

Phenomenological picture of fluctuations in branching random walks

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We propose a picture of the fluctuations in branching random walks, which leads to predictions for the distribution of a random variable that characterizes the position of the bulk of the particles. We also interpret the $1/\sqrt{t}$ correction to the average position of the rightmost particle of a branching random walk for large times $t \gg 1$, computed by Ebert and Van Saarloos, as fluctuations on top of the mean-field approximation of this process with a Brunet-Derrida cutoff at the tip that simulates discreteness. Our analytical formulas successfully compare to numerical simulations of a particular model of a branching random walk.

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I. INTRODUCTION

The goal of this work is to better understand the distribution of the particles generated by a branching random-walk process after some large evolution time. Our initial motivation for addressing this problem comes from particle physics [1] (for a review, see [2]). In the context of the scattering of hadrons at large energies, high-occupation quantum fluctuations dominate some of the scattering cross sections currently measured, for example, at the LHC. These quantum fluctuations can be thought of as being built up, as the hadrons are accelerated, by the successive branchings first of their constituent quarks into quark-gluon pairs, and then of the gluons into pairs of gluons, with some diffusion in their momenta. The dynamics of these gluons is actually exactly the kind of branching diffusion process that we are going to address in this work. Therefore, results that do not depend on the detailed properties of the particular branching random walk considered may be transposed to particle physics, and give quantitative insight into hadronic scattering cross sections.

Of course, the applications of branching random walks are much wider than particle physics. Branching random walks may, for example, generate Cayley trees, which would represent the configuration space of directed polymers in random media [3].

Although our discussion will be very general, for definiteness we shall consider a simple model for a branching random walk (BRW) in continuous time t and one-dimensional space x , defined by two elementary processes: Each particle diffuses independently of the others with diffusion constant 1, and may split into two particles at rate 1, in such a way that the mean particle density $\langle n(t, x) \rangle$ obeys the equation

$$\partial_t \langle n(t, x) \rangle = \partial_x^2 \langle n(t, x) \rangle + \langle n(t, x) \rangle. \quad (1)$$

A particular realization of this BRW is represented in Fig. 1.

Several properties of BRW are known. In particular, in any given realization of the stochastic process, for large enough times, the forward part of the distribution of the particles looks like an exponential e^{-x} (scaled by an appropriate

time-dependent constant, also depending on the particular realization considered) up to fluctuations effectively concentrated at its low-density tip. We shall call this exponential part the “front.”

Then, one can also establish rigorously [4,5] that the probability $Q(t, x)$ that all particles sit at a position smaller than x obeys a nonlinear partial differential equation that reads

$$\partial_t Q(t, x) = \partial_x^2 Q(t, x) - Q(t, x) + Q^2(t, x). \quad (2)$$

This is a version of the Fisher-Kolmogorov-Petrovsky-Piscounov (FKPP) equation [6,7]. (For an extensive review, see Ref. [8], and for more applications of the FKPP equation, see, e.g., Ref. [9]). If the BRW starts at time $t = 0$ with a single particle located at $x = 0$, then the initial condition is $Q(t = 0, x) = \theta(x)$.

With such an initial condition, the solution of the FKPP equation tends to a so-called “traveling wave.” The position of a FKPP traveling wave, which is related to the average position of the rightmost particle in the BRW, is known in the large-time limit:

$$\begin{aligned} [\text{FKPP front position}] &= 2t - \frac{3}{2} \ln t + \text{const} + \frac{C_X}{\sqrt{t}} + \dots \\ \text{with } C_X &= -3\sqrt{\pi}, \end{aligned} \quad (3)$$

where the last term was found by Ebert and Van Saarloos [10]. (The additive constant depends on the way one defines the position of the front. It is uninteresting for our purpose.) Note that the Ebert–Van Saarloos term is a decreasing but *positive* contribution to the front velocity. Equation (3) may easily be extended to different branching diffusion models by appropriately replacing some numerical constants (see below, Sec. V).

More generally, if $N(t)$ is the number of particles at time t , and $\{x_i(t)\}$ is the set of their positions in a given realization, then

$$G_t(x) \equiv \left\langle \prod_{i=1}^{N(t)} f[x - x_i(t)] \right\rangle \quad (4)$$

for any given function f satisfies the same FKPP equation as Q , the initial condition being the function $f(x)$ itself in the case of a BRW starting with one single particle at the origin. If f is a monotonous function of x such that $f(x) \xrightarrow{x \rightarrow -\infty} 0$

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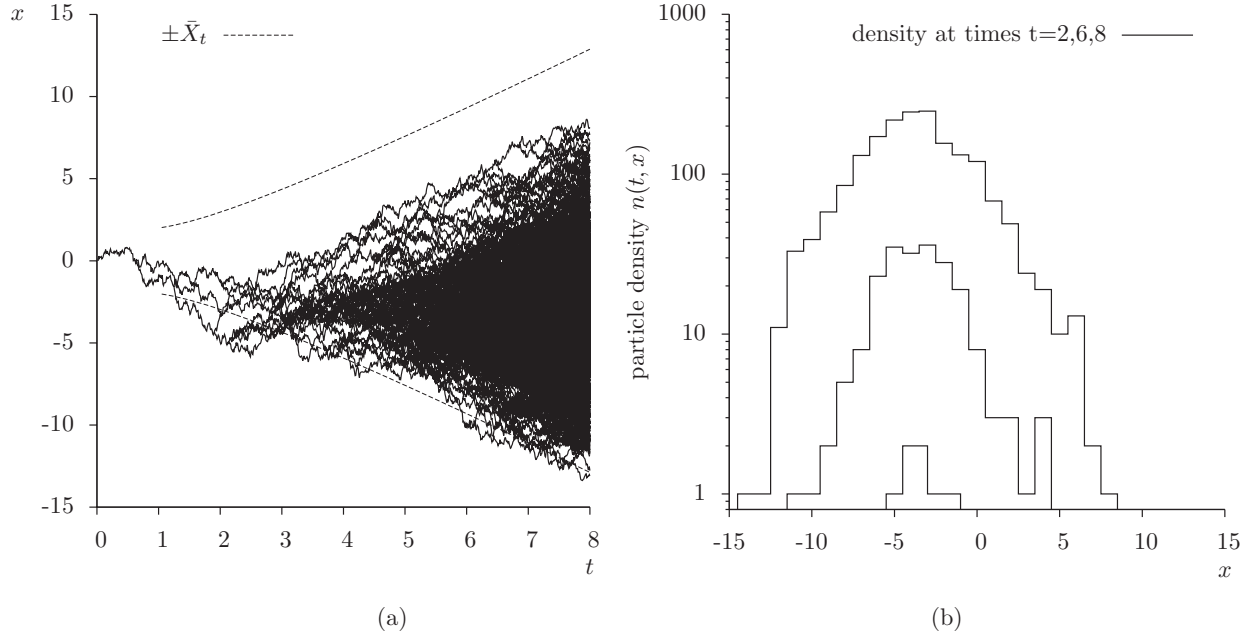


FIG. 1. (a) One realization of the continuous BRW up to time $t = 8$. To guide the eye, we also plot the theoretical (truncated) mean position of the boundaries of the BRW, namely $\pm\bar{X}_t = \pm 2t \mp \frac{3}{2} \ln t$ (dashed lines). (b) Distribution of the particles at times $t = 2, 6$, and 8 for this particular realization in bins of size 1 (log₁₀ scale on the vertical axis). We see the bulk building up a smoother (more “deterministic”) distribution as time elapses, while the low-density tails remain noisy. Also, for this realization, the distribution is skewed toward negative values of x , due to an accidentally large drift in the initial stages, whose memory is kept throughout the evolution.

and $f(x) \xrightarrow{x \rightarrow +\infty} 1$, and if f reaches 1 fast enough, namely $1 - f(x) \underset{x \rightarrow +\infty}{\sim} e^{-\gamma x}$ with $\gamma > 1$, then the traveling wave solution holds and the front position is still given by Eq. (3). Another interesting particular case is the “critical case” when f is such that $\gamma = 1$ exactly. Then,

$$[\text{FKPP front position}]_{\text{critical}} = 2t - \frac{1}{2} \ln t + \text{const} + \frac{C_Y}{\sqrt{t}} + \dots, \quad (5)$$

where C_Y is a constant that we shall determine later on (see Sec. IV).

There also exists a theorem established by Lalley and Sellke [11] that gives the asymptotic (large time) shape of the distribution of the position of the rightmost particle in a frame whose origin is at position $\ln Z$, where Z is some random variable that depends on the realization and may be thought of as a characterization of the position of the bulk of the particles in the BRW. (Its precise definition will be given later on.) More recently [12–14], the distribution of the distances between the foremost particles was derived with the help of the solution to the FKPP equation with some peculiar initial condition.

In this paper, we propose a phenomenological picture of the fluctuations in BRW, and we derive within this picture some statistical properties of a random variable similar to $\ln Z$. (Appendix B also lists some properties of $\ln Z$ itself.)

In Sec. II, we shall introduce our phenomenological picture for branching random walks. Section III is devoted to deriving the quantitative predictions of this model for a particular random variable that can characterize the early-time fluctuations of the branching random walk. The computation of a few free constant parameters requires us to solve deterministic

equations: This is explained in Sec. IV. Numerical checks are in order since our analytical results are based on conjectures: This is done in Sec. V. In light of our phenomenological model, we shall then come back to the discussion of the Ebert–Van Saarloos result on the $1/\sqrt{t}$ correction to the position of FKPP fronts (Sec. VI). Conclusions are given in Sec. VII.

II. PHENOMENOLOGICAL DESCRIPTION OF BRANCHING RANDOM WALKS

A. Picture

The picture of the fluctuations in branching random walks (BRWs) that we have in mind is the following. There are essentially three types of fluctuations that may affect the position of the front or of the foremost particle.

(i) First, there are fluctuations occurring at very early times ($t \sim 1$), when the system consists in a few particles. They have a large (of order 1) and lasting impact on the position of the front or of the rightmost particle. The main effect is given by the random waiting time of the first particle before it splits into two particles, during which it diffuses, but the subsequent waiting times of the latter two particles also contribute, etc., until the system contains a large enough number of particles that makes it partly “deterministic.” We do not believe that there is a simple way to compute the effect of these fluctuations, since there is no large parameter in the problem that would allow for sensible approximations.

(ii) Once the system contains many particles, which happens, say, at time $t'_0 \gg 1$, it enters a “mean-field” regime: In a first approximation, its particle density obeys a deterministic evolution with a moving absorptive boundary at a position that we shall call \bar{X}_t (and symmetrically at $-\bar{X}_t$), set in such a way

that the particle density is 1 at some fixed distance of order 1 to the left or to the right of this boundary, respectively. These boundaries simulate the discreteness of the particles. This is the Brunet-Derrida cutoff [15], and it was shown to correctly represent the leading effect of the noise on the position of the front in the context of the stochastic FKPP equation.

From now on, we shall focus on the right boundary. (The right and the left halves of the BRW essentially decouple once the system has grown large enough.) The large-time expression of the shape of the particle density near the right boundary reads, in such a model,

$$\psi_{\bar{X}_t}(x,t) = [\alpha(x - \bar{X}_t) + \beta] \exp\left(\bar{X}_t - x - \frac{(\bar{X}_t - x)^2}{4t}\right) \times \theta(\bar{X}_t - x) \quad (6)$$

in the region $1 \ll \bar{X}_t - x \lesssim \sqrt{t}$, where

$$\bar{X}_t = 2t - \frac{3}{2} \ln t + \frac{C_{\bar{X}}}{\sqrt{t}} \quad (7)$$

is, up to an uninteresting nonuniversal additive constant, the position of the tip of the front, namely of the right boundary. The Heaviside step function θ enforces the fact that the particle density is 0 to the right of \bar{X}_t . $C_{\bar{X}}$, $\alpha < 0$, and β are constants undetermined at this stage. ψ is essentially a decreasing exponential supplemented with a Gaussian and a linear prefactor. The t dependence enters explicitly as the width of the Gaussian, and implicitly through the position \bar{X}_t of the absorptive boundary. (There are corrections to the shape of the front at order $1/\sqrt{t}$, namely to the function $\psi_{\bar{X}}$ itself, but it turns out that we do not need to take them into account in our model, except for the determination of one overall numerical constant: We will come back to the derivation of Eq. (6) and of its corrections in Sec. IV.)

The fluctuations on top of this essentially deterministic front we have just described must take place in the tip region, where the particle density is low. We shall assume that a single fluctuation effectively gives the dominant correction to the deterministic evolution, and that the distribution $p(\delta)$ of the position δ of this fluctuation with respect to the tip of the front is exponential:

$$p(\delta) = C_1 e^{-\delta}. \quad (8)$$

We have found (see below) that these fluctuations bring a contribution of order $1/\sqrt{t}$ to the average position both of the front and of the rightmost particle in the BRW.

(iii) Finally, there are tip fluctuations occurring at very late times, say between $t - \bar{t}_0$ and t , where \bar{t}_0 is of order 1. They are also distributed as $e^{-\delta}$. They obviously add noise to the position of the tip of the front, but they do not have an effect on the bulk of the particle distribution since they do not have time to develop their own front at time t .

This picture is parallel to the phenomenological model for front fluctuations proposed in Ref. [16] in the context of the stochastic FKPP problem.

B. Variables

To arrive at quantitative predictions for the behavior of the BRW, we need to introduce random variables that characterize the realizations. We shall discuss the following ones:

(i) X_t , the position of the rightmost particle.

(ii) $Y_t = \sqrt{t} \sum_i e^{x_i(t)-2t}$, where the sum goes over all the particles in the system.

(iii) $Z_t = \sum_i [2t - x_i(t)] e^{x_i(t)-2t}$.

Throughout, we shall denote by $\langle A \rangle$ the statistical average (over realizations) of a given variable A in the full stochastic model, and by \bar{A} the value of this variable in a mean-field approximation of the same model with a discreteness cutoff at the tip. [These notations have already been used in Eq. (1) and Eqs. (6) and (7), respectively.] Discrete sums over the particles will often be replaced by integrals wherever the particle density is large enough.

Let us briefly comment on the random variables we have just introduced.

(i) As already mentioned, $\langle X_t \rangle$ is related to the solution of the FKPP equation with the step function as an initial condition.

(ii) The average $\langle \ln Y_t \rangle$ tends to a constant at large t . In addition, in any given event, the random variable $\ln Y_t$ itself tends to a constant, which has some distribution (which we do not know how to compute) that may be used to characterize the early-time fluctuations. Note that an appropriate generating function of the moments of

$$\tilde{Y}_t \equiv \sum_i e^{x_i(t)} = Y_t \times \frac{e^{2t}}{\sqrt{t}} \quad (9)$$

also obeys the FKPP equation, but with the ‘‘critical’’ initial condition discussed in the Introduction. We will come back to the latter fact in Sec. IV. Also, in the context of directed polymers in random media, \tilde{Y}_t is the partition function and $\langle \ln \tilde{Y}_t \rangle$ is the free energy averaged over the disorder [3].

(iii) Z_t is the variable used by Lalley and Sellke in the theorem alluded to in the Introduction. However, we are not going to focus on this variable in the body of this paper, since we found it has many drawbacks for our purpose. First, a practical drawback: Although Z_t tends almost surely to a positive constant when $t \rightarrow +\infty$ [11], it takes negative values at finite times, with finite probability; $\ln Z_t$ is then undefined in these particular realizations. Second, a theoretical drawback: It turns out that the finite-time corrections to the moments of $\ln Z_t$ are very sensitive to the initial fluctuations, the ones that are not computable analytically. We shall nevertheless quote a few results on the distribution of $\ln Z_t$ in the Appendix B.

In some intuitive sense, $\ln Y_t$ and $\ln Z_t$ characterize the position of the ‘‘front’’ of a particular realization of the evolution at time t .

The variables $\ln Y_t$ and $\ln Z_t$ keep the memory of the initial fluctuations. Therefore, we shall not attempt to compute the distribution of the fluctuations in $\ln Y$ accumulated over the whole history of the BRW, but instead the fluctuations of this variable between two large times t_0 and t , in order to have a quantity that is independent of the very early times at which there is no mean-field regime.

III. STATISTICS OF $f \equiv \ln Y_t - \ln Y_{t_0}$ IN THE PHENOMENOLOGICAL PICTURE

Here, starting from the phenomenological model defined in Sec. II, we shall deduce results on the distribution $p(f)$ of the variable $f \equiv \ln Y_t - \ln Y_{t_0}$ (and on its moments) for

$t, t_0, t - t_0 \gg 1$, up to one single constant for the moments of order larger than 2, and up to an additional constant for the first moment. Throughout, we shall aim at the accuracy $O(1/\sqrt{t_0}, 1/\sqrt{t}, 1/\sqrt{t-t_0})$ for $p(f)$ and neglect higher powers of these expansion variables.

A. Effect of a fluctuation on $\ln Y$

We first compute \bar{Y}_t , namely the variable Y_t in the mean-field approximation with the cutoff in the tail. Using the definition of the variable Y_t in Sec. II B and using Eqs. (6) and (7), we find

$$\bar{Y}_t = \sqrt{t} \int_{-\infty}^{+\infty} dx \psi_{\bar{X}_t}(x, t) e^{x-2t} = -2\alpha \left(1 + \frac{C_{\bar{Y}}}{\sqrt{t}} \right), \quad (10)$$

at first order in $1/\sqrt{t}$. $C_{\bar{Y}}$ is a constant literally equal to $C_{\bar{X}} - \frac{\beta\sqrt{\pi}}{2\alpha}$ in this calculation, but there would also be other contributions to $C_{\bar{Y}}$ that we cannot get from the large- t asymptotic shape of the front exhibited in Eq. (6). We shall postpone the full calculation of $C_{\bar{Y}}$ to Sec. IV.

It turns out that the term of order $1/\sqrt{t}$ in Eq. (10) generates $O(1/t)$ contributions to the distributions and to the moments that we shall address. Hence it is enough for our purpose to keep no more than the constant term, namely we write

$$\bar{Y}_t \simeq -2\alpha. \quad (11)$$

We perform a more complete calculation in Appendix A, keeping the subleading terms, in order to demonstrate that this *a priori* approximation is indeed accurate enough.

Let us consider a fluctuation occurring at time $t_1 \gg 1$ at a distance δ from the tip of the deterministic front. From Eq. (7), at time $t > t_1$ such that $t - t_1 \gg 1$, this fluctuation has developed its own front whose tip sits at position

$$\bar{X}_{\delta, t} = \bar{X}_{t_1} + \delta + \bar{X}_{t-t_1} = \bar{X}_t + \delta - \frac{3}{2} \ln \frac{t_1(t-t_1)}{t}. \quad (12)$$

There would of course be terms proportional to $1/\sqrt{t_1}$, $1/\sqrt{t}$, and $1/\sqrt{t-t_1}$ also here, but again we anticipate that they would eventually lead to corrections of higher order to the quantities of interest. We refer the reader to Appendix A for the details.

The shape of the front generated by this fluctuation will eventually have the form $C \times \psi_{\bar{X}_{\delta, t}}(x, t - t_1)$, where C is a constant that we cannot determine since it is related to some ‘‘average’’ shape of the fluctuation. With this extra fluctuation, Y_t has the following expression:

$$Y_t = \sqrt{t} \int_{-\infty}^{+\infty} dx \psi_{\bar{X}_t}(x, t) e^{x-2t} + C\sqrt{t} \int_{-\infty}^{+\infty} dx \psi_{\bar{X}_{\delta, t}}(x, t - t_1) e^{x-2t}. \quad (13)$$

The first term is just \bar{Y}_t : We replace it by Eq. (11). The second term is integrated in the same way as the first one, using the expression (12) for $\bar{X}_{\delta, t}$. We find

$$Y_t = -2\alpha \left(1 + C \frac{e^\delta}{t_1^{3/2}} \sqrt{\frac{t}{t-t_1}} \right). \quad (14)$$

Thus the forward shift in $\ln Y_t$ induced at time t by such a fluctuation occurring at time t_1 reads

$$\delta \ln Y_t = \ln Y_t - \ln \bar{Y}_t = \ln \left(1 + C \frac{e^\delta}{t_1^{3/2}} \sqrt{\frac{t}{t-t_1}} \right). \quad (15)$$

Note that in the asymptotic limit of interest, at first glance, the nontrivial term in this expression seems to be of order $1/t_1^{3/2}$, thus, if $t_1 \sim t_0$, it is smaller than our accuracy goal. However, it is enhanced by the e^δ factor, which turns out to be large.

B. Probability distribution and moments

We may convert the conjectured probability of a forward fluctuation of size δ [Eq. (8)] into the probability distribution of the difference of $\ln Y$ between two times t_0 and $t > t_0$ by simple changes of variables. We first discuss the variable

$$\delta f \equiv \delta \ln Y_t - \delta \ln Y_{t_0}. \quad (16)$$

The fundamental observation is that a fluctuation may essentially have two opposite effects on $\delta f \equiv \delta \ln Y_t - \delta \ln Y_{t_0}$. If it occurs after time t_0 , then it gives a positive contribution. If instead it occurs before t_0 , it generates a negative δf . Now we observe that the difference between δf and f reads $\ln \bar{Y}_t / \bar{Y}_{t_0}$, which is of order $1/\sqrt{t}, 1/\sqrt{t_0}$, and thus we may trade δf for f (see Appendix A for more details).

Let us first address the case in which the fluctuation occurs between t_0 and t . Using Eq. (15) together with the distribution (8), the probability that the size of the shift in δf induced by a fluctuation at time t_1 is less than some F reads

$$P(f < F; t_1) = C_1 \left(1 - \frac{C}{t_1^{3/2}} \sqrt{\frac{t}{t-t_1}} \frac{e^{-F}}{1-e^{-F}} \right). \quad (17)$$

We shall always assume that F is finite, and the ordering $t, t_1, t - t_1 \gg 1$. The probability distribution of f then reads

$$p(f; t_1) = \frac{\partial P(f < F; t_1)}{\partial F} \Big|_{F=f} = \frac{CC_1}{t_1^{3/2}} \sqrt{\frac{t}{t-t_1}} \frac{e^{-f}}{(1-e^{-f})^2}. \quad (18)$$

The rate of the fluctuations is assumed constant in time, thus the distribution of f results from a simple integration over t_1 from t_0 to t with uniform measure. It reads

$$p(f) = 2CC_1 \sqrt{\frac{1}{t_0} - \frac{1}{t}} \frac{e^{-f}}{(1-e^{-f})^2} \quad \text{for } f > 0. \quad (19)$$

Exactly in the same way, we may compute the probability distribution of f when the fluctuation occurs at a time smaller than t_0 . In this case, the effect on f of a fluctuation of size δ reads

$$f = \ln \frac{1 + C \frac{e^\delta}{t_1^{3/2}} \sqrt{\frac{t}{t-t_1}}}{1 + C \frac{e^\delta}{t_1^{3/2}} \sqrt{\frac{t_0}{t_0-t_1}}}. \quad (20)$$

Using the same method, we find

$$p(f) = 2CC_1 \sqrt{\frac{1}{t_0} - \frac{1}{t}} \frac{e^f}{(1 - e^f)^2} \times \left(1 - \sqrt{\frac{1 - e^f}{1 + e^f}} \right) \quad \text{for } f < 0. \quad (21)$$

The integral over t_1 which has to be performed to arrive at these expressions is dominated by values of t_1 of the order of t_0 . This helps to understand *a posteriori* why we were allowed to drop terms of order $1/\sqrt{t}$ and $1/\sqrt{t_1}$ in Eqs. (10) and (12), respectively, although we were aiming at such accuracy for $p(f)$.

The probability distribution given in Eqs. (19) and (21) cannot be normalized, and the first moment $\langle f \rangle$ is also divergent. We shall compute the latter separately in the next section.

An analytic continuation of the generating function for the moments of f can be obtained from Eqs. (19) and (21) by a direct calculation. We get

$$\langle e^{\nu f} \rangle = 2CC_1 \sqrt{\frac{1}{t_0} - \frac{1}{t}} \left\{ -\nu \psi(-\nu) + \nu \psi(\nu) + \sqrt{\pi} \left[\frac{\Gamma(\frac{1}{2} + \frac{\nu}{2})}{\Gamma(\frac{\nu}{2})} + \frac{\Gamma(1 + \frac{\nu}{2})}{\Gamma(\frac{1}{2} + \frac{\nu}{2})} \right] \right\}, \quad (22)$$

keeping in mind that this formula can be used only for moments of second order or higher.

The analytical structure of Eq. (22) is particularly simple. There are poles on the positive real ν axis, fully contained in the first term $-\nu \psi(-\nu)$: They correspond to positive values of f . All the other poles, contained in the remaining terms, are located on the real negative axis and correspond to negative values of f .

A comment is in order on the conjectured probability distribution (8) of the tip fluctuations that we used in the above derivation. Actually, we omitted a time-dependent Gaussian factor of the form $e^{-\delta^2/(4t_1)}$, which would cut off the exponential distribution of δ at a distance $2\sqrt{t_1}$ ahead of the tip of the front, and thus modify the distribution of f for large positive f . However, numerically, we do not find evidence for such a modification: It seems that Eq. (19) has a more general validity. We do not have a good explanation for this surprising fact in the context of our phenomenological model for fluctuations. But it turns out that a different calculation of the positive f fluctuations outlined in Appendix C does not have such limitations.

C. Correction to the first moment of f due to the fluctuations

Since it is not possible to use Eq. (22) to get the first moment of f , we shall arrive at its expression through a direct calculation. We must take into account the expansion (keeping terms at least as large as $1/\sqrt{t}$, $1/\sqrt{t_0}$, $1/\sqrt{t-t_0}$) of the density of particles in the deterministic limit with a discreteness cutoff, and, in addition, the effect of the fluctuations which intermittently speed up the evolution. We have already guessed that there is an $O(1/\sqrt{t})$ contribution to the deterministic evolution [see Eq. (10)], but a full calculation

will eventually be needed. Here, we shall simply denote by $C_{\bar{Y}}$ its coefficient.

The average of $f = \ln Y_t - \ln Y_{t_0}$ over realizations has thus a mean-field contribution, and a contribution from the fluctuations which in turn can be decomposed in positive and negative contributions μ_1^+ and μ_1^- , respectively. We shall evaluate the latter in this section.

We write

$$\mu_1 = \langle \ln Y_t - \ln Y_{t_0} \rangle = C_{\bar{Y}} \left(\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{t_0}} \right) + \mu_1^+ - \mu_1^-. \quad (23)$$

Using Eqs. (8) and (15), we get the expression

$$\mu_1^+ = \int_{t'_0}^t dt_1 \int_0^{+\infty} d\delta C_1 e^{-\delta} \ln \left(1 + C \frac{e^\delta}{t_1^{3/2}} \sqrt{\frac{t}{t-t_1}} \right) \quad (24)$$

for the positive part of the contribution at t of the fluctuations, and

$$\mu_1^- = \int_{t'_0}^{t_0} dt_1 \int_0^{+\infty} d\delta C_1 e^{-\delta} \ln \left(1 + C \frac{e^\delta}{t_1^{3/2}} \sqrt{\frac{t_0}{t_0-t_1}} \right) \quad (25)$$

subtracts the effect at t_0 of the fluctuations occurring at $t_1 < t_0$. We have introduced a time t'_0 of order 1 as a lower bound in these integrals in order to make these expressions finite. The physical meaning of this cutoff is clear: Before t'_0 , there is no mean-field regime because the whole system consists in a few particles only.

Let us start with the computation of μ_1^+ . It is useful to perform the change of variables

$$\lambda = \frac{t_1}{t}, \quad u_\delta = e^{-\delta} \frac{t^{3/2}}{C} \lambda^{3/2} \sqrt{1-\lambda}, \quad (26)$$

which leads to the following expression of μ_1^+ :

$$\mu_1^+ = \frac{CC_1}{\sqrt{t}} \int_{t'_0/t}^1 \frac{d\lambda}{\lambda^{3/2}} \frac{1}{\sqrt{1-\lambda}} \int_0^{u_0(\lambda)} du_\delta \ln \left(1 + \frac{1}{u_\delta} \right), \quad (27)$$

where $u_0(\lambda) = \frac{t^{3/2}}{C} \lambda^{3/2} \sqrt{1-\lambda}$. u_0 is large compared to 1, except when $1-\lambda$ is of order $1/t^3$. But the contribution of the region $[1-1/t^3, 1]$ in the λ integration is smaller than $\sim 1/t^{3/2}$, and hence negligible. So we may always assume $u_0 \gg 1$.

The integral over u_δ is performed analytically, and the large- u_0 limit may eventually be taken:

$$\int_0^{u_0} du_\delta \ln \left(1 + \frac{1}{u_\delta} \right) = (1 + u_0) \ln(1 + u_0) - u_0 \ln u_0 \underset{u_0 \gg 1}{\sim} \ln u_0. \quad (28)$$

The remainder reads

$$\mu_1^+ = \frac{CC_1}{\sqrt{t}} \left[\ln \left(\frac{t^{3/2}}{C} \right) \mathcal{I}_0 + \mathcal{I}_1 \right], \quad (29)$$

where

$$\begin{aligned} \mathcal{I}_0 &= \int_{t'_0/t}^1 d\lambda \lambda^{-3/2} (1-\lambda)^{-1/2}, \\ \mathcal{I}_1 &= \int_{t'_0/t}^1 d\lambda \lambda^{-3/2} (1-\lambda)^{-1/2} \ln(\lambda^{3/2} \sqrt{1-\lambda}). \end{aligned} \quad (30)$$

\mathcal{I}_0 and \mathcal{I}_1 can be performed with the help of the change of variable $\lambda = \sin^2 \theta$:

$$\mathcal{I}_0 = 2 \int_{\arcsin \sqrt{\frac{t'_0}{t}}}^{\frac{\pi}{2}} \frac{d\theta}{\sin^2 \theta} = 2 \cot \left(\arcsin \sqrt{\frac{t'_0}{t}} \right) = 2 \sqrt{\frac{t}{t'_0} - 1}, \quad (31)$$

while for \mathcal{I}_1 , a further integration by parts is needed to get rid of the log. We eventually arrive at the following exact expression for (29):

$$\mu_1^+ = 2CC_1 \left\{ \sqrt{\frac{1}{t'_0} - \frac{1}{t}} \left[\ln \left(\frac{t_0^{3/2}}{C} \sqrt{1 - \frac{t'_0}{t}} \right) + 3 \right] - \frac{4}{\sqrt{t}} \arccos \sqrt{\frac{t'_0}{t}} \right\}. \quad (32)$$

The term μ_1^- is the same as the term μ_1^+ except for the replacement $t \rightarrow t_0$.

Since we are neglecting terms of relative order t_0/t , t'_0/t_0 , and higher, we may expand the expressions for μ_1^+ and μ_1^- . The difference $\mu_1^+ - \mu_1^-$ then reads

$$\mu_1^+ - \mu_1^- = 4\pi CC_1 \left(\frac{1}{\sqrt{t_0}} - \frac{1}{\sqrt{t}} \right). \quad (33)$$

Equation (23) eventually leads to the following expression for μ_1 :

$$\mu_1 = (4\pi CC_1 - C_{\bar{Y}}) \left(\frac{1}{\sqrt{t_0}} - \frac{1}{\sqrt{t}} \right). \quad (34)$$

We note a very important property of this result: It does not depend on t'_0 . If it did, then we would lose predictivity because t'_0 is the arbitrary time after which we declare that the fluctuations are small enough for our calculation to apply. (This would not be true at the next order in $1/\sqrt{t}$, $1/\sqrt{t_0}$.)

IV. DETERMINISTIC CALCULATIONS

In this section, we first review the Ebert–Van Saarloos method [10] to compute the order $1/\sqrt{t}$ correction to the mean position of the rightmost particle in the BRW $\langle X_t \rangle$. We extend the method to the position of the right boundary in the deterministic model with discreteness cutoffs \bar{X}_t , and eventually adapt it to $\langle \ln Y_t \rangle$.

The calculations presented here will enable us to determine the remaining unknown constants, namely $C_{\bar{X}}$ [see Eq. (7)], $C_{\bar{Y}}$ [Eq. (10)], and CC_1 . The latter two constants appear in particular in Eqs. (19), (21), (22), and (34).

A. Ebert–Van Saarloos calculation and its extension

The original calculation of Ebert and Van Saarloos aimed at finding properties of the solutions to the FKPP equation,

$$\partial_t \phi(t, x) = \partial_x^2 \phi(t, x) + \phi(t, x) - \phi^2(t, x) \quad (35)$$

for $\phi \ll 1$, with a steep enough initial condition, e.g., $\phi(t=0, x) = \theta(-x)$. This equation is actually the same as Eq. (2), with the correspondence $\phi(t, x) = 1 - Q(t, x)$. The nonlinearity can essentially be viewed as a moving absorptive boundary on a linear partial differential equation, the position

of the boundary being set in such a way that ϕ has a maximum at a fixed height.

To determine the value of the constants $C_{\bar{X}}$ and $C_{\bar{Y}}$, we need to address a branching diffusion equation with a nonlinearity that forces ϕ to go to zero over a distance of order 1 at the right of the point at which $\phi(t, x) = 1$, and which therefore acts as a tip cutoff. In terms of a smooth equation, we may write, for example,

$$\partial_t \phi(t, x) = \partial_x^2 \phi(t, x) + \frac{\phi^2(t, x)}{1 + \phi(t, x)} \quad (36)$$

and study the properties of the solutions to this equation in the region $\phi \gg 1$.

In both cases, the equation can be linearized in the respective domain of interest, and one gets

$$\partial_t \phi(t, x) = \partial_x^2 \phi(t, x) + \phi(t, x). \quad (37)$$

We shall assume that the nonlinear term is equivalent to a right-moving absorptive boundary at the accuracy at which we want to address the problem. (This assumption was better motivated by Ebert and Van Saarloos in their discussion of what they call the ‘‘interior expansion’’ [10]). In the first case, we study the function ϕ to the right of the boundary; in the second case, we study the function to the left.

1. Solution to the linearized equation with an appropriate boundary condition

Near the boundary, at large times, the function ϕ reads

$$\phi(t, x) \sim (\alpha \xi + \beta) e^{-\xi}, \quad (38)$$

where $\xi = x - (\text{position of the cutoff}) \sim x - 2t + \frac{3}{2} \ln t + \dots$ and this is valid for $1 \ll \xi \ll \sqrt{t}$. According to Ebert–Van Saarloos [10], the large- t corrections to this shape are of the form $1/t$ (there is no term of order $1/\sqrt{t}$). All these features should not depend on whether we address Eq. (35) or Eq. (36) above, except for the signs of α and ξ .

We write

$$\phi(t, x) = e^{-\xi - z} g(t, z), \quad (39)$$

where $z = \frac{\xi^2}{4t}$, and the following *Ansätze* are taken:

$$\xi = x - 2t + \frac{3}{2} \ln t + \frac{2c}{\sqrt{t}} + \dots, \quad (40)$$

$$g(t, z) = \sqrt{t} g_{-\frac{1}{2}}(z) + g_0(z) + \dots$$

The variable ξ may be positive in the linear domain (this is the case for the usual Ebert–Van Saarloos solution) or negative: Therefore, we write $\xi = \varepsilon \sqrt{4t} z$, where $\varepsilon = \pm 1$. The *Ansatz* for the front position contained in ξ already incorporates the two known [5] dominant terms at large t , namely $2t - \frac{3}{2} \ln t$. The $-\frac{2c}{\sqrt{t}}$ term was new in Ref. [10].

Thanks to these *Ansätze*, the original equation splits into a hierarchy of equations for the functions g . The first two equations of this set read

$$\begin{aligned} z g''_{-\frac{1}{2}} + \left(\frac{1}{2} - z\right) g'_{-\frac{1}{2}} + \frac{1}{2} g_{-\frac{1}{2}} &= 0, \\ z g''_0 + \left(\frac{1}{2} - z\right) g'_0 + g_0 &= c g_{-\frac{1}{2}} - \varepsilon^3 \frac{3}{2} \sqrt{z} (g_{-\frac{1}{2}} - g'_{-\frac{1}{2}}). \end{aligned} \quad (41)$$

The first equation of the hierarchy is the Kummer equation

$$z \frac{d^2 w}{dz^2} + (b - z) \frac{dw}{dz} - aw = 0 \quad (42)$$

with $w = g_{-\frac{1}{2}}$, $a = -\frac{1}{2}$ and $b = \frac{1}{2}$. Two independent solutions are, for example, the two Kummer functions (or ${}_1F_1$ hypergeometric functions)

$$M(a, b, z) \text{ and } z^{1-b} M(a - b + 1, 2 - b, z), \quad (43)$$

namely, in our case,

$$M\left(-\frac{1}{2}, \frac{1}{2}, z\right) \text{ and } \sqrt{z} M\left(0, \frac{3}{2}, z\right). \quad (44)$$

The latter is just the elementary function \sqrt{z} , while the former diverges like $-e^z/(2z)$ for large z , and thus has to be discarded. Hence the solution reads

$$g_{-\frac{1}{2}} = 2\alpha' \sqrt{z} \quad (45)$$

where α' is a constant, arbitrary at this stage.

As for the second equation in Eqs. (41), whose solution is the function g_0 , it is an inhomogeneous Kummer differential equation. A basis for the solutions of the homogeneous part is, for example, the set of the two functions

$$M\left(-1, \frac{1}{2}, z\right) = 1 - 2z \text{ and } \sqrt{z} M\left(-\frac{1}{2}, \frac{3}{2}, z\right). \quad (46)$$

We need to find a particular solution of the full equation. We define $y \equiv \sqrt{z}$; the equation for g_0 then reads

$$\frac{d^2 g_0}{dy^2} - 2y \frac{dg_0}{dy} + 4g_0 = 8\alpha' \left(-\frac{3}{2} \varepsilon y^2 + cy + \frac{3}{4} \varepsilon \right) \quad (47)$$

and we may look for solutions in terms of a series:

$$g_0(y) = \sum_{k=0}^{+\infty} a_k y^k. \quad (48)$$

Inserting this expression into the differential equation (47), we get the following relations between the coefficients of the series:

$$a_{k+2} = \frac{2(k-2)}{(k+1)(k+2)} a_k \text{ for } k \geq 3, \quad (49)$$

$$a_2 = -2a_0 + 3\alpha' \varepsilon, \quad a_3 = -\frac{a_1}{3} + \frac{4\alpha' c}{3}, \quad a_4 = -\alpha' \varepsilon.$$

The free parameters are a_0 and a_1 . We may choose them in such a way that $a_{2,3} = 0$: We therefore set $a_0 = \frac{3}{2} \alpha' \varepsilon$ and $a_1 = 4\alpha' c$. Then

$$a_{2n} = -\frac{3}{2} \sqrt{\pi} \alpha' \varepsilon \frac{\Gamma(n-1)}{\Gamma(n+1)\Gamma(n+1/2)} \text{ for } n \geq 2, \quad (50)$$

$$a_2 = 0, \text{ and } a_{2n+1} = 0 \text{ for } n \geq 1,$$

where we used the duplication formula $\Gamma(2n+1) = \frac{2^{2n}}{\sqrt{\pi}} \Gamma(n + \frac{1}{2}) \Gamma(n+1)$. Switching back to the variable z , the final expression for the particular solution reads

$$g_0^{\text{sp}}(z) = \frac{3}{2} \alpha' \varepsilon + 4\alpha' c \sqrt{z} - \frac{3\alpha' \varepsilon}{2} F_2(z), \quad (51)$$

$$\text{where } F_2(z) = \sqrt{\pi} \sum_{n=2}^{\infty} \frac{\Gamma(n-1)}{\Gamma(n+1/2)\Gamma(n+1)} z^n,$$

which, except for the sign factors ε , is the Ebert–Van Saarloos result [10]. Following again Ref. [10], we write the solution for g_0 as

$$g_0(z) = \frac{3}{2} \alpha' \varepsilon + 4\alpha' c \sqrt{z} - \frac{3\alpha' \varepsilon}{2} F_2(z) + k_0(1 - 2z) + l_0 \sqrt{z} M\left(-\frac{1}{2}, \frac{3}{2}, z\right), \quad (52)$$

and inserting (52) together with (45) into (40) and (38), we would get the expression of ϕ up to the constants α', c, k_0, l_0 . We are now going to determine them from a matching procedure.

2. Matching conditions

We now match with the shape of the so-called ‘‘interior’’ region at $z \ll 1$. This means that ϕ just obtained should have the same small- z expansion as the limiting form of ϕ in Eq. (38). Hence we need to impose

$$g_{-\frac{1}{2}}(z) \underset{z \ll 1}{\sim} 2\alpha \sqrt{z} \text{ and } g_0(z) \underset{z \ll 1}{\sim} \beta + O(z). \quad (53)$$

The first constraint is solved by setting $\alpha' = \alpha$. As for the second one, it means in particular that there should be no term proportional to \sqrt{z} in $g_0(z)$. This requirement leads to the equations

$$\frac{3\alpha \varepsilon}{2} + k_0 = \beta, \quad 4\alpha c + l_0 = 0. \quad (54)$$

Now we must also check the behavior at $z \rightarrow +\infty$. We need the expansion of the functions M and F_2 for $z \rightarrow \infty$. Let us start with M . We shall use the integral representation

$$M(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 du e^{zu} u^{a-1} (1-u)^{b-a-1}. \quad (55)$$

We change the variable for u to $1-u$ in the integral, and we expand the $(1-u)^{a-1}$ factor near $u=0$:

$$M(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} e^z \int_0^1 du e^{-zu} u^{b-a-1} \times \sum_{k=0}^{+\infty} \frac{\Gamma(1-a+k)}{\Gamma(1-a)\Gamma(1+k)} u^k. \quad (56)$$

We then notice that we may extend the integral to $+\infty$ without adding exponentially enhanced terms. Finally, we perform the remaining integration over u . The result reads

$$M(a, b, z) = e^z z^{a-b} \frac{\Gamma(b)}{\Gamma(a)} \sum_{k=0}^{+\infty} \frac{z^{-k}}{\Gamma(1+k)} \frac{\Gamma(1-a+k)\Gamma(b-a+k)}{\Gamma(1-a)\Gamma(b-a)} + o(e^z) = e^z z^{a-b} \frac{\Gamma(b)}{\Gamma(a)^2} F_0(1-a, b-a; 1/z) + o(e^z). \quad (57)$$

Setting $a = -1/2$ and $b = 3/2$, we write

$$\sqrt{z} M\left(-\frac{1}{2}, \frac{3}{2}, z\right) \sim -\frac{1}{4} e^z z^{-3/2} \sum_{k=0}^{+\infty} \frac{\Gamma(\frac{3}{2}+k)}{\Gamma(\frac{3}{2})} (1+k) z^{-k} = -\frac{1}{4} e^z z^{-3/2} {}_2F_0\left(\frac{3}{2}, 2; \frac{1}{z}\right). \quad (58)$$

We now turn to F_2 . We write the following integral representation:

$$\frac{1}{2}F_2(z) = \lim_{\eta \rightarrow 0} \left[\int_0^1 du e^{zu} u^{-2+\eta} \sqrt{1-u} - \frac{\sqrt{\pi}}{2} \frac{\Gamma(\eta-1)}{\Gamma(\eta+\frac{1}{2})} - \frac{\sqrt{\pi}}{2} \frac{\Gamma(\eta)}{\Gamma(\eta+\frac{3}{2})} z \right]. \quad (59)$$

This representation may be checked by expanding the exponential in the integral and performing the integration over u . For large z , the two rightmost terms do not play any role since they are not exponentially enhanced. We may now treat the first term exactly in the same way as in the case of the Kummer function M . After taking the $\eta \rightarrow 0$ limit, which is finite once all nonexponentially enhanced terms have been discarded, we get

$$F_2 \sim 2e^z z^{-3/2} \sum_{k=0}^{+\infty} \Gamma\left(k + \frac{3}{2}\right) (k+1) z^{-k}. \quad (60)$$

Up to an overall constant, all terms are identical to the ones in the expansion of the M function.

Requiring the cancellation of these exponentially enhanced terms in the expression (52) for g_0 leads to the equation

$$\frac{3}{2}\alpha\varepsilon\sqrt{\pi} + \frac{l_0}{4} = 0. \quad (61)$$

Using this equation and the second equation in (54), one determines the value of c :

$$c = \frac{3}{2}\varepsilon\sqrt{\pi}. \quad (62)$$

Hence this constant is *positive* for the Ebert–Van Saarloos solution of the FKPP equation, but *negative* when one computes the position of the tip of a front with a discreteness cutoff.

3. Matched solution

All in all, we get

$$\phi(t, x) = e^{-\xi-z} \left\{ \alpha\xi + \beta + (3\alpha\varepsilon - 2\beta)z + 6\alpha\varepsilon\sqrt{\pi z} \times \left[1 - M\left(-\frac{1}{2}, \frac{3}{2}, z\right) \right] - \frac{3\alpha\varepsilon}{2} F_2(z) \right\} \quad (63)$$

with

$$\xi = x - 2t + \frac{3}{2} \ln t + 3\varepsilon \sqrt{\frac{\pi}{t}}. \quad (64)$$

The first two terms in ϕ , namely $e^{-\xi-z}(\alpha\xi + \beta)$, give back Eq. (6). The next terms are finite-time corrections.

Identifying ξ with $x - \langle X_t \rangle$ and setting $\varepsilon = +1$, we recover the value of

$$C_X = -3\sqrt{\pi} \quad (65)$$

already derived by Ebert and Van Saarloos [see Eq. (3)]. With $\xi = x - \bar{X}_t$ and $\varepsilon = -1$, we read off this formula the value of the constant

$$C_{\bar{X}} = 3\sqrt{\pi}. \quad (66)$$

We can also deduce the value of $C_{\bar{Y}}$ by using the definition of the variable Y_t given in Sec. II B and the shape of the mean-field particle distribution (63):

$$\bar{Y}_t = \sqrt{t} \int_{-\infty}^{+\infty} dx \phi(t, x) e^{x-2t}, \quad (67)$$

which, after replacement by the expression (63) and setting $\varepsilon = -1$, becomes

$$\bar{Y}_t = e^{3\sqrt{\pi/t}} \left\{ -2\alpha - \frac{1}{\sqrt{t}} \int_0^{+\infty} \frac{dz}{\sqrt{z}} e^{-z} \left[\beta(1-2z) - 3\alpha z - 6\alpha\sqrt{\pi z} \left\{ 1 - M\left(-\frac{1}{2}, \frac{3}{2}, z\right) \right\} + \frac{3\alpha}{2} F_2(z) \right] \right\}. \quad (68)$$

The term proportional to β is zero after the integration, and the other terms give numerical constants. We finally find, at order $1/\sqrt{t}$,

$$\ln \bar{Y}_t = \ln(-2\alpha) + \frac{C_{\bar{Y}}}{\sqrt{t}} \quad \text{with} \quad C_{\bar{Y}} = \frac{3}{2}\sqrt{\pi}. \quad (69)$$

B. Solution of the deterministic FKPP equation with the critical initial condition

Let us consider a generating function of the moments of the \tilde{Y}_t variable:

$$G_t(x) = \langle e^{-\tilde{Y}_t e^{-x}} \rangle = \left\langle \prod_{i=1}^{N(t)} e^{-e^{-(x-x_i(t))}} \right\rangle. \quad (70)$$

Defining $f(x) = e^{-e^{-x}}$, $G_t(x)$ has exactly the form shown in Eq. (4) and thus $\phi(t, x) \equiv 1 - G_t(x)$ solves the FKPP equation (35), $\partial_t \phi = \partial_x^2 \phi + \phi - \phi^2$. If the initial condition for the underlying branching random walk is a single particle at position $x = 0$,

$$\phi(t = 0, x) = 1 - e^{-e^{-x}}, \quad (71)$$

and then the position of the FKPP traveling wave is given by Eq. (5). In this section, we shall address this case using the Ebert–Van Saarloos method in order to obtain the $1/\sqrt{t}$ correction to the latter and some analytic features of ϕ . Indeed, from the expression of ϕ , we may in principle compute the moments of $\ln \tilde{Y}_t$, using the identity

$$\langle \tilde{Y}_t^v \rangle = -\frac{1}{\Gamma(1-v)} \int_{-\infty}^{+\infty} dx e^{vx} \frac{\partial \phi(t, x)}{\partial x}. \quad (72)$$

1. General solution of the linearized equation in a moving frame

Following Ebert–Van Saarloos, we define

$$\xi = x - 2t - \chi_t \quad \text{and} \quad \phi(t, x) = e^{-\xi} \psi(t, \xi). \quad (73)$$

The linearized FKPP equation for ψ reads

$$\partial_t \psi(t, \xi) = \partial_\xi^2 \psi(t, \xi) + \dot{\chi}_t (\partial_\xi - 1) \psi(t, \xi). \quad (74)$$

Next, we take the *Ansatz* $\chi_t = -\frac{1}{2} \ln t - \frac{2c}{\sqrt{t}}$, and we introduce the variable $z = \frac{\xi^2}{4t}$. The function $g(t, z)$ is $\psi(t, \xi)$ expressed with the help of z , and we look for solutions in the form

$$g(t, z) = \sqrt{t} g_{-\frac{1}{2}}(z) + g_0(z). \quad (75)$$

We are led to the following hierarchical set of equations [compare to Eq. (41)]:

$$\begin{aligned} z g''_{-\frac{1}{2}}(z) + \left(z + \frac{1}{2}\right) g'_{-\frac{1}{2}}(z) &= 0, \\ z g''_0(z) + \left(z + \frac{1}{2}\right) g'_0(z) + \frac{1}{2} g_0(z) &= c g_{-\frac{1}{2}}(z) + \frac{1}{2} \sqrt{z} g'_{-\frac{1}{2}}(z). \end{aligned} \quad (76)$$

The solution reads

$$\begin{aligned} g_{-\frac{1}{2}}(z) &= b + a \sqrt{\pi} \operatorname{erf}(\sqrt{z}), \\ g_0(z) &= \left[\frac{c_1 \sqrt{\pi}}{2} e^{-z} \operatorname{erfi}(\sqrt{z}) + c_2 e^{-z} \right] \\ &\quad + 2c [b + a \sqrt{\pi} \operatorname{erf}(\sqrt{z})] + a z e^{-z} {}_2F_2\left(1, 1; \frac{3}{2}, 2; z\right), \end{aligned} \quad (77)$$

where erf , erfi are the error functions defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt e^{-t^2}, \quad \operatorname{erfi}(x) = -i \operatorname{erf}(ix), \quad (78)$$

and a, b, c_1, c_2 are integration constants to be determined. The terms in the first square brackets in Eq. (77) correspond to the general solution of the homogeneous equation for g_0 , while the next two terms represent a particular solution of the full equation, as can easily be checked.

2. Matching conditions

Because of the initial condition, the tail of the front at $\xi \rightarrow \infty$ has the exact shape

$$\phi(t, x \gg 2t \gg 1) = e^{-(x-2t)} \quad (79)$$

at any time. In particular, there is no overall constant. Comparing to Eqs. (73) and (75), this condition means that

$$g(t \rightarrow \infty, z \rightarrow \infty) = \sqrt{t} + 2c + O(1/\sqrt{t}). \quad (80)$$

Let us expand our solution (77) for $g(t, z)$ for large t, z :

$$\begin{aligned} g(t \rightarrow \infty, z \rightarrow \infty) &= \sqrt{t} [b + a \sqrt{\pi} + O(e^{-z})] + 2c(b + a \sqrt{\pi}) \\ &\quad + \frac{1}{2} (c_1 + a \sqrt{\pi}) \left(\frac{1}{\sqrt{z}} + \frac{1}{2z^{3/2}} + \dots \right) + O(e^{-z}). \end{aligned} \quad (81)$$

The identification with the expected asymptotic form leads to the conditions

$$b + a \sqrt{\pi} = 1, \quad 2c(b + a \sqrt{\pi}) = 2c. \quad (82)$$

The second condition is trivial once the first one is satisfied.

We also impose that all terms that are not exponentially suppressed cancel, which is realized by setting

$$c_1 + a \sqrt{\pi} = 0. \quad (83)$$

We turn to the limit $z \rightarrow 0$. The condition (38) reads, in terms of the g function,

$$g(t \rightarrow \infty, z \rightarrow 0) = \alpha \sqrt{tz} + \beta, \quad (84)$$

which, in particular, forbids constant terms and terms proportional to \sqrt{z} . Since the small- z expansion of our solution reads

$$\begin{aligned} g(t \rightarrow \infty, z \rightarrow 0) &= \sqrt{t} [b + 2a \sqrt{z} + O(z)] + 2cb + c_2 \\ &\quad + (4ac + c_1) \sqrt{z} + O(z), \end{aligned} \quad (85)$$

we see that b needs to be set to 0 and $c = -c_1/(4a)$.

Putting everything together, we find that all constraints are solved by the choice

$$a = \frac{1}{\sqrt{\pi}}, \quad b = 0, \quad c = \frac{\sqrt{\pi}}{4}, \quad c_1 = -1, \quad c_2 = \beta. \quad (86)$$

Note that the coefficient α in Eq. (84) is determined to be $\alpha = 2/\sqrt{\pi}$, while in the noncritical case, it is a free parameter.

3. Matched solution

All in all, our solution reads

$$\begin{aligned} \phi(t, x) &= e^{-\xi} \left[\sqrt{t} \operatorname{erf}(\sqrt{z}) + e^{-z} \left\{ \beta + \frac{z}{\sqrt{\pi}} {}_2F_2\left(1, 1; \frac{3}{2}, 2; z\right) \right. \right. \\ &\quad \left. \left. + \frac{\sqrt{\pi}}{2} [e^z \operatorname{erf}(\sqrt{z}) - \operatorname{erfi}(\sqrt{z})] \right\} \right], \end{aligned} \quad (87)$$

with

$$\xi = x - 2t + \frac{1}{2} \ln t + \frac{\sqrt{\pi}}{2} \frac{1}{\sqrt{t}}, \quad z = \frac{\xi^2}{4t}. \quad (88)$$

The $1/\sqrt{t}$ term is identical to the one in the ‘‘pushed front’’ calculation of Ref. [10] [see Appendix G, Eq. (G18) therein], although the front solution chosen in that work is different [see Eq. (G7)].

We can now deduce from this calculation the average value μ_1 of $\ln Y_t - \ln Y_{t_0}$ by expanding the exact formula Eq. (72) in powers of ν and keeping the coefficient of the term of order ν :

$$\begin{aligned} \langle \ln \tilde{Y}_t \rangle &= \langle \ln Y_t \rangle + 2t - \frac{1}{2} \ln t \\ &= -\psi(1) - \int_{-\infty}^{+\infty} dx x \frac{\partial}{\partial x} \phi(t, x). \end{aligned} \quad (89)$$

We find

$$\mu_1 = \langle \ln Y_t - \ln Y_{t_0} \rangle = \frac{\sqrt{\pi}}{2} \left(\frac{1}{\sqrt{t_0}} - \frac{1}{\sqrt{t}} \right). \quad (90)$$

Identifying the latter equation in Eq. (34) and taking into account the value of $C_{\bar{Y}}$ already computed in Eq. (69), we finally obtain a determination of CC_1 :

$$CC_1 = \frac{1}{2\sqrt{\pi}}. \quad (91)$$

V. COMPLETE RESULTS AND NUMERICAL CHECKS

Since the results we have obtained rely in an essential way on a model for fluctuations and hence on a set of conjectures, we need to check them with the help of numerical simulations in order to get confidence in the validity of our picture. In the first part of this section, we shall list the formulas we have obtained, but extending them to more general BRW models. Then, we define a model that is convenient for numerical

implementation in Sec. VB, and we test our results against numerical simulations of this particular model in Secs. VC and VD.

A. Parameter-free predictions for a general branching diffusion

We now extend our results to general branching diffusion kernels. In the continuous case, we write the equation for the average particle density as

$$\partial_t \langle n(t, x) \rangle = \chi(-\partial_x) \langle n(t, x) \rangle, \quad (92)$$

where $\chi(-\partial_x)$ is the operator that represents the branching diffusion. The eigenfunctions are the exponential functions $e^{-\gamma x}$, and the corresponding eigenvalues are $\chi(\gamma)$. In the case discussed in the previous sections, $\chi(-\partial_x) = \partial_x^2 + 1$ and $\chi(\gamma) = \gamma^2 + 1$. We introduce γ_0 , which solves $\chi'(\gamma_0) = \chi(\gamma_0)/\gamma_0$. Then in the case studied so far, $\gamma_0 = 1$ and $\chi(\gamma_0) = \chi'(\gamma_0) = \chi''(\gamma_0) = 2$.

We can also address the discrete time and space case, which is useful in particular for numerical simulations. We write

$$\frac{\langle n(t + \Delta t, x) \rangle - \langle n(t, x) \rangle}{\Delta t} = \chi(-\delta_x) \langle n(t, x) \rangle, \quad (93)$$

where now δ_x is some finite-difference operator, such as

$$\delta_x f(x) = \frac{f(x + \Delta x) - f(x)}{\Delta x}. \quad (94)$$

In this case, t and x take their values on lattices of respective spacing Δt and Δx . Again, the eigenfunctions of the kernel are the exponential functions.

The generalization of our previous results to an arbitrary BRW relies on the fact that at large times, the ‘‘wave number’’ γ_0 dominates and the kernel eigenvalue $\chi(\gamma)$ may be expanded to second order around γ_0 [8]. We then essentially use dimensional analysis to put in the appropriate process-dependent factors. We list here the generalized expressions without detailed justifications.

With the general kernel, the FKPP front position reads [see Eq. (3)]

$$\langle X_t \rangle = \chi'(\gamma_0)t - \frac{3}{2\gamma_0} \ln t + \text{const} - \frac{3}{\gamma_0^2} \sqrt{\frac{2\pi}{\chi''(\gamma_0)}} \frac{1}{\sqrt{t}} + \dots \quad (95)$$

The position of the tip of the front in the mean-field model with a discreteness cutoff reads instead

$$\bar{X}_t = \chi'(\gamma_0)t - \frac{3}{2\gamma_0} \ln t + \text{const} + \frac{3}{\gamma_0^2} \sqrt{\frac{2\pi}{\chi''(\gamma_0)}} \frac{1}{\sqrt{t}} + \dots \quad (96)$$

This expression generalizes Eq. (7) with C_X computed in Sec. IV A [see Eq. (66)].

The relevant variable that characterizes the fluctuations of the position of the bulk of the particles is $\frac{1}{\gamma_0} \ln Y_t$. We have computed its value in the deterministic model with a tip cutoff:

$$\frac{1}{\gamma_0} \ln \bar{Y}_t = \text{const} + \frac{3}{2\gamma_0^2} \sqrt{\frac{2\pi}{\chi''(\gamma_0)}} \frac{1}{\sqrt{t}}. \quad (97)$$

This equation generalizes Eq. (69).

The stochasticity that we found tractable analytically is related to the fluctuations of the difference of this variable at two distinct large times t_0 and t :

$$f = \frac{1}{\gamma_0} (\ln Y_t - \ln Y_{t_0}). \quad (98)$$

Its first moment reads

$$\mu_1 = \langle f \rangle = \frac{1}{2\gamma_0^2} \sqrt{\frac{2\pi}{\chi''(\gamma_0)}} \left(\frac{1}{\sqrt{t_0}} - \frac{1}{\sqrt{t}} \right). \quad (99)$$

The probability distribution of the fluctuations reads

$$p(f) = \begin{cases} \sqrt{\frac{2}{\pi \chi''(\gamma_0)}} \sqrt{\frac{1}{t_0} - \frac{1}{t}} \frac{e^{-\gamma_0 f}}{(1 - e^{-\gamma_0 f})^2} & \text{if } f > 0, \\ \sqrt{\frac{2}{\pi \chi''(\gamma_0)}} \sqrt{\frac{1}{t_0} - \frac{1}{t}} \frac{e^{\gamma_0 f}}{(1 - e^{\gamma_0 f})^2} \left[1 - \sqrt{\frac{1 - e^{\gamma_0 f}}{1 + e^{\gamma_0 f}}} \right] & \text{if } f < 0. \end{cases} \quad (100)$$

This formula is the generalized form of Eqs. (19) and (21). A generating function of the moments of order larger than 2 can be written as

$$\langle e^{\gamma_0 v f} \rangle = \frac{1}{\gamma_0} \sqrt{\frac{2}{\pi \chi''(\gamma_0)}} \sqrt{\frac{1}{t_0} - \frac{1}{t}} \left\{ -\nu \psi(-\nu) + \nu \psi(\nu) + \sqrt{\pi} \left[\frac{\Gamma(\frac{1}{2} + \frac{\nu}{2})}{\Gamma(\frac{\nu}{2})} + \frac{\Gamma(1 + \frac{\nu}{2})}{\Gamma(\frac{1}{2} + \frac{\nu}{2})} \right] \right\}. \quad (101)$$

For example, expanding this generating function, we find that the moments of order $k \geq 2$ read

$$\mu_k = \frac{1}{\gamma_0^{k+1}} \sqrt{\frac{2}{\pi \chi''(\gamma_0)}} \sqrt{\frac{1}{t_0} - \frac{1}{t}} m_k, \quad (102)$$

where the m_k 's are numerical constants. The first ones read

$$\begin{aligned} m_2 &= \frac{7\pi^2}{12} - \pi \ln 2 + \ln^2 2, \\ m_3 &= \frac{3}{2} \zeta(3) + \frac{\pi^3}{8} - \frac{\pi^2}{4} \ln 2 + \frac{3\pi}{2} \ln^2 2 + \ln^3 2, \\ m_4 &= 3(2 \ln 2 - \pi) \zeta(3) + \frac{119\pi^4}{240} - \frac{\pi^3}{2} \ln 2 \\ &\quad - \frac{\pi^2}{2} \ln^2 2 - 2\pi \ln^3 2 + \ln^4 2, \end{aligned} \quad (103)$$

or in numbers, $m_2 = 4.06013\dots$, $m_3 = 6.56570\dots$, $m_4 = 26.9902\dots$

B. Model suitable for a numerical implementation

For simplicity of the implementation, we considered a discretized branching diffusion model. At each time step, a particle on lattice site x (with lattice spacing $\Delta x = 1$) has the probability Δt to give birth to another particle on the same site, Δt to move to the site $x + 1$, Δt to move to the site $x - 1$, and $1 - 3\Delta t$ to stay unchanged at the same site. The eigenfunctions of the corresponding diffusion kernel are the exponential functions $e^{-\gamma x}$, and the eigenvalues read

$$\chi(\gamma) = \frac{1}{\Delta t} \ln[1 + \Delta t(e^\gamma + e^{-\gamma} - 1)]. \quad (104)$$

The discretization in time is chosen to be $\Delta t = 0.01$. The relevant parameters for this model are

$$\begin{aligned} \gamma_0 &= 0.91338\dots, & \chi'(\gamma_0) &= 2.05412\dots, \\ \chi''(\gamma_0) &= 2.79893\dots \end{aligned} \quad (105)$$

C. Check of the deterministic analytical results

We solve the equivalent of the deterministic FKPP equation with the critical initial condition. For our discretized model, the FKPP equation becomes the finite-difference equation

$$l_{x+1}(t + \Delta t) = l_x(t) + \ln\{1 + \Delta t[e^{l_{x+1}(t)-l_x(t)} + e^{l_{x-1}(t)-l_x(t)} - 1 - e^{l_x(t)}]\} \quad (106)$$

with the initial condition $l_x(t = 0) = \ln[1 - \exp(-e^{-\gamma_0 x})]$. Here x is an integer that labels the sites of the lattice. $l_x(t)$ is the logarithm of the equivalent of ϕ defined in Sec. IV. The use of a logarithmic variable avoids problems with numerical accuracy in the region $\phi \rightarrow 0$, upon which the solution depends crucially.

First, we integrate the solution according to Eq. (89) in order to get $\frac{1}{\gamma_0} \ln Y_t$. The analytical expectation for the model that is implemented is given in Eq. (99) with the numerical inputs (105):

$$\frac{1}{\gamma_0} \ln Y_t = \text{const} - \frac{0.8969\dots}{\sqrt{t}}. \quad (107)$$

The numerical calculation is shown in Fig. 2, and is in perfect agreement with the analytical formula. To estimate more quantitatively the quality of this agreement, we fit a function of the form

$$f(t) = c_0 + \frac{c_{\frac{1}{2}}}{\sqrt{t}} + \frac{c_1}{t} + \frac{c_{\frac{3}{2}}}{t^{3/2}}, \quad (108)$$

where the c 's are the free parameters. The value of $c_{\frac{1}{2}}$ that we get from the fit is $c_{\frac{1}{2}} = 0.8918$, which is very close to the expected value from our analytical calculation.

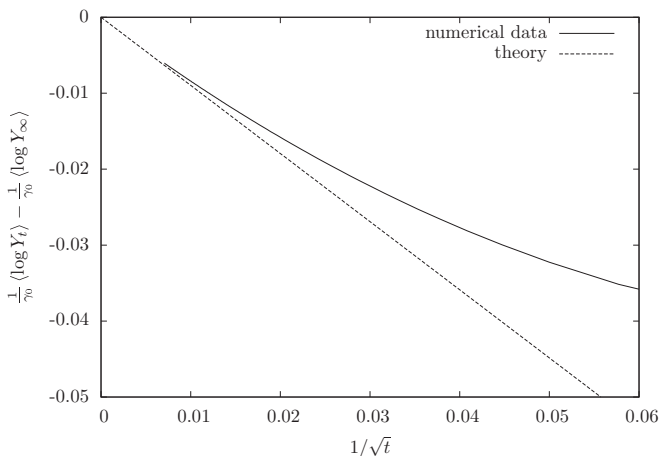


FIG. 2. $\frac{1}{\gamma_0} \ln Y_t$ from the numerical solution of the FKPP equation with the ‘‘critical’’ initial condition, as a function of $1/\sqrt{t}$. (The constant term is subtracted.) One sees that it converges to the analytical result Eq. (99) (with $t_0 \rightarrow +\infty$; straight line) for $t \rightarrow +\infty$.

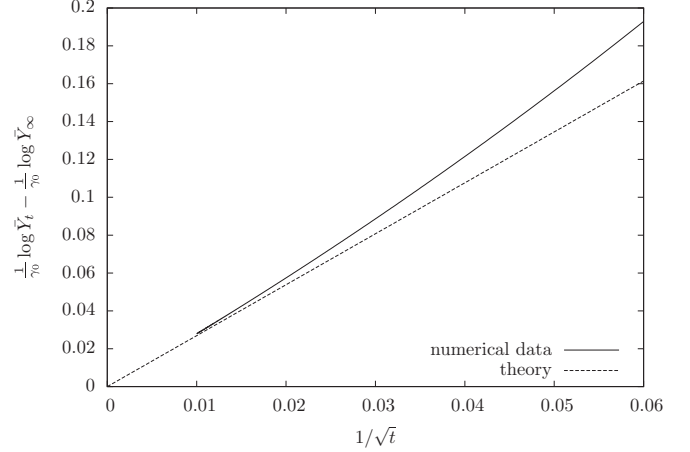


FIG. 3. $\frac{1}{\gamma_0} \ln \bar{Y}_t$ from the numerical solution of the branching diffusion equation with a cutoff as a function of $1/\sqrt{t}$. (The constant is subtracted.) Again, the numerical solution converges to the analytical result [Eq. (97); straight line] as t gets large.

Next, we solve the deterministic FKPP equation with a tip cutoff. In practice, the latter cutoff is implemented as a smooth nonlinearity, as in Eq. (36). More precisely, the equation we solve numerically is the following:

$$l_x(t + \Delta t) = l_x(t) + \ln \left\{ 1 + \Delta t \left[e^{l_{x+1}(t)-l_x(t)} + e^{l_{x-1}(t)-l_x(t)} - 2 + \frac{1}{1 + e^{-l_x(t)}} \right] \right\} \quad (109)$$

with $l_x(t = 0) = -|x|$. Here, $l_x(t)$ is the logarithm of the number of particles on site x at time t . The logarithmic scale for the evolved function is useful here because of the exponential growth of the number of particles with time. Also in this case, the result is in excellent agreement with the analytical expectation (see Fig. 3), which, for the considered model, should read [see Eq. (97)]

$$\frac{1}{\gamma_0} \ln \bar{Y}_t = \text{const} + \frac{2.6909\dots}{\sqrt{t}}. \quad (110)$$

The fit of the same function $f(t)$ as before to the numerical data gives $c_{\frac{1}{2}} = 2.7120$, which again is very close to the analytical estimate.

D. Check of the statistics of f

We now use a Monte Carlo implementation of the stochastic model of a branching random walk described above in order to test the probability distribution of f given in Eq. (100). The implementation is quite straightforward, except maybe that after a few time steps, the number of particles n_x in the central bins [typically $|x| \leq \chi'(\gamma_0)t$] becomes very large. To handle such large particle numbers, we further evolve these bins in a deterministic way. (In practice in the code, we set the limit between stochastic and deterministic evolution at $n_x = 10^6$.) Such an approximate treatment was tested before in a similar context; see, e.g., Refs. [17–19]. As in the deterministic case discussed above, we also switch to logarithmic variables, $l_x \equiv \ln n_x$, in order to be able to handle the large particle numbers

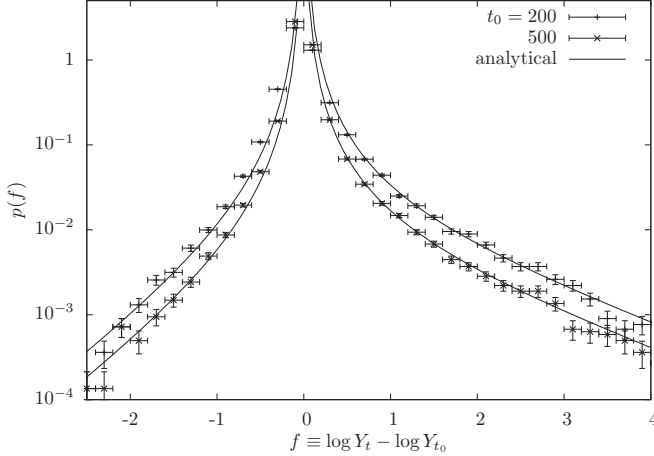


FIG. 4. Distribution of f for $t = 1000$ and two different values of t_0 . The numerical data (points with statistical error bars and bin width) are compared to Eq. (100) (continuous lines) (\log_{10} scale on the vertical axis).

in a standard double-precision computer representation. Of course, the low-density tails of the system are treated fully stochastically.

The result for the distribution of f is displayed in Fig. 4 compared to the analytical formulas (100). We see an excellent agreement between the outcome of our model and the numerical data.

We can also compute numerically the first few moments of the variable f and plot them against t_0 (Fig. 5). Here again, there is good agreement between the analytical result and the numerical calculation, although more statistics would be needed in order to reach a good accuracy for the moments of order 3 and 4.

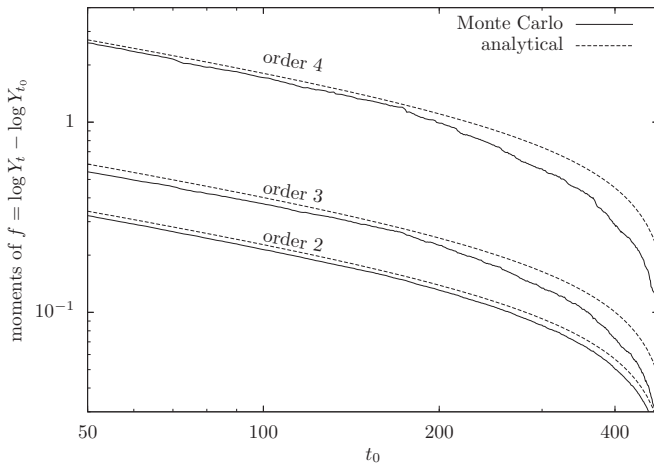


FIG. 5. Moments of $f = \ln Y_t / Y_{t_0}$ of order 2 to 4 for $t = 500$ as a function of t_0 in logarithmic scales. The numerical data (full lines) are compared to the analytical calculation in Eq. (102) (dashed lines) (\log_{10} scale on both axes). There are about 4×10^5 realizations in the statistical ensemble used to perform the averages.

VI. STOCHASTIC INTERPRETATION OF THE $1/\sqrt{t}$ CORRECTIONS TO THE POSITION OF FKPP FRONTS

So far, we have essentially discussed the statistics of the $\ln Y$ random variable in light of our phenomenological picture of BRW. We are now going to address the average of the position of the rightmost particle $\langle X_t \rangle$, which is also the position of the FKPP front, and whose expression at order $1/\sqrt{t}$ was obtained in Ref. [10].

A. Correction to $\langle X_t \rangle$ due to fluctuations

As in the case of $\langle f \rangle$, in our picture, the average value of the position of the front is given by the deterministic evolution of the bulk of the particles, supplemented by a contribution from fluctuations in the low-density region. We may write

$$\begin{aligned} \mu'_1 \equiv \langle X_t \rangle - \langle X_{t_0} \rangle &= 2(t - t_0) - \frac{3}{2} \ln \frac{t}{t_0} + C_{\bar{x}} \left(\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{t_0}} \right) \\ &+ \mu_1^{'+} - \mu_1'^-. \end{aligned} \quad (111)$$

In this section, we shall compute $\mu_1^{'+} - \mu_1'^-$. $\mu_1^{'+}$ is the contribution at time t of fluctuations that occur all over the range of time, and $\mu_1'^-$ is the contribution at time t_0 of fluctuations that have occurred before t_0 :

$$\mu_1^{'+} = \int_0^t dt_1 \int_0^{+\infty} d\delta \delta X_t, \quad \mu_1'^- = \int_0^{t_0} dt_1 \int_0^{+\infty} d\delta \delta X_{t_0}, \quad (112)$$

where the appropriate regulators will be introduced later. δX_t is the contribution to the shift of the position of the tip of the front at time t of a fluctuation of size δ occurring at t_1 . Let us now evaluate δX_t .

When a fluctuation occurs at time t_1 at position δ ahead of the tip \bar{X}_{t_1} of the regular front, then it develops its own front by independent branching diffusion. The resulting density of particles at time $t \gg t_1$ becomes the sum of two terms, and therefore has the shape

$$\psi_{\bar{X}_t + \delta X_t}(x, t) = \psi_{\bar{X}_t}(x, t) + C \psi_{\bar{X}_{\delta, t}}(x, t), \quad (113)$$

where ψ is given by Eq. (6) and $\bar{X}_{\delta, t}$ by Eq. (12). Using the latter equations, one is led to

$$\delta X_t = \ln \left[1 + C e^{\delta} \left(\frac{t}{t_1(t - t_1)} \right)^{3/2} \right]. \quad (114)$$

The calculation of $\mu_1^{'+}$ and $\mu_1'^-$ proceeds exactly as in the case of μ_1^+ and μ_1^- in Sec. III C. $\mu_1^{'+}$ is still given by an equation of the form of (27), but with the replacements $\mathcal{I}_0 \rightarrow \mathcal{I}'_0$ and $\mathcal{I}_1 \rightarrow \mathcal{I}'_1$, where now

$$\begin{aligned} \mathcal{I}'_0 &= \int_{t'_0/t}^{1-\bar{t}_0/t} d\lambda \lambda^{-3/2} (1 - \lambda)^{-3/2}, \\ \mathcal{I}'_1 &= \int_{t'_0/t}^{1-\bar{t}_0/t} d\lambda \lambda^{-3/2} (1 - \lambda)^{-3/2} \ln[\lambda^{3/2} (1 - \lambda)^{3/2}]. \end{aligned} \quad (115)$$

Note that in the present case, late times need to be cut off in order to ensure the convergence of the integrals: We pick some arbitrary $\bar{t}_0 \ll t$, say of order 1.

The same change of variable as before may be used: $\lambda = \sin^2 \theta$, then

$$\begin{aligned} \mathcal{I}'_0 &= 8 \int_{\arcsin \sqrt{1-\frac{t'_0}{t}}}^{\arcsin \sqrt{1-\frac{\bar{t}_0}{t}}} \frac{d\theta}{\sin^2 2\theta}, \\ \mathcal{I}'_1 &= 24 \int_{\arcsin \sqrt{\frac{t'_0}{t}}}^{\arcsin \sqrt{1-\frac{\bar{t}_0}{t}}} \frac{d\theta}{\sin^2 2\theta} \ln \frac{\sin 2\theta}{2}. \end{aligned} \quad (116)$$

After performing the integrals and expanding in the limit of small \bar{t}_0, t'_0 compared to t , one gets

$$\begin{aligned} \mu_1^{'+} &= 2CC_1 \left\{ \frac{1}{\sqrt{\bar{t}_0}} \left(\ln \frac{\bar{t}_0^{3/2}}{C} + 3 \right) \right. \\ &\quad \left. + \frac{1}{\sqrt{t'_0}} \left(\ln \frac{t_0^{3/2}}{C} + 3 \right) - \frac{6\pi}{\sqrt{t}} \right\}. \end{aligned} \quad (117)$$

The main difference with respect to Eq. (32) (once the relevant expansions have been performed) is the presence of \bar{t}_0 -dependent terms and of an extra factor 3 in the last term. As before, μ_1^{-} is deduced from the above formula by replacing t by t_0 . We then see that in the difference $\mu_1^{'+} - \mu_1^{-}$, the \bar{t}_0 and t'_0 dependences cancel.

As for the moments of order $n \geq 2$, they are found to depend on \bar{t}_0 , that is, on the late-time fluctuations, as they should, since X_t is the position of the rightmost particle, which experiences a stochastic motion of size 1 over time scales of order 1.

B. Recovering the Ebert–Van Saarloos term

Putting everything together, namely the value of $C_{\bar{X}}$ from Eq. (66) and the value of $\mu_1^{'+} - \mu_1^{-}$ just computed, we find the interesting expression

$$\begin{aligned} \langle X_t - X_{t_0} \rangle &= \mu_1' = 2(t-t_0) - \frac{3}{2} \ln \frac{t}{t_0} + \left[3\sqrt{\pi} \left(\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{t_0}} \right) \right] \\ &\quad - \left[6\sqrt{\pi} \left(\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{t_0}} \right) \right]. \end{aligned} \quad (118)$$

The terms that grow with t and t_0 are the usual deterministic terms from Bramson's classical solution [5]. Then, the next terms, under the square brackets, are, respectively, the deterministic correction to the position of the discreteness cutoff in the mean-field model, and the correction due to fluctuations. We see that the latter is exactly twice the former, with a minus sign. The sum of these two terms gives back the Ebert–Van Saarloos correction for $\langle X_t - X_{t_0} \rangle$; see Eq. (3).

In other words, the mismatch between \bar{X}_t , the position of the tip of the front in the deterministic model with a cutoff and $\langle X_t \rangle$, the mean position of the rightmost particle in the full stochastic model, is exactly due to the very fluctuations we have been analyzing in this paper.

VII. CONCLUSIONS

Some time ago, we proposed a model for the fluctuations of stochastic pulled fronts [16], which are realizations of the stochastic FKPP (sFKPP) equation (for a review, see Ref. [20]). Equations in the class of the sFKPP equation may be thought of, for instance, as representing the dynamics of

the particle number density in a branching-diffusion process in which there is in addition a nonlinear selection/saturation process that effectively limits the density of particles. The realizations of such equations are stochastic traveling waves. The stochasticity comes from the discreteness of the number of particles. In this context, the (deterministic) FKPP equation represents the mean-field (or infinite number of particle) limit of the full dynamics.

Expansions about the mean-field solution were considered already a long time ago; see, e.g., Ref. [21], where the so-called Ω expansion (see Ref. [22]) was applied to study fluctuations in the context of reaction-diffusion processes. Later, we could obtain new analytical results thanks to a phenomenological model [16]. The picture encoded in our model was the following: Most of the time, the traveling wave front propagates deterministically, obeying the ordinary deterministic FKPP equation supplemented with a cutoff in the tail, accounting for discreteness by making sure that the number density of particles reaches 0 rapidly whenever it drops below 1. Brunet and Derrida had shown [15] that such a cutoff correctly represents the main effect of the noise on the velocity of the front. On top of that, in our model, there are some rare fluctuations consisting in a few particles randomly sent far ahead of the tip of the front, which upon further evolution build up a new front that completely takes over the old one. A positive correction to the front velocity was found, and the cumulants of the front position were computed (see Ref. [16]).

In the present paper, we have considered a simple branching random walk, without any selection mechanism. We have used exactly the same ingredients as the ones conjectured in the model for stochastic fronts, namely deterministic evolution with a cutoff and fluctuations consisting in a few particles randomly sent ahead the tip of the front at a distance distributed exponentially. We were also able to arrive at a quantitative characterization of the fluctuations of the front in these processes.

There are, however, a few important differences between the branching random walk and the stochastic FKPP front. First, the initial fluctuations are never “forgotten” in the BRW case. This is because of the absence of a selection mechanism able to “kill” the front and let it be periodically regenerated by fluctuations. Therefore, we could only compute the effect of the fluctuations on the front position between two large times t_0 and t . Next, while it was quite straightforward to define a proper front position in the sFKPP case (as, for example, the integral of the normalized particle density from position say 0 to $+\infty$), it is more tricky for the simple branching random walk. We were led to consider the variables $\ln Y$ and $\ln Z$ (introduced in Sec. II B). Our main result is the distribution of the variable $\ln Y_t/Y_{t_0}$ given in Eq. (100), where t_0 and t are two large times such that $t_0, t, t-t_0 \gg 1$. Interestingly enough, the distribution of the positive values of this variable is identical (up to an overall factor) to the distribution of the front fluctuations in the sFKPP case. The same holds true for the distribution of $\ln Z_t/Z_{t_0}$, to which we dedicate Appendix B.

We were also able to discuss the average of the position of the rightmost particle, but not its higher moments, since they are sensitive to the very late-time fluctuations, which are not properly described in our model. As for the average position,

we could nevertheless propose an appealing interpretation of the $O(1/\sqrt{t})$ correction to the front position computed by Ebert and Van Saarloos in Ref. [10].

There are still many open questions. Maybe the most outstanding one on the technical side would be to try and compute the statistics of $\ln Y_t/Y_{t_0}$ (and of $\ln Z_t/Z_{t_0}$) exactly, instead of relying on a phenomenological picture involving conjectures. We outlined such a calculation in Appendix C, based on the evaluation of a generating function, but without being able to complete it.

ACKNOWLEDGMENTS

The idea at the origin of Appendix C is due to B. Derrida. We thank him also for very helpful discussions, and for his reading of the manuscript. We acknowledge support from ‘‘P2IO Excellence Laboratory,’’ and from the Office of Science of the U.S. Department of Energy, Grant No. DE-FG02-92ER40699.

APPENDIX A: DETAILS OF THE CALCULATION OF THE PROBABILITY DISTRIBUTION OF f

In this appendix, we go back to the calculations that lead to Eqs. (15), (19), and (21), but keeping the subleading terms that we neglected *a priori* in Sec. III A in order to simplify the presentation.

The exact evaluation of \bar{Y}_t starting from its definition given in Eq. (10), in which one inserts Eqs. (6) and (7), makes use of the basic Gaussian integral

$$\int_{-\infty}^0 dx (\alpha x + \beta) e^{-\frac{x^2}{4t}} = -2\alpha t + \beta\sqrt{\pi t}. \quad (\text{A1})$$

We immediately arrive at Eq. (10), which may also be rewritten at order $1/\sqrt{t}$ as

$$\bar{Y}_t = -2\alpha e^{\frac{C_{\bar{Y}}}{\sqrt{t}}}. \quad (\text{A2})$$

We now add a fluctuation occurring, say, at time t_1 . It develops a front whose tip sits, at time t , at position

$$\begin{aligned} \bar{X}_{\delta,t} &= \bar{X}_t + \delta - \frac{3}{2} \ln \frac{t_1(t-t_1)}{t} \\ &+ C_{\bar{X}} \left(\frac{1}{\sqrt{t_1}} + \frac{1}{\sqrt{t-t_1}} - \frac{1}{\sqrt{t}} \right), \end{aligned} \quad (\text{A3})$$

which is Eq. (7) supplemented with the subleading terms. Keeping all of the latter, we see that Eq. (15) just needs to be replaced by

$$\delta \ln Y_t = \ln \left[1 + C \frac{e^\delta}{t_1^{3/2}} \sqrt{\frac{t}{t-t_1}} e^{\frac{C_{\bar{X}}}{\sqrt{t_1}} + C_{\bar{Y}} \left(\frac{1}{\sqrt{t-t_1}} - \frac{1}{\sqrt{t}} \right)} \right]. \quad (\text{A4})$$

As for the probability distribution of the fluctuations in Eq. (18), it becomes

$$p(\delta f; t_1) = \frac{CC_1}{t_1^{3/2}} \sqrt{\frac{t}{t-t_1}} e^{\frac{C_{\bar{X}}}{\sqrt{t_1}} + C_{\bar{Y}} \left(\frac{1}{\sqrt{t-t_1}} - \frac{1}{\sqrt{t}} \right)} \frac{e^{-\delta f}}{(1 - e^{-\delta f})^2}, \quad (\text{A5})$$

which has to be integrated over t_1 . We recall that after integration over t_1 , the obtained expression will be correct at order $1/t_1$, $1/(t-t_1)$, $1/t$, hence only the first nontrivial terms are relevant in the expansion of the exponential.

In the absence of $O[1/(t-t_1)]$ terms, the integration region could be chosen to be $[t_0, t]$ as in Sec. III. Now, however, we have a nonintegrable singularity at $t_1 = t$ that needs to be cut off. Hence we write

$$\begin{aligned} p(\delta f) &= \int_{t_0}^{t-\bar{t}_0} dt_1 p(\delta f; t_1) = CC_1 e^{-\frac{C_{\bar{Y}}}{\sqrt{t}}} \frac{e^{-\delta f}}{(1 - e^{-\delta f})^2} \\ &\times \int_{t_0}^{t-\bar{t}_0} \frac{dt_1}{t_1^{3/2}} \sqrt{\frac{t}{t-t_1}} e^{\frac{C_{\bar{X}}}{\sqrt{t_1}} + \frac{C_{\bar{Y}}}{\sqrt{t-t_1}}}, \end{aligned} \quad (\text{A6})$$

where \bar{t}_0 is an arbitrary time interval whose length is of the order of 1.

Let us compute the integral

$$\mathcal{J} \equiv \int_{t_0}^{t-\bar{t}_0} \frac{dt_1}{t_1^{3/2}} \sqrt{\frac{t}{t-t_1}} e^{\frac{C_{\bar{X}}}{\sqrt{t_1}} + \frac{C_{\bar{Y}}}{\sqrt{t-t_1}}} \quad (\text{A7})$$

appearing in the previous expression. We expand the exponential to lowest order, and hence we get the three terms

$$\mathcal{J} = \mathcal{J}_0 + C_{\bar{X}} \mathcal{J}_1^{(1)} + C_{\bar{Y}} \mathcal{J}_1^{(2)}, \quad (\text{A8})$$

where \mathcal{J}_0 [which is essentially the same integral as \mathcal{I}_0 in Eq. (30)] gives back the lowest-order result in Eq. (19):

$$\begin{aligned} \mathcal{J}_0 &= \int_{t_0}^{t-\bar{t}_0} \frac{dt_1}{t_1^{3/2}} \sqrt{\frac{t}{t-t_1}} = 2\sqrt{\frac{t-t_0}{tt_0}} - 2\sqrt{\frac{\bar{t}_0}{t(t-\bar{t}_0)}} \\ &\simeq 2\sqrt{\frac{t-t_0}{tt_0}} + O(1/t). \end{aligned} \quad (\text{A9})$$

As for the two other terms,

$$\begin{aligned} \mathcal{J}_1^{(1)} &= \sqrt{t} \int_{t_0}^{t-\bar{t}_0} \frac{dt_1}{t_1^2} \frac{1}{\sqrt{t-t_1}}, \\ \mathcal{J}_1^{(2)} &= \sqrt{t} \int_{t_0}^{t-\bar{t}_0} \frac{dt_1}{t_1^{3/2}} \frac{1}{t-t_1} \end{aligned} \quad (\text{A10})$$

are new contributions that are subleading, as is easy to demonstrate from an exact calculation of these integrals. We start with the computation of $\mathcal{J}_1^{(1)}$:

$$\begin{aligned} \mathcal{J}_1^{(1)} &= \sqrt{\frac{t-t_0}{t}} \left(\frac{1}{t_0} + \frac{1}{t} \operatorname{arctanh} \sqrt{\frac{t-t_0}{t}} \right) \\ &- \sqrt{\frac{\bar{t}_0}{t}} \left(\frac{1}{t-\bar{t}_0} + \frac{1}{t} \operatorname{arctanh} \sqrt{\frac{\bar{t}_0}{t}} \right). \end{aligned} \quad (\text{A11})$$

Since $\operatorname{arctanh} \sqrt{1-x} \underset{x \rightarrow 0}{\sim} -\frac{1}{2} \ln x$, it is clear that the largest terms in $\mathcal{J}_1^{(1)}$ are at most of order $\ln(t/t_0)/t$ and $1/t_0$. As for $\mathcal{J}_1^{(2)}$,

$$\begin{aligned} \mathcal{J}_1^{(2)} &= 2 \left(\frac{1}{\sqrt{tt_0}} - \frac{1}{\sqrt{t(t-\bar{t}_0)}} \right) \\ &+ \frac{2}{t} \left(\operatorname{arctanh} \sqrt{1 - \frac{\bar{t}_0}{t}} - \operatorname{arctanh} \sqrt{\frac{\bar{t}_0}{t}} \right). \end{aligned} \quad (\text{A12})$$

The second term is divergent for $\bar{t}_0 \rightarrow 0$. It gives the dominant contribution at large t : $\mathcal{J}_1^{(2)} \sim \ln(t/\bar{t}_0)/t$. The other terms are also subleading, of order $1/\sqrt{tt_0}$ and $1/t$.

Hence we see that at order $O(1/\sqrt{t}, 1/\sqrt{t_0}, 1/\sqrt{t-t_0})$, \mathcal{J} boils down to the first term in the expansion of \mathcal{J}_0 in Eq. (A9).

Lastly, we have already noticed in Sec. III that

$$f - \delta f = \ln \frac{\bar{Y}_t}{\bar{Y}_{t_0}} \simeq C_{\bar{Y}} \left(\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{t_0}} \right), \quad (\text{A13})$$

thus replacing δf by f in $p(\delta f)$ brings about only subleading contributions.

All in all, we have justified the approximations that led to Eq. (19). From a very similar calculation, we would also recover Eq. (21).

APPENDIX B: STATISTICS OF $f_Z \equiv \ln Z_t - \ln Z_{t_0}$

In the same way as for the variable $f = \ln Y_t - \ln Y_{t_0}$, we may try to get the statistics of $f_Z = \ln Z_t - \ln Z_{t_0}$ from our phenomenological model. The variable Z_t is of interest since it is used in a mathematical theorem to characterize what we call the position of the front in each realization, however, as we shall see, we cannot obtain full analytical formulas for the first moment of f_Z as in the case of f . Moreover, as was already commented above, the Z_t variable has properties that make it awkward for numerical simulations.

The first step is to compute Z_t in the mean-field approximation with a tip cutoff. The result reads

$$\bar{Z}_t = -2\alpha\sqrt{\pi} \left(1 + \frac{3 \ln t}{2\sqrt{\pi t}} \right). \quad (\text{B1})$$

This formula is analogous to Eq. (10), but there is now a slightly stronger t dependence, $\propto \ln t / \sqrt{t}$, which we are able to determine completely from the leading-order shape of the particle distribution. We have dropped terms of order $1/\sqrt{t}$ and higher.

The effect of a fluctuation occurring at time t_1 on $\ln Z_t$ is

$$\delta \ln Z_t = \ln \left[1 + C \frac{e^\delta}{t_1^{3/2}} \frac{1 + \frac{\ln \{ [t_1(t-t_1)]^{3/2} e^{-\delta} \}}{\sqrt{\pi(t-t_1)}}}{1 + \frac{\ln t^{3/2}}{\sqrt{\pi t}}} \right] \quad (\text{B2})$$

[compare to Eq. (15)]. The following approximate formula can now be written for the distribution of f_Z :

$$p(f_Z) = \int dt_1 \int_0^{+\infty} d\delta p(\delta) \delta[f_Z - (\delta \ln Z_t - \delta \ln Z_{t_0})], \quad (\text{B3})$$

where $\delta \ln Z$ is given by Eq. (B2), while $p(\delta)$ is the probability distribution (8). The bounds on the integral over t_1 depend on whether f_Z is positive or negative. Indeed, positive values of f_Z are generated by fluctuations occurring at t_1 between t_0 and t , while fluctuations before t_0 (namely between the times t'_0 at which we declare that the system contains a large number of particles and t_0) give rise to negative values of f_Z .

The distribution of positive f_Z is quite easy to compute. It is enough to recognize that the terms of order $1/\sqrt{t-t_1}$ and $1/\sqrt{t}$ inside the square brackets give subleading contributions to $p(f_Z)$. It turns out that the final result is very similar to $p(f)$ [see Eq. (19)], except for the detailed form of the t_0 and t dependence:

$$p(f_Z > 0) = 2CC_1 \left(\frac{1}{\sqrt{t_0}} - \frac{1}{\sqrt{t}} \right) \frac{e^{-f_Z}}{(1 - e^{-f_Z})^2}. \quad (\text{B4})$$

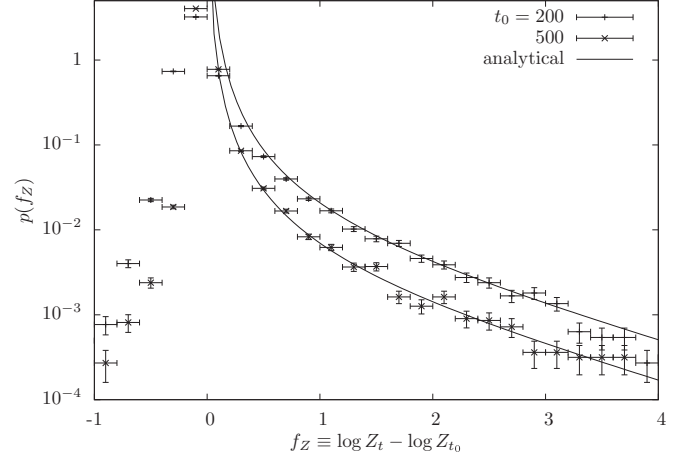


FIG. 6. Distribution of f_Z for $t = 1000$ and two different values of t_0 . The numerical data (points with statistical error bars and bin width) are compared to Eq. (B5) (continuous lines). (log₁₀ scale on the vertical axis.)

Inserting the value of the constant CC_1 previously determined [see Eq. (91)] and going to a general branching diffusion kernel, we get

$$p(f_Z > 0) = \sqrt{\frac{2}{\pi \chi''(\gamma_0)}} \left(\frac{1}{\sqrt{t_0}} - \frac{1}{\sqrt{t}} \right) \frac{e^{-f_Z}}{(1 - e^{-f_Z})^2}. \quad (\text{B5})$$

Negative values of f_Z are more complicated to deal with since we can no longer neglect the $1/\sqrt{t-t_1}$ term in Eq. (B2) *a priori*. Performing the change of variable $u \equiv t_1^{3/2}/(Ce^\delta)$ and expanding for large t and t_0 , the equation for $p(f_Z)$ simplifies to

$$p(f_Z < 0) = CC_1 \int_{t'_0}^{t_0} \frac{dt_1}{t_1^{3/2}} \int_0^{t_1^{3/2}/C} du \delta \left[f_Z - \frac{1}{\sqrt{\pi}} \frac{\ln(Cu)}{1+u} \left(\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{t_0}} \right) \right]. \quad (\text{B6})$$

Due to the Dirac δ function, we see that $p(f_Z) = 0$ as soon as $f_Z < \frac{1}{\sqrt{\pi}u_0} \left(\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{t_0}} \right)$, where u_0 solves $\ln(Cu_0) = 1 + \frac{1}{u_0}$, and hence is of order 1. This means that $p(f_Z)$ is of higher order in powers of $1/\sqrt{t}$ and $1/\sqrt{t_0}$ when $f_Z < 0$.

Our formula for the distribution, Eq. (B5), successfully compares to the numerical data; see Fig. 6. We also see that the distribution of negative values of f_Z is indeed sharply suppressed (compare to the distribution of f in Fig. 4).

As for the mean of f_Z , we found that it depends on the arbitrary time t'_0 roughly as $1/\sqrt{t'_0}$, and thus is not calculable.

APPENDIX C: GENERATING FUNCTION FOR THE MOMENTS

In this section, we are going to find the form of the large positive f fluctuations from a generating function, hence from a deterministic calculation.

1. General framework and exact formulas

We can write the following identity:

$$\left\langle \left(\frac{\tilde{Y}_t}{\tilde{Y}_{t_0}} \right)^v \right\rangle = \frac{\sin \pi v}{\pi v} \int_0^{+\infty} d\tilde{u} d\tilde{v} \left(\frac{\tilde{v}}{\tilde{u}} \right)^v \frac{\partial^2}{\partial \tilde{u} \partial \tilde{v}} \langle e^{-\tilde{u}\tilde{Y}_t - \tilde{v}\tilde{Y}_{t_0}} \rangle, \quad (\text{C1})$$

see Eq. (9) for the definition of \tilde{Y} . This equation follows from the integral representation of the Γ function, and it is suitable for series expansions in v , which eventually lead to the moments of $\ln \tilde{Y}_t/\tilde{Y}_{t_0}$. For some calculations outlined below, it will prove useful to change \tilde{u} and \tilde{v} to the variables

$$u = \tilde{u} \frac{e^{2t}}{\sqrt{t}}, \quad v = \tilde{v} \frac{e^{2t_0}}{\sqrt{t_0}} \quad (\text{C2})$$

since Eq. (C1) then holds in the very same form (just up to the replacements $\{\tilde{u}, \tilde{v}, \tilde{Y}_t, \tilde{Y}_{t_0}\} \rightarrow \{u, v, Y_t, Y_{t_0}\}$) directly for the moments of $\ln Y_t/Y_{t_0} = f$.

Let us introduce the generating function

$$G_{t_0}(x) = \langle e^{-(\tilde{u}\tilde{Y}_t + \tilde{v}\tilde{Y}_{t_0})e^{-x}} \rangle. \quad (\text{C3})$$

It is the value of this function at zero, $G_{t_0}(0)$, from which one computes the generating function in Eq. (C1), which reads

$$\left\langle \left(\frac{\tilde{Y}_t}{\tilde{Y}_{t_0}} \right)^v \right\rangle = \frac{\sin \pi v}{\pi v} \int_0^{+\infty} d\tilde{u} d\tilde{v} \left(\frac{\tilde{v}}{\tilde{u}} \right)^v \frac{\partial^2 G_{t_0}(0)}{\partial \tilde{u} \partial \tilde{v}}. \quad (\text{C4})$$

The function $G_{t_0}(x)$ may also be written as

$$G_{t_0}(x) = \left\langle \prod_{i=1}^{N(t_0)} g_\tau[x - x_i(t_0)] e^{-\tilde{v}e^{-[x-x_i(t_0)]}} \right\rangle, \quad (\text{C5})$$

where $\tau \equiv t - t_0$ is a parameter in this equation, and

$$g_\tau(x) \equiv \langle e^{-\tilde{u}\tilde{Y}_\tau e^{-x}} \rangle. \quad (\text{C6})$$

In this form, it is clear that $G_{t_0}(x)$ obeys the FKPP equation (with time variable t_0), with the initial condition $g_\tau(x)e^{-\tilde{v}e^{-x}}$. But $g_\tau(x)$ may also be written as

$$g_\tau(x) = \left\langle \prod_{i=1}^{N(\tau)} e^{-\tilde{u}e^{-[x-x_i(\tau)]}} \right\rangle, \quad (\text{C7})$$

which makes it obvious that it also obeys the FKPP equation (with time variable τ), with the initial condition $g_0(x) = e^{-\tilde{u}e^{-x}}$.

So far, these formulas are exact and should in principle enable the computation of the moments of f , from some hopefully limited knowledge of the properties of the solutions to the FKPP equation.

We have not been able to fully compute the generating function. However, we can use the systematic solution to FKPP for the evolution of g , and a mean-field approximation for G : Interestingly enough, this turns out to be enough to compute the positive fluctuations of f .

2. Approximate solution: Moments of $f > 0$

In this section, we shall consider the stronger limit $t \gg t_0 \gg 1$. Let us treat the evolution from the initial time $t = 0$ to time t_0 in the mean-field approximation with a tip cutoff: This means that we assume a distribution of particles at time t_0 given by Eq. (6). Then the product over the particles in Eq. (C5) becomes the exponential of an integral over the spatial coordinate weighted by the particle density:

$$G_{t_0}(x) = \exp \left[- \int_{-\infty}^{\bar{X}_{t_0}} dx' \alpha(x' - \bar{X}_{t_0}) e^{-(x' - \bar{X}_{t_0}) - \frac{(x' - \bar{X}_{t_0})^2}{4t_0}} \times \{ \tilde{v} e^{-(x-x')} - \ln[g_{t-t_0}(x-x')] \} \right], \quad (\text{C8})$$

where $\bar{X}_{t_0} = 2t_0 - \frac{3}{2} \ln t_0$. We have dropped the β term in the form of the particle distribution as well as the $1/\sqrt{t_0}$ term in \bar{X}_{t_0} since they would eventually give subleading contributions, of order $1/t_0$, at large t_0 . We see that the Gaussian under the integral makes sure that the range of integration in the variable $x' - \bar{X}_{t_0}$ is effectively $[-2\sqrt{t_0}, 0]$.

We turn to the g_{t-t_0} . We know that it obeys the FKPP equation with the critical initial condition. Hence the solution can be deduced from Eq. (87). However, since $t_0 \ll t$, defining $\xi = x - \ln u - 2(t - t_0) + \frac{1}{2} \ln(t - t_0)$, we may expand the solution for $1 \ll \xi \ll \sqrt{t - t_0}$, namely

$$1 - g_{t-t_0}(\xi) \simeq \frac{1}{\sqrt{\pi}} \xi e^{-\xi}. \quad (\text{C9})$$

We have dropped the term of order $1/\sqrt{t - t_0}$ in ξ .

We shall now proceed with the integration in Eq. (C8). Keeping only the term of order $1/\sqrt{t_0}$ and switching to the u, v variables, we find

$$G_{t_0}(0) = e^{-2(u+v)} \left(1 + u \frac{2 \ln u - 3 \ln t_0}{\sqrt{\pi t_0}} \right). \quad (\text{C10})$$

Inserting this expression into Eq. (C4) (with \tilde{u}, \tilde{v} being replaced by u, v), we now perform the integrals over u and v . The exact result is

$$\langle e^{vf} \rangle = 1 + \frac{1}{\sqrt{\pi t_0}} \left[-v \psi(-v) + 1 + v \left(\frac{3}{2} \ln t_0 + \ln 2 \right) \right]. \quad (\text{C11})$$

Remarkably, if we invert this equation for the probability distribution of f by performing an appropriate contour integration over v , we exactly recover Eq. (100) for the case $f > 0$ [in the limit $t \rightarrow +\infty$, and up to replacements of the parameters in (100): $\gamma_0 \rightarrow 1$, $\chi''(\gamma_0) \rightarrow 2$]. Note that the constant CC_1 that appeared in the phenomenological model is determined without any further calculation in the present approach. The case $f < 0$, however, cannot be obtained unless we were able to release the mean-field approximation for the evolution between $t = 0$ and $t = t_0$.

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