Complete spectral scaling of time series: Towards a classification of $1/f$ noise

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The standard spectral scaling, $S(f) \sim 1/f^{\beta}$, has been traditionally used as a correlation measure characterizing the dynamical behavior of time series. The ubiquity of $1/f$ and $1/f^2$ spectra in many processes certainly suggests the existence of universal mechanisms, but also gives rise to the suspicion that some important features are not included in this scaling. In this paper we argue that a complete spectral scaling, including as a main variable the size of the series, $S(f,T) \sim T^{\eta}/f^{\beta}$, which is usually considered irrelevant, gives an insight into this problem. Using synthetically generated series we show that, in general, the scaling exponent β is too generic, while the exponent associated with the size, η, gives a more specific information. Hence, we propose the use of both exponents in a scheme to classify series into different universality classes. In this way many of the processes appearing in the literature could be better identified, and much of the ambiguity that surrounds the standard spectral scaling could be clarified.

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I. INTRODUCTION

Fluctuating time series are observed and analyzed in all fields of science, from rheology and finance [1,2] to electronics and psychology [3,4]. A first aim of such an analysis consists of the characterization of the inherent dynamics of the process for the sake of a subsequent model formulation. Among other possibilities, the power spectra analysis is one of the most important methods for the dynamical characterization of a time series. In many cases spectra exhibit scaling as $S(f) \sim 1/f^{\beta}$ and the exponent β becomes a quantifier of correlation. The observation of universal exponents with $\beta \sim 1,2..$ is a recurrent fact in almost all fields of physics. It has been reported in many experiments and observations in a surprising variety of fields, electronics [3], music [5], condensed matter [7], atmospheric sciences [8], medical sciences [9], astronomy [10], etc.

In the last twenty years this ubiquity has motivated a large number of studies in order to find universal mechanisms able to explain such a regularity. There have been interesting proposals, but without a commonly accepted conclusion. While some authors try to find such a universal mechanism [11,12], others assume that this diversity implies that the physical origin of the noise cannot be universal [3,13]. The discussion continues in general with emphasis on the old questions of how much of the explanation is universal and how much fits into any clear theoretical picture [7]. Several possible mechanisms have been postulated as being the origin of this noise. Among the most well-known it is worth mentioning the self-organizing criticality (SOC) mechanism [11], the existence of broad distribution of relaxation times [10], linear processes [14], correlated pulses [12], fractal renewal [15], and linear diffusion [16]. On the other hand, authors not favoring the idea of universality claim the need to use statistical quantifiers distinct from the spectrum [13]. In this paper we argue that these positions are compatible if we assume a more complete spectral scaling. Our proposal is also to use the size of the series, T, as a relevant variable, obtaining a different scaling of the spectrum $S(f,T) \sim T^{\eta}/f^{\beta}$, with a different exponent that, as shown in the paper, depends on the underlying dynamics. The importance of this exponent in the analysis of time series has only been recognized in a recent paper [17]. With two relevant and independent exponents a more complete classification of processes can be established and a better identification of models is feasible. In this way it is possible to continue with the idea of universal behaviors, but with a greater variety of classes, and also to have a more precise guide for the formulation of a theoretical model. This procedure is akin to the generic scaling used in the growth analysis of interfaces where the spatial behavior is accounted for by considering the scaling of one local and one global variable [18]. This analysis has been widely used in experiments of interface growth in condensed matter, erosion, pattern formation in geophysics, etc. [19]. An application of the scaling method in surfaces to time series can be seen in [20]. The organization of the paper is as follows. Conditions of the time series to be analyzed and the proposal of a complete spectral scaling in terms of standard exponents are given in Sec. II. An essential classification based on this scaling follows in Sec. III, where the main classes are defined. Sections IV and V are devoted to illustrating the existence of these classes by using numerical generators of series. Focusing on the ubiquitous $1/f$ series we show how well-known numerical generators of these series belong to, at least, four universality classes. Finally, a discussion on the application of this analysis to experimental data concludes this paper.

II. SPECTRAL SCALING

Let us consider a time series, $Y(t) \equiv \{y_{t1}, y_{t2}, \ldots y_{tn}\}$, representing some numerical or empirical process in discrete or continuous time. Let us assume that, as result of some experiment or simulation, we have an ensemble of series $\{Y\}$ with a given preparation. As we have mentioned above, an important variable in our analysis is the size of the series, so we consider intervals of a given length T $\{Y_T\}$. We proceed by separating time mean values, $\overline{Y_T}$, and fluctuations, $\delta_T = Y_T - \overline{Y_T}$. The analysis of fluctuations is performed by means of the spectral density, $S_T(f) = \frac{1}{T}\widetilde{\delta_T(f)}\widetilde{\delta_T(-f)}$, where $\widetilde{\sharp}$ means Fourier transform (discrete or continuous depending on the variable) and $\overline{\sharp}$ is a time average. As the spectral density of a single realization is too noisy, an ensemble average (represented by $\langle\#\rangle$) is needed to smooth its shape. In many cases the averaged spectral density, $S(f,T) = \langle S_T(f)\rangle$, exhibits scaling as $S(f,T) = \frac{A}{f^\beta}$. This is the standard spectral scaling observed in so many situations. If we now take series with several sizes T, we can observe a new scaling of the spectrum amplitude as $A(T) \sim T^\eta$. Here it is worth remarking how the ensemble average works. If the ensemble is composed of series intervals extracted from only one series of a given experiment, which can be strongly interdependent, the shape of the averaged spectrum becomes smooth, but the amplitude maintains a strong dependence on the original series, producing a value $\eta \sim 0$. On the other hand, if the ensemble is formed of statistically independent samples, for any initial preparation the amplitude of spectra depends only on the size of the series, and the exponent η gives information of the system dynamics. This is the case we are interested in, so we consider an averaging with independent samples and take several sizes to have enough variation in the variable T. Then the sample averaged spectral density obtained in this way will exhibit scaling in both variables f and T as $S(f,T) \sim T^\eta/f^\beta$. Note that the complete spectral scaling can only be observed in cases where the sampling is abundant, that is, in experiments and simulations. Many processes of the real world are unique and other indirect methods should be applied to find, where possible, the complete scaling.

The aim of this paper is to show that this complete scaling is necessary to have the essential knowledge to identify a model and, in addition, that in many cases the exponent η, not used before, carries the specific dynamical information of the time series. Before doing this it is convenient to connect the spectral exponents β and η with other better known exponents coming from the scaling of widths and correlations. The standard scaling is usually written as $1/f^{2\alpha_s+1}$, α_s being the spectral exponent which accounts for the local scaling. For $\alpha_s \in (0,1/2)$ it coincides with the exponent of local width in fractal curves, $\alpha_{\rm loc}$, defined by the scaling of local widths as

$$w(\tau,T) = \langle\overline{[Y_T(t+\tau) - Y_T(t)]^2}\rangle^{1/2} \sim \tau^{\alpha_{\rm loc}}, \qquad (1)$$

and also with the roughness exponent, Hurt's exponent, etc. The global scaling is well characterized by the variance (or squared width), $W(T)^2 = \langle\overline{\delta_T(t)^2}\rangle$, which scales with the size as $W(T)^2 \sim T^{2\alpha}$. From now on α is the global exponent. Its connection with the original spectral exponents is easily obtained by the Parcival identity, $W(T)^2 = \int S(f,T)df$, obtaining $\eta = 2(\alpha - \alpha_s)$ when $\alpha_s > 0$ and $\eta = 2\alpha$ when $\alpha_s < 0$. Hence, we can write for the scaling of the spectra in terms of α and α_s:

$$S(f,T) \sim \frac{T^{2(\alpha-\alpha_s)}}{f^{2\alpha_s+1}}, \quad \alpha_s \geqslant 0, \qquad (2)$$

$$S(f,T) \sim \frac{T^{2\alpha}}{f^{2\alpha_s+1}}, \quad \alpha_s < 0. \qquad (3)$$

Note that this complete scaling can be also formulated in terms of local widths as

$$w(\tau,T) \sim \tau^{\alpha_{\rm loc}} T^{\alpha-\alpha_{\rm loc}}, \qquad (4)$$

where we have used (1) with $w(T,T) \sim W(T)$.

III. AN ESSENTIAL CLASSIFICATION

The form of this spectral scaling suggests a first classification of time series. On the one hand the spectral scaling with $\alpha_s < 0$ and $\alpha_s > 0$ are distinct, which suggests the classification of time series into noises, with $\alpha_s < 0$, and fractal curves, with $\alpha_s > 0$. The pure $1/f$ process with $\alpha_s = 0$ is an intermediated process that preserves the characteristics of both types. On the other hand the dependence of the spectra on the size T is another relevant fact since it implies the existence of one or two main scales in the scaling. Series whose spectra are independent of the size, with $\eta = 0$, possess a unique scale characterized by the spectral exponent α_s. In the case of noise ($\alpha = 0$, $\alpha_s < 0$) they are stationary in a strict sense, that is, with finite variance ($\alpha = 0$) and single spectral scaling. In the case of fractal curves ($\alpha = \alpha_s > 0$) they are self-affine, that is, with the same local and global scaling ($\alpha = \alpha_s$). Then, we define the class of stationary noise (SN) for the former case and the class of self-affine curves (SA) for the latter.

On the other hand, series with spectral size dependence have two relevant scales characterized by two exponents α and α_s. In this paper we are going to consider only classes concerned with the $1/f$ process, so we focus our analysis on the interval $\alpha_s \in [-\frac{1}{2},\frac{1}{2}]$ and, besides the two previous classes with single scaling SN and SA, we are going to consider two other classes with double scaling. One is the case with $\alpha = 0$ and $\alpha_s > 0$ that we call stationary fractal class (SF) and the other is the class of pure $1/f$ processes (1F), with $\alpha_s = 0, \alpha < 0$. Note that now the stationarity is only in a weak sense, that is with finite variance $\alpha = 0$ but double spectral scaling. Most of the noise generators existing in the literature belong to one of these four classes. A diagram with this classification in an α versus α_s plot is shown in Fig. 1.

The classes involving a single scale (SN and SA classes) should have a trivial form in their scale transformation. If $\xi_c(t)$ is a given process of these classes, with $c = \alpha_s$, the scaling $\xi_c(\lambda t) \sim \lambda^c \xi_c(t)$ can be obviously assumed. The cases with $c = -0.5$ and $c = 0$ correspond respectively to

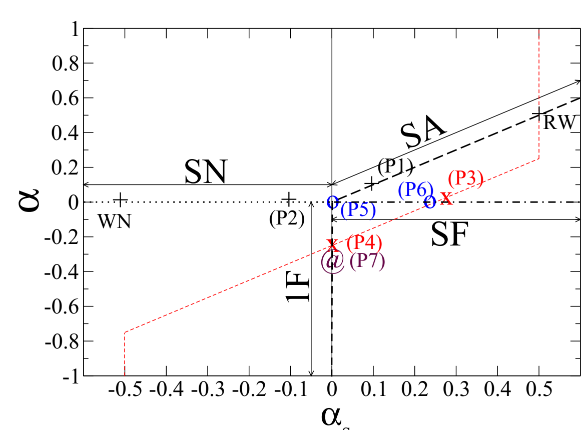

FIG. 1. (Color online) Classification of time series by their complete scaling spectra showing the main classes: stationary noises (SN) and stationary fractal curves (SF), self-affine curves (SA), and pure $1/f$ processes (1F). Points $P1$–$P8$ correspond to examples shown in the figures: (+, black) additive and (x, red) multiplicative processes, (o, blue) uncorrelated and (@, brown) correlated pulses.

stationary white (WN) and $1/f$ noises. The case with $c = 0.5$ corresponds to the random walk (RW) process. In contrast, the classes with two relevant scales should have a nontrivial scaling. As we will see throughout the paper, simulations of diverse time series suggest the scaling $Y_T(\lambda t) \sim \lambda^\alpha Y_T(t)$, which means that, surprisingly, the exponent related with the scale transformation, and therefore with the underlying mechanisms, is the global exponent α. The measure of such an exponent has been traditionally ignored. In this paper we use time series synthetically generated to illustrate our theory. We have considered two common ways of generating time series, by adding pulses and by simulating stochastic equations. Samples of each ensemble are independently generated. Their number should be enough to assure a good spectral resolution (100 samples for additive processes and 1000 samples for the remainder). In this way the averaged spectrum plotted in a log-log representation is nearly a straight line, as can be seen in the figures. However, with a more reduced sampling (one order of magnitude less), the complete spectral scaling is feasible, obviously with more error in the determination of exponents. Initial conditions are arbitrary, simply they should be adequate to measure the increase (when $\alpha > 0$) or decrease (when $\alpha < 0$) of variance with T in the range of analysis. Our aim is not to study in detail generation methods but only to explain why distinct features produce distinct classes. Hence, we use scaling arguments, which are not mathematically rigorous, but our results are always corroborated by numerical simulations. Moreover, besides the analysis of spectra and widths, which are essential to formulate the complete spectral scaling, and for the sake of illustration, we also plot single samples of the generated process to visually compare similitudes and differences between samples of the same class.

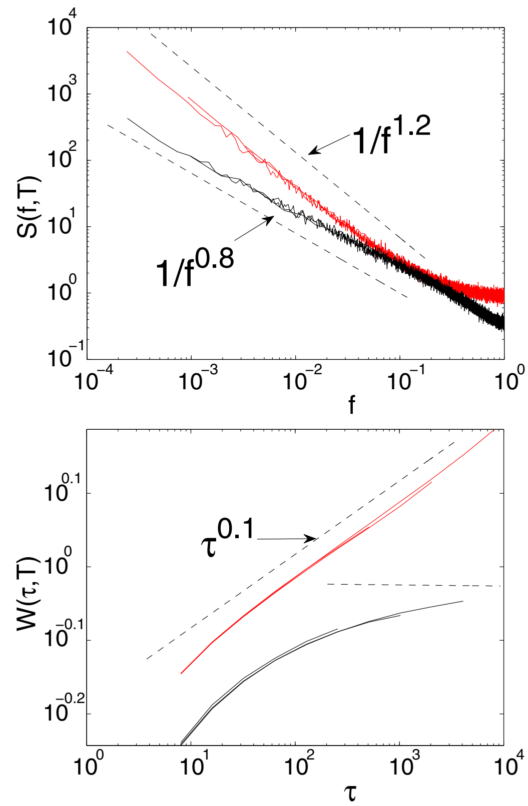

FIG. 2. (Color online) Spectra (upper) and local widths (lower) of SA ($P1$, $\alpha = \alpha_{\rm loc} = \alpha_s = 0.1$) and SN ($P2$, $\alpha = \alpha_{\rm loc} = 0$, $\alpha_s = -0.1$) series generated from a fractional-Brownian method as explained in the text. The initial condition is a Gaussian with $\sigma^2 = 2.14$, the number of samples is 100, and the sizes of series are $T = 2^9, 2^{11}, 2^{13}$.

IV. ADDITIVE AND MULTIPLICATIVE PROCESSES AS GENERATORS OF SINGLE AND DOUBLE SPECTRAL SCALING CLASSES RESPECTIVELY

Equations with additive noise or, more generally, the use of linear transformations of single scale series produce processes with single scaling. The simplest example is the derivative (or integration) of series. Derivation of series in the SA class with $\alpha = \alpha_{\rm loc}$ gives series in the SN class with $\alpha'_s = \alpha_{\rm loc} - 1$, $\alpha' = \alpha - \alpha_{\rm loc} = 0$. Another example is the convolution. In general we have

$$\xi_{c'}(t) \sim \int g(t - t')\xi_c(t')dt', \qquad (5)$$

with $c' = c + \theta$, when the scaling of the kernel is $g(\lambda t) \sim \lambda^\theta g(t)$. Usually $\xi_c(t)$ is a white random noise, $c = -0.5$, and then a kernel with $\theta = 0.5$ is necessary to have an $1/f$ noise [10]. Processes known as fractional Brownian motion (FBM) are also generated using convolutions with white noise. When this noise is Gaussian the generated process is also Gaussian. To generate self-affine series with exponents in the interval $\alpha = \alpha_{\rm loc} \in (0,1)$ we have used the Levinson algorithm implemented in the MATLAB framework, which is, basically, a discretization of (5). This is a standard method to get FBM processes. In Fig. 2 the averaged spectra and widths for series with several sizes T and value $\alpha_s = \alpha_{\rm loc} = 0.1$ are plotted. Series in the SN class can be generated from the derivative of series in the SA class. In our case we obtain series with $\alpha_s = -0.1$, $\alpha = 0$ from the derivative of SA series with $\alpha_{\rm loc} = \alpha = 0.9$. It can be seen in this figure that in both cases the spectra and widths for distinct sizes do not shift but collapse to a unique curve. This is the sign of single scaling. Series with this property are plotted in Fig. 3. They represent a transition from a fractal curve ($P1$) to a correlated noise ($P2$), the stationary $1/f$ process ($P6$) being a frontier between them. It is worth remarking that this process is an unattainable limit within the FBM methods. Note that we have used $\alpha_{\rm loc}$ instead of α_s in some cases. This is because in the FBM the spectral exponent collapses to $\alpha_s = 1/2$ for $\alpha > 1/2$ and then the correct measure of local scaling is the local exponent $\alpha_{\rm loc}$. In this paper we have focused our analysis on the case $\alpha_{\rm loc} < 1/2$ where the spectral and local exponent coincides, so the spectral exponent serves to characterize the local scaling. A more complete analysis going beyond this case will be shown in a future paper.

On the other hand, multiplicative processes typically produce series with a double scaling in the spectra. Let us consider a generalized multiplicative process given by the stochastic equation

$$\dot{Y}(t) = (\mu - \epsilon\kappa)Y^{2\mu-1} + \xi(t)Y(t)^\mu, \qquad (6)$$

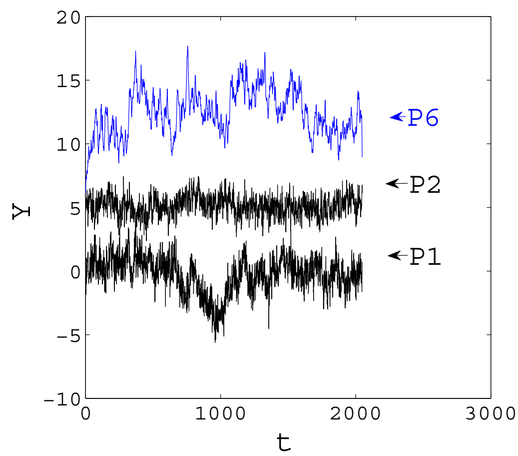

FIG. 3. (Color online) Plot of individual series (with $T = 2^{11}$) showing cases with single scaling: In the SA class, $P1$ ($\alpha = \alpha_s = 0.1$), stationary noise, $P2$ ($\alpha = \alpha_s = -0.1$), and stationary $1f$ process, $P6$ ($\alpha = \alpha_s = 0$). Fractality of $P1$ and (less evident) $P6$ can be observed by eye. $P2$ and $P6$ are displaced upwards by 5 and 10 units respectively.

with $\mu \neq 1$, $\xi(t)$ being a Gaussian white noise and taking $\epsilon = 1/2$ in the Ito prescription and $\epsilon = 1$ in the Stratonovich prescription. With the appropriate boundary conditions the process becomes positive and stationary with a probability density $P(Y) \sim Y^{-\kappa}$. In a free evolution from an initial condition one can see that in a region of Y the probability density evolves as $P(Y,t) \sim Y^{-\kappa} F(t)$. If we now apply our scaling laws, taking $Y(\lambda t) \sim \lambda^{\alpha} Y(t)$, substituting this law into Eq. (6), with the scaling of the white noise $\xi(\lambda t) \sim \lambda^{-1/2}\xi(t)$, and equating powers, we obtain for the global exponent

$$\alpha = \frac{1}{2(1-\mu)}. \tag{7}$$

This important result means that the global scaling of stochastic equations driven by multiplicative white noise only depends on the multiplicative power μ. The local exponent α_s cannot be deduced from scaling arguments but it can be obtained from geometrical considerations. On the one hand it can be seen that the local exponent is limited to values in the interval $[-0.5, 0.5]$, becoming saturated at the extreme values. On the other hand it is also observed that when the local scaling is not saturated, $-0.5 < \alpha_s < 0.5$, the shape of the probability density $\sim Y^{-\kappa}$ and the exponents α and α_s are mutually dependent. Hence, it can be expected that critical changes in this shape, which occur for $\kappa = 1$ and $\kappa = 3$, would define the limits between classes which occur for $\alpha = \alpha_s$ and $\alpha_s = 0$. One simple formula that accounts for these conditions would be $\alpha_s = \frac{(3-\kappa)}{2}\alpha$. Substituting α by (7) the formula for the spectral exponent then reads

$$\alpha_s = \frac{3-\kappa}{4(1-\mu)}, \tag{8}$$

and coincides with the formula obtained in [21] by a direct calculation of the spectra of series with correlated pulses. It is worth remarking that in general (excepting the case $\kappa = 1$) a multiplicative process will exhibit a double scaling spectrum. The role played by the exponents α, α_s, and κ is also important

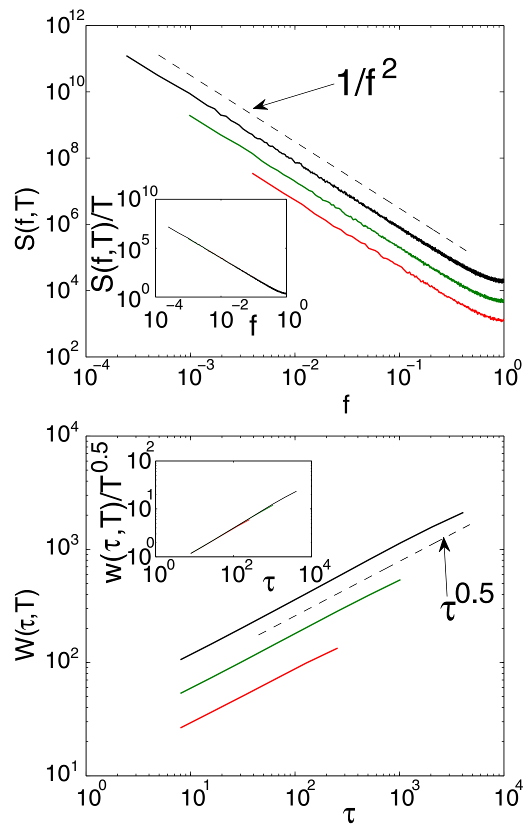

FIG. 4. (Color online) Spectra (upper) and local widths (lower) with their respective collapses (insets) of a multiplicative process ($P3$, $\alpha = 1$, $\alpha_{\rm loc} = \alpha_s = 0.5$) generated as the $q = 2$ power of a random walk (RW). The initial condition is Gaussian with $\sigma = 1$, the number of samples is 1000, and the sizes of series are $T = 2^9, 2^{11}, 2^{13}$.

to remark. While α depends exclusively on the multiplicative exponent μ, α_s, and κ seem to be more dependent on details. Our simulations corroborate these results. The simplest example is a pure multiplicative process in the Stratonovich prescription, $\kappa = \mu$, that can be numerically obtained as the q power of a positive RW process. Another multiplicative process with $\mu = 1 - 1/q$ and $\kappa = \mu = 1 - 1/q$ is then obviously generated. In this case the algorithm is easy to implement and numerically stable for any q and μ. We simply use the map $Y_{i+1} = Y_i + \xi_i, \xi$ being a Gaussian normal process (Y is a Wiener process) and then we take $|Y|^q$. We have obtained the predicted values of $\alpha_s = \frac{2q+1}{4}$ for $q \in [-3/2, 1/2]$ and $\alpha = q/2$ in the complete range of validity (dashed red line in Fig. 1). Note that the case with $q = 0$, that would produce stationary processes (series in the SF and SN classes), is an unattainable limit ($\mu \to -\infty$) for a multiplicative process. For $q > 1/2$ α_s saturates to $1/2$ but $\alpha = q/2$ as given by (7). In Fig. 4 we show a spectral analysis and an analysis with local widths for the case $q = 2$ ($\alpha_s = 1/2$, $\alpha = 1$). The shift with the size, observed in both figures, is the sign of double scaling. The local exponent α_s, or $\alpha_{\rm loc}$ in each case, is calculated from the slope of the corresponding curves. As expected, we have $\alpha_s = \alpha_{\rm loc} = 1/2$. To get the global exponent α we plot $S(f,T)T^{-2(\alpha-\alpha_s)}$ against f and $w(\tau,T)T^{-(\alpha-\alpha_{\rm loc})}$ against τ respectively. According to the scaling formulas (3)

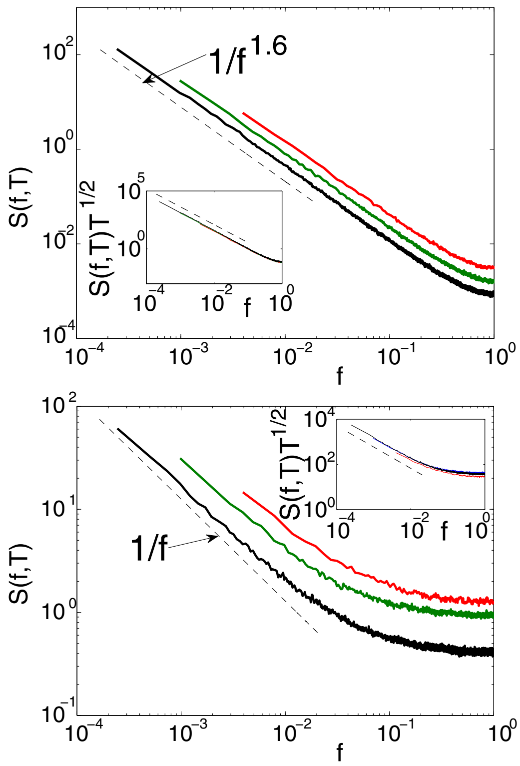

FIG. 5. (Color online) Spectra and collapses (insets) of two multiplicative processes generated as the q power of a RW with $\sigma_0 = 1$ as initial condition. Upper with $q = 0.1$ ($P4$, $\alpha = 0.05$, $\alpha_s = 0.3$) in the limit of the SF class. Lower with $q = -1/2$ ($P5$, $\alpha = -1/4$, $\alpha_s = 0$) in the 1F class. As in the previous figure $T = 2^9, 2^{11}, 2^{13}$ with 1000 samples.

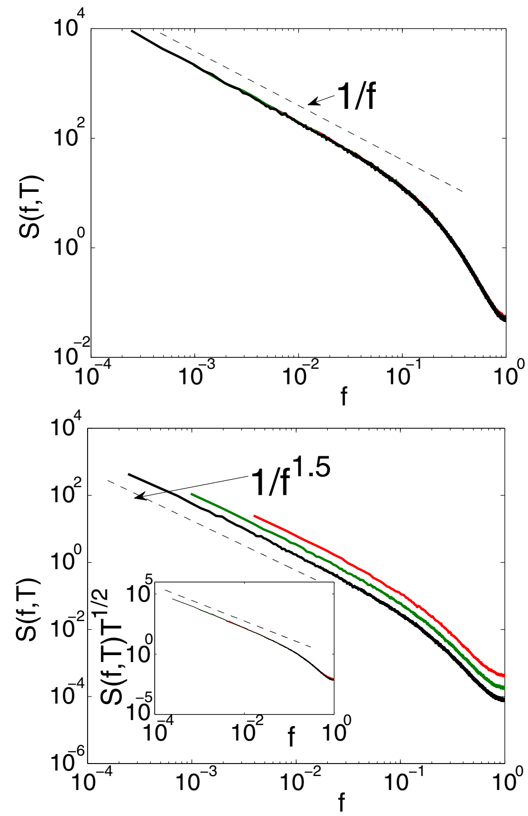

FIG. 6. (Color online) Spectra and collapse (inset) of series of uncorrelated pulses. Upper, a typical example of stationary 1F noise ($P6$, $\alpha = \alpha_s = 0$) generated from exponential pulses with a uniform distribution of widths and a Poissonian renewal time. Lower, an example of the SF class ($P7$, $\alpha = 0$, $\alpha_s = 1/4$) generated from exponential pulses with the same width than their renewal time, which is power law distributed. The same sizes and number of samples as in the previous figures.

and (4) curves of distinct size should be coincident in this plot. Then we take $\alpha_s = \alpha_{\text{loc}} = 1/2$ with varying α until we obtain the best possible collapse of curves with distinct size. This situation is observed for $\alpha = 1$ as expected. We show this collapse in the inset of both figures. This method will be followed throughout the paper where, for the sake of conciseness, only the spectral analysis will be presented. In Fig. 5 we show the spectral analysis in two interesting situations: First (upper) in the limit to the SF class with $q = 0.1$ ($\alpha = 0.05$) and second (lower) a new way of generating $1/f$ processes, with $q = -1/2$ ($\alpha_s = 0$, $\alpha = -1/4$).

V. SERIES OF UNCORRELATED AND CORRELATED PULSES

Time series in the case of uncorrelated pulses can be written as $Y(t) = \sum_l h(t - t_l, \theta_l)$, $h(\tau, \theta)$ being the shape of the pulse or relaxation process centered in $\tau = 0$, t_l the times where the pulse takes place, and θ a parameter quantifying the shape of the pulse. The spectral density can be directly calculated as

$$S_T(f) = \frac{1}{T} \sum_{l,m} \exp\left[-if(t_l - t_m)\right] \tilde{h}(f, \theta_l) \tilde{h}(-f, \theta_m). \quad (9)$$

Here we have only treated short ranged pulses whose finite Fourier transform is not dependent on the position t_l and integration range T. Considering point processes, where the renewal times, $\tau_l = t_{l+1} - t_l$, are uncorrelated, so as the shape parameters θ_l, and ignoring the off diagonal ($l \neq m$) terms, assuming that the coefficient $\exp[-if(t_l - t_m)]$ acts as a random phase [17], we have

$$S(f,T) \sim \frac{N(T)}{T} \langle |\tilde{h}(f,\theta)|^2 \rangle. \quad (10)$$

$N(T)$ is the mean number of pulses in the interval T. If the mean renewal time $\overline{\tau}$ is well defined and finite, $N(T) = \frac{T}{\overline{\tau}}$ and the size T is irrelevant in the spectra. Thus we have a case with single scaling whose spectral exponent is determined by the shape and distribution of pulses. For instance, if the shapes of pulses scale as $|h(f,\theta)| \sim \theta^{-a} g(f/\theta)$ and they are distributed as a power law with $P(z\theta) \sim z^{-b} P(\theta)$, the spectral exponent becomes $\alpha_s = a + b/2 - 1$. The generation of a stationary $1/f$ process ($\alpha = \alpha_s = 0$) with exponential pulses, $h(t,\theta) = \pm\exp(-\theta|t|)$ ($a = 1$), then requires a uniform distribution $P(\theta) = cte$ ($b = 0$). In Fig. 6 we show the spectral analysis of such a case using a Poissonian distribution of renewal times, with $P(\tau) = \exp(-\tau)$, a random amplitude of pulses with value ± 1 and an uniform probability, $P(\theta) = 1$, $\theta \in (1/T, 1/T + 1)$. This is an old way of simulating an

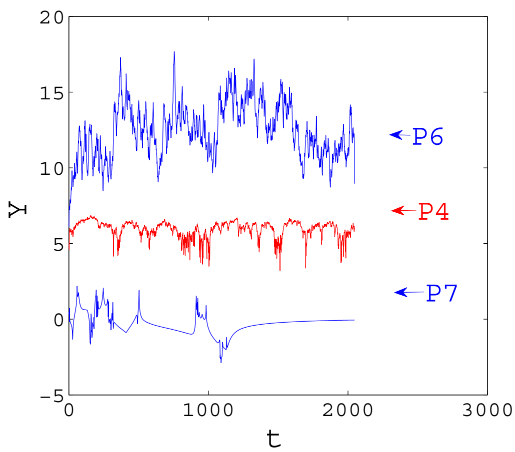

FIG. 7. (Color online) Plot of individual series (with $T = 2^{11}$) showing samples nearly of the SF class ($\alpha \sim 0$, $\alpha_s \geqslant 0$) with distinct spectral exponents: $\alpha_s = 0.25$ ($P7$), $\alpha_s = 0.3$ ($P4$), $\alpha_s = 0$ ($P6$). $P4$ and $P7$ have closed exponents but their shape is very different. $P4$ and $P6$ are displaced upwards by five and ten units respectively. $P4$ is amplified by a factor of 5.

effective stationary $1/f$ process with ($\alpha = \alpha_s = 0$) for series of finite size [22]. On the other hand if the mean renewal time is dependent on T we have a case of double scaling in the spectra. As an example we can consider the case with a power law in the probability of waiting times $P(\tau) = (d-1)\tau^{-d}$ with $1 < d < 2, \tau \in (1,\infty)$, where $\overline{\tau} = \frac{d-1}{2-d}(T^{2-d} - 1)$. In this case, comparing (3) with (10), we have $2(\alpha - \alpha_s) = d - 2$, and if, in addition, the process is stationary ($\alpha = 0$) the spectral scaling would read

$$S(f,T) = \frac{T^{d-2}}{f^{3-d}}. \quad (11)$$

This is a good example of the SF class with $\alpha = 0, \alpha_s = 1 - d/2 \in (0,1/2)$. It has been used for studies of $1/f$ noise in [17] using flat pulses with value ± 1. In Fig. 6 we show the spectral scaling of a similar case, now with exponential pulses with the same width that their renewal time, $h(t,\tau_i) = \pm\exp(-|\frac{t}{\tau_i}|)$, and a value of $d = 1.5$. In Fig. 7 a plot of this generated series is shown ($P7$), and can be compared with a series generated with a pure multiplicative process ($P4$). Although both examples have closed exponents the visual differences are appreciable since, as already mentioned in the latter section, multiplicative processes are critical in $\alpha \sim 0$.

Series of correlated pulses are often used as generators of $1/f$ processes. In a series of correlated pulses the shape of the pulses becomes irrelevant at long times (see Fig. 8) and the scaling of spectra is only determined by the process of the interpulse time $\{\tau_k\}$. When this process comes from a multiplicative process with parameters μ and κ it is easy to see [21] that series of flat pulses with amplitude τ_k^{-1} are in fact multiplicative processes with $\mu' = 5/2 - \mu$ and $\kappa' = 3 - \kappa$ (in the Ito prescription). This analogy of correlated pulses with multiplicative processes allows us to calculate the scaling exponents. Hence, if we use pure multiplicative processes in the Ito prescription ($\mu = \kappa/2$) for the generation of interpulse times, and we want to generate $1/f$ series, that is $\alpha_s' = 0, \kappa' = 3$, we would require $\mu = \kappa = 0$, that is, a RW as time generator. The value of α' is obtained substituting $\mu' = 5/2$ in (3) to get $\alpha' = -1/3$. In Fig. 8 we corroborate this result. We use a Wiener process, $\tau_{k+1} = \tau_k + \xi_k$ (ξ_t being a Gaussian process), as generator of interpulse times, $\{|\tau_k|\}$, and exponential pulses $\exp(-|t|)$ with uniformly distributed amplitude in (0,1). This is a good example of a series in the 1F class that is often used because of the perfect $1/f$ shape obtained. In Fig. 9 several series with a perfect $1/f$ shape are plotted. It is illustrative to see how a pure multiplicative process ($P5$) seems to be formed of correlated pulses as in $P8$. However, they are not stationary ($\alpha < 0$) as $P6$ is ($\alpha = 0$). Note that this property cannot be detected by visual inspection. To have stationarity an Oerstein-Uhlenbeck process as $\tau_{k+1} = (1 - \frac{1}{T_s})\tau_k + \xi_k$ is currently used in the generation of correlated pulses [23], but then, at times where the process becomes stationary, $T > T_s$, the spectrum saturates to a WN with $\alpha_s = -0.5$, $\alpha = 0$.

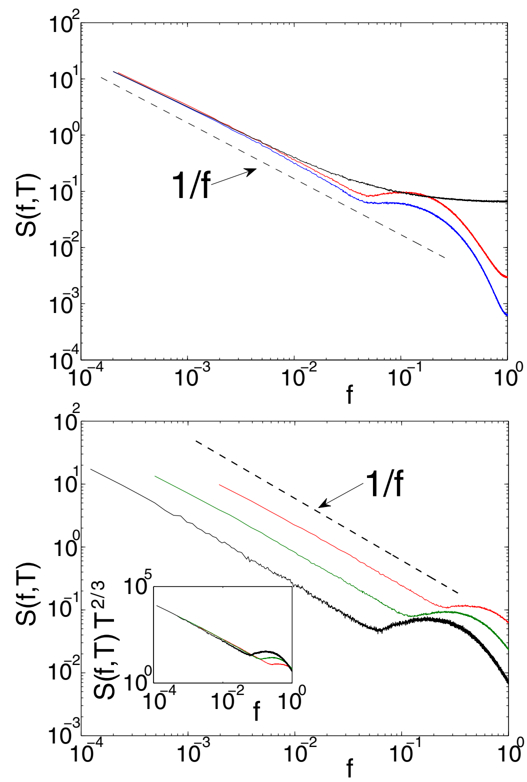

FIG. 8. (Color online) Spectra of series of correlated pulses as examples of the 1F class ($P8, \alpha = -1/3, \alpha_s = 0$). Upper, with a positive Wiener process with $\sigma_0 = 1$ for the interevent times and distinct shapes of pulses: flat (black), exponential (red), and Lorentzian (blue). Lower, for series of exponential pulses with distinct sizes, from top to bottom $T = 2^9, 2^{11}, 2^{13}$. The inset shows the collapse of spectra with $\alpha = -1/3$. In all cases each spectrum is an average of 1000 samples.

VI. CONCLUSIONS AND DISCUSSION

A more complete spectral scaling is here proposed for the analysis of time series. Two independent exponents associated to a local and a global scaling allow a classification of time series in different universality classes. For the illustration of the

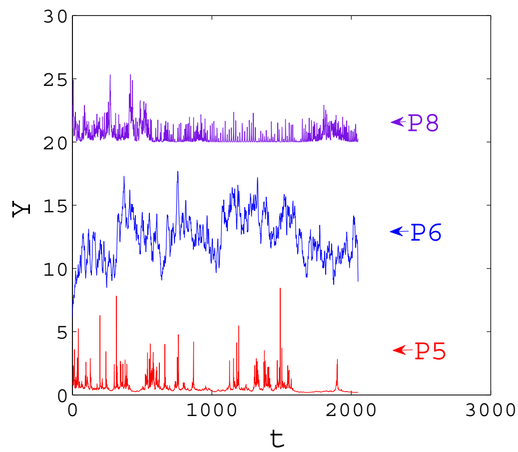

FIG. 9. (Color online) Plot of individual series showing samples in the 1F class ($\alpha_s = 0$) with distinct global exponent: $\alpha = -0.25$ ($P5$), $\alpha = -0.33$ ($P8$), and $\alpha = 0$ ($P6$). Despite the distinct nature of the generating process $P5$ and $P8$ look similar. $P6$ and $P8$ are displaced upwards by 10 and 20 units respectively. $P8$ is amplified by a factor of 5.

theory we use synthetically generated time series. Concerning research on $1/f$ noise, series in four main universality classes, with single and double scaling, are generated and analyzed with our complete spectral scaling to support our findings.

The application of the complete scaling analysis to empirical series certainly would significantly improve the current situation achieved by the standard method. It is clear that the existence of one more independent exponent and of several universality classes provides a better description of the possible underlying mechanism acting on the system dynamics and clarifies the true ambiguity occurring in many of the fluctuation analyses appearing in the References.

When the empirical time series to be analyzed is unique, which occurs frequently in natural processes, the global scaling cannot be directly measured by the exponent α since the extracted pieces of series of size T can be strongly correlated. This is not a limitation of this method but a kind of statistical indetermination which is intrinsic in series with strong correlation. So, no other direct method exists to deal with this scaling. Then another kind of analysis, which indirectly provides information about it, should be applied. For instance, if the series comes from a multiplicative process it is possible to measure the exponent μ which, if the noise is white, is directly related to the exponent α (7). If it is a series of pulses the interpulse statistics gives an indirect way of estimation of the global scaling. Other cases have to be specifically studied.

Note that our concept of class is more generic than the concept of mechanism, although in some cases both coincide. For instance, linear mechanisms of any kind, coming from space or time processes [14,16], can be always associated to single scaling classes since, in essence, they come from convolutions with noise. In contrast, SOC type mechanisms, coming from the existence of internal or external thresholds, produce, in general, correlated [24] or uncorrelated [25] relaxation processes, and therefore they generate series in single or double scaling classes. So, a class represents a more essential concept since it is associated to a more evident symmetry.

Ambiguity occurs since the underlying mechanism cannot be deduced by the value of the spectral exponent only. As shown in the paper, it is just the global exponent which best informs us about the system dynamics. Hence, in many cases the choice of model is more a consequence of intuition than of information. There are many examples in the literature where the same processes are related to distinct mechanisms, that is, possibly to the distinct universal classes in our theory. Without being exhaustive we mention here some examples. Earthquakes are modeled either by SOC models with uncorrelated [26] (SN class) or correlated [24] (1F class) relaxation times. Musical series exhibiting $1/f$ fluctuations have been modeled by analogical devices [5], numerical algorithms [27], and also as a process of correlated pulses (1F class) [6]. Stock market activity (volatility) has been seen as a FBM process (SA class) [2], a kind of SOC model (probably in the SN class) [28], and also as a process of correlated pulses (1F class) [29,30]. Physical systems exhibiting intermittency have been associated to a multiplicative process (1F class) [31] and also to a renewal process (SF class) [17].

Finally, we would like to remark that, using this method, not only is a better classification possible, but also a more precise estimation of the underlying mechanism is feasible. As a representative example we take the model of correlated pulses with a RW interpulse process [12], which is very often used [6,23,29,30], probably by the perfect $1/f$ shape obtained. As we show in the paper this is a specific case of the 1F class with $\alpha = -1/3$, that is, with a strong correlation in the interpulse process. We guess that, in general, empirical series of this class have to exhibit more or less correlation, that can be quantified by the exponent α. A more detailed model could be so formulated once the exponent α is known. Note that this exponent is more robust and therefore more essential for the dynamics than κ, which is the common fitted exponent. A paper showing these ideas is now in preparation.

Not only in these cases, but in general is it clear that a better analysis of the empirical series such as the one proposed in this paper will determine the true class. Therefore it provides a more precise model of the investigated phenomena. So, we conclude by encouraging experimentalists to use this method when the necessary sample preparation is feasible and theoreticians to develop methods for the cases of empirical series without sampling.

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