tag{65}$$

Finally, we use the above result in (62) and obtain

$$\overline{\lambda}_N = (1+\sigma^2\kappa\Psi_N)\big(\sigma^2\lambda_N^{(\xi)} + \lambda_N^{(\zeta)}\big). \tag{66}$$

This result can be generalized to the block diagonal ξ and ζ with dimensionally the same blocks where each block of ξ is represented by an equal-cross-correlation matrix while the corresponding ζ block is rank 1. For this setup one can generalize results (64) and (66), replacing the subscript N by k for the kth separated eigenvalue.

Solving the above equation for an equal-cross-correlation matrix ξ and a rank-1 matrix ζ we obtain the ensemble averaged mean position for the separated eigenvalue as

$$\overline{\lambda}_N = \frac{(N\Delta^2+\sigma^2)(N\Delta^2+\sigma^2\kappa)}{N\Delta^2}, \Delta^2 = \mu_0^2\sigma^2 + \mu^2, \tag{67}$$

where the above result is valid for $N\Delta^2 > \sqrt{\kappa}$. This result is an interesting generalization of the corresponding results for the CWE and nc-WE. Here the bulk density is described by the Marčenko-Pastur density with a rescaled variance $\sigma^2(1-\mu_0^2)$ as in Eq. (38).

B. A nontrivial example

For nontrivial and noncommuting ξ and ζ, Eq. (55) has to be solved numerically. Thus one has to extend the numerical algorithm of Ref. [18] for two equations of two variables, viz.

$$\begin{aligned}f_1\big(\overline{\mathbf{g}_{\mathbb{W}}^{(n)}}(z),\overline{\mathbf{g}_{\mathbb{W};\xi}^{(n)}}(z)\big) - \overline{\mathbf{g}_{\mathbb{W}}^{(n)}}(z) &= 0,\\ f_2\big(\overline{\mathbf{g}_{\mathbb{W}}^{(n)}}(z),\overline{\mathbf{g}_{\mathbb{W};\xi}^{(n)}}(z)\big) - \overline{\mathbf{g}_{\mathbb{W};\xi}^{(n)}}(z) &= 0,\end{aligned} \tag{68}$$

where $f_1(\overline{\mathbf{g}_{\mathbb{W}}^{(n)}}(z),\overline{\mathbf{g}_{\mathbb{W};\xi}^{(n)}}(z))$ and $f_2(\overline{\mathbf{g}_{\mathbb{W}}^{(n)}}(z),\overline{\mathbf{g}_{\mathbb{W};\xi}^{(n)}}(z))$ are the rhs of (55) respectively with $L = \mathbf{1}_N$ and $L = \xi$. Next, we start with initial guesses $\overline{\mathbf{g}_{\mathbb{W}}^{(0)}}(z)$ and $\overline{\mathbf{g}_{\mathbb{W};\xi}^{(0)}}(z)$ for a given z and use Newton's method to obtain the solution in the machine precision.

To illustrate the result we solve Eq. (55) for $\xi_{jk} = \delta_{jk} + (1-\delta_{jk})\mu_0^{(|j-k|)}$, where $\mathsf{B}_{j\nu} = \mu^{|j-\nu|}$ with $\mu = 0.5$ and μ_0 is varied as $\mu_0 = 0.1$, 0.3, and 0.5. Also we choose $\sigma^2 = 0.25$ and $N = 512$ with $T = 2N$. The result is shown in Fig. 2 where open circles represent the histogram data obtained from the Monte Carlo simulation of C and solid lines represent the numerical solution of the theory (55) for $N = 512$. As shown in the figure, the theory reasonably explains numerical results. In this figure we also compare theory for $\mu = 0.5$ for the corresponding CWOE ($\mu = 0$). As can be seen in this figure, the nonzero mean not only changes the density profile but also shifts nontrivially the spectrum.

VII. SUMMARY AND DISCUSSIONS

We have studied nc-CWE and obtained an exact result for the spectral density at large matrix dimension. The derivation

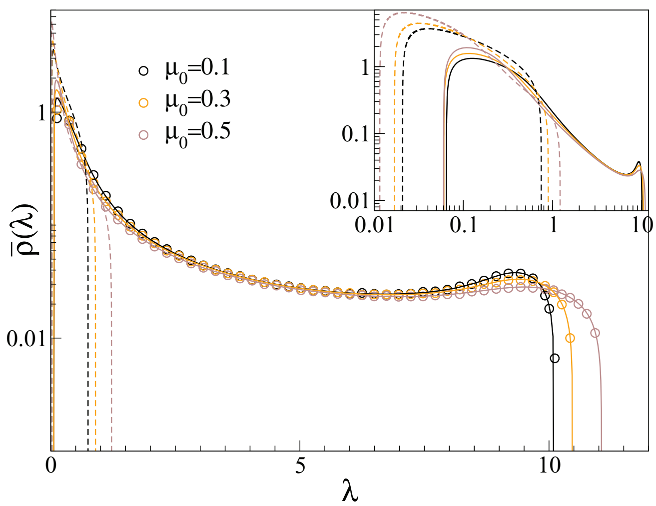

FIG. 2. (Color online) Spectral density for nc-CWOE where $\xi_{jk} = \delta_{jk} + (1-\delta_{jk})\mu_0^{(|j-k|)}$ and $\mathsf{B}_{j\nu} = \mu^{|j-\nu|}$ with $\mu = 0.5$ and $\sigma^2 = 0.25$. In this figure we have used $N = 512$ and $T = 2N$. In the main figure we show the spectral density on semilog plot for $\mu_0 = 0.1$, 0.3, and 0.5, respectively with black, orange, and brown colors. Solids lines represent the theory (55) for nc-CWE and dashed lines represent the corresponding $\mu = 0$ cases. We use the same color code for μ_0 in the inset where we compare only theory for $\mu = 0$ and $\mu = 0.5$ on the log-log scale.

is formalized in two steps, viz. first we obtain the loop equation for X, which eigenvalues are closely related with those of $\mathbb{W}$, and second we derive the Pastur equation for $\mathbb{W}$ from the loop equation. With this formalism we have derived Pastur equations for CWE, nc-WE, and nc-CWE. For all three cases, we have exploited a linear relation between the averaged quantities $u\overline{g}_{11}$ and $u\overline{g}_{22}$. We notice that in the first two cases the Pastur equation depends on the eigenvalues of positive definite symmetric matrices, ξ or $\zeta = \mathsf{B}\mathsf{B}^\dagger/T$. We have shown that in general, unlike CWE and nc-WE, the spectral density for nc-CWE does not depend simply on the spectra of ξ and ζ, rather more intricately on the matrices.

From the Pastur equation, we have worked out the ensemble-averaged mean position of the separated eigenvalues for the nc-WE. For the CWE this has been worked out in Refs. [17,18,42]. Following Ref. [18], we have derived the result for a general ζ. In the nc-CWE case the Pastur equation is more complicated. However, we have been able to work out the ensemble-averaged mean position of the separated eigenvalues for some special cases. As for CWE and nc-WE, for more general cases of nc-CWE we have used Newton's method to solve the Pastur equation numerically. We have supplemented our theoretical result with numerics for some nontrivial examples.

Finally, it would be interesting to extend this generalization for the Wishart model of nonsymmetric correlation matrices, those as dealt with in Refs. [51,52]. Another important extension of this work is related to short time series, meaning the situation where the number of time series is larger than the length of time series. This situation is often encountered in the correlation analysis of multivariate complex systems. In these examples, $N \gg T$ resulting in a correlation matrix which is singular with significantly many zero eigenvalues. In Refs. [55,56] the power map method has been proposed and used recently in Ref. [28] as a tool to get rid of this degeneracy. This method results in a spectrum emerging from the zero eigenvalues when the exponent is very close to 1. It has been shown in Ref. [27] that the so emerging spectrum is very sensitive to correlations. We believe that study of the emerging spectra corresponding to the nc-CWE is very important in the context of short time series.

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