PHYSICAL REVIEW E **90**, 041001(R) (2014)

Transition from dissipative to conservative dynamics in equations of hydrodynamics

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(Received 25 March 2014; published 6 October 2014)

We show, by using direct numerical simulations and theory, how, by increasing the order of dissipativity (α) in equations of hydrodynamics, there is a transition from a dissipative to a conservative system. This remarkable result, already conjectured for the asymptotic case $\alpha \to \infty$ [U. Frisch *et al.*, Phys. Rev. Lett. **101**, 144501 (2008)], is now shown to be true for any large, but finite, value of α greater than a crossover value $\alpha_{\mathrm{crossover}}$. We thus provide a self-consistent picture of how dissipative systems, under certain conditions, start behaving like conservative systems and hence elucidate the subtle connection between equilibrium statistical mechanics and out-of-equilibrium turbulent flows.

DOI: 10.1103/PhysRevE.90.041001 PACS number(s): 47.27.Gs, 05.20.−y, 47.10.ad

Since the pioneering work of Hopf [1] and Lee [2] over 60 years ago, physicists have tried to understand the strongly out-of-equilibrium, dissipative turbulent flows by using tools of classical equilibrium statistical mechanics. What makes such attempts difficult is that although, from a microscopic point of view, fluid motion can be modeled via a Hamiltonian formulation, with statistically steady states having an invariant Gibbs measure, a self-consistent macroscopic approach inevitably leads to a dissipative hydrodynamical description with an irreversible energy loss. In the last few years, however, significant work has gone into our understanding of the interplay between equilibrium statistical mechanics and turbulent flows [3–9]. In particular, thermalized solutions to the Galerkin-truncated equations of hydrodynamics, such as the three- (3D) or two-dimensional (2D) Euler [3,10], Gross-Pitaevskii [11], and magnetohydrodynamic [12] equations and the one-dimensional (1D) Burgers equation, have been studied extensively by several authors [5,9]. Thus we can obtain a conservative dynamical system, which obeys Gibbsian statistical mechanics, for hydrodynamical equations of an ideal fluid where only a finite number of Fourier modes are retained via Galerkin truncation [3,9,13] whose existence was shown by Cichowlas *et al.* [3], for the incompressible, truncated 3D Euler equations, and the explanation of how thermalization sets in such systems was given by Ray *et al.* [9] through the phenomenon of *tygers*. In this Rapid Communication we now show, analytically and numerically, that thermalization can set in not only for inviscid equations but for viscous equations of hydrodynamics through a resonance effect triggered by waves generated in certain boundary layers.

Since thermalized states are discussed for finite-dimensional, conservative systems obeying a Liouville theorem, it is important to ask if there are connections between such states and dissipative, turbulent flows described by viscous Navier-Stokes-like equations. Frisch *et al.* [14] showed that the energy spectrum bottleneck, a bump in the spectrum between the inertial and dissipation ranges, is due to aborted thermalization. By using direct numerical simulations (DNSs) of the hyperviscous Burgers equation (HBE) and eddy-damped-quasinormal-Markovian calculations [15] of the 3D hyperviscous Navier-Stokes equation, it was shown that if we replace the usual viscous operator $\nu\nabla^2\mathbf{u}$ by the hyperviscous operator $-\nu(-\nabla^2)^\alpha\mathbf{u}$, where ν is the coefficient of viscosity, α is the order of hyperviscosity (dissipativity), and $\mathbf{u}$ the velocity field, the bottleneck becomes stronger with increasing α. The authors observed that for extremely large values of α, the bottleneck is due to partial thermalization [3] and that the large α limit yields thermalized states.

The large α limit [14] is extremely important from the point of view of our understanding of hydrodynamical equations. However in most DNSs much smaller values of $\alpha \leqslant 16$ are typically used, which, nevertheless produce significant bottlenecks. Recently, a more complete explanation of this effect was given in [16] where it was shown that this bottleneck has its origins in oscillations in the velocity correlation function. This mechanism is, *apparently*, very different from the aborted thermalization for large α proposed in [14]. For other detailed studies on the bottleneck effect, see [17–23].

Although some of the previous work on the hyperviscous equations sought to explain bottlenecks, in this Rapid Communication we answer an entirely different question, namely, can the *apparent paradox*, when going from small to large values of α, be resolved? We show how, by increasing α, a crossover from one regime [14] to another [16] occurs and thus resolve the paradox. This remarkable result was already conjectured in [14] for the asymptotic case $\alpha \to \infty$; in this Rapid Communication we show, by using both DNSs and theory, that this is already the case for a large, but finite, value of α. We thus provide a self-consistent picture of how dissipative systems can start behaving like conservative systems and thus elucidate the subtle connection between equilibrium statistical mechanics and out-of-equilibrium turbulent flows.

The Burgers equation has had a long history of being a testing ground for such ideas related to fluid dynamics [24], and more recently the chaotic behavior in conservative systems [9]. Therefore, we begin with the 1D, unforced, HBE:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = -\nu\left(-\frac{1}{k_d^2}\frac{\partial^2}{\partial x^2}\right)^\alpha u, \qquad (1)$$

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where u is the velocity field, x and t are the space and time variables, respectively, ν is the coefficient of kinematic hyperviscosity, and k_d a reference wave number [14,16]. In the limit $\nu \to 0$, with $\alpha \geqslant 2$, the solution to Eq. (1) develops oscillations in the boundary layer around the shock. These oscillations—which have been studied by using boundary-layer-expansion techniques [16]—are localized in the neighborhood of the shock and decay exponentially as one moves away. The wavelength $\lambda_\alpha^{\rm th}$ and the decay rate $K_\alpha^{\rm th}$ of these oscillations are

$$\lambda_\alpha^{\rm th} = 2\pi \nu^\beta k_d^{-2\alpha\beta}\{2^\beta \sin[(2n_\star + 1)\beta\pi]\}^{-1}, \tag{2}$$

$$K_\alpha^{\rm th} = 2^\beta \nu^{-\beta} k_d^{2\alpha\beta} \cos[(2n_* + 1)\beta\pi]; \tag{3}$$

$\beta = \frac{1}{2\alpha - 1}$ and n_* is an integer, $0 \leqslant n_* \leqslant 2\alpha - 2$, whose value is obtained via linearization and boundary-layer analysis [16].

For extremely large values of $\alpha \geqslant 500$, the solution of Eq. (1) starts thermalizing [14] and, at long times, becomes indistinguishable from the solution $v(x,t)$ of the associated inviscid, *conservative* Galerkin-truncated Burgers equation (GTB) [9]: $\frac{\partial v}{\partial t} + {\rm P}_{K_{\rm G}} \frac{1}{2}\frac{\partial v^2}{\partial x} = 0$; the Galerkin projector ${\rm P}_{K_{\rm G}}$ is a low-pass filter which sets to zero all Fourier components with wave numbers $|k| > K_{\rm G}$.

At this stage, it behooves us to ask what happens for intermediate values of α? Furthermore, is there a mechanism which can self-consistently describe the transition from a turbulence regime to a thermalized state in equations of hydrodynamics?

The onset of thermalization, in the finite-dimensional Euler or the inviscid Burgers equation, is due to the birth of localized oscillatory structures called *tygers*. These are caused [9] by the motion of fluid particles interacting resonantly with the waves generated, because of truncation, by small-scale features, such as shocks. The special points $x_{\rm s}$ in space where tygers appear in the case of the GTB, are points which have the same velocity as the shock(s) and a positive local gradient. Is there another way, apart from truncation waves in inviscid systems, for waves to be generated at the stagnation points in a fluid for similar resonant interactions leading to an onset of thermalization? We show that for α greater than a crossover value $\alpha_{\rm crossover}$, a significant fraction of the oscillations, governed by Eqs. (2) and (3), which start from the boundary layer near the shock, must reach $x_{\rm s}$ and trigger tygerlike structures leading to thermalization.

In order to answer these questions we first perform pseudospectral DNSs of Eq. (1) on a 2π periodic line, with a second-order Runge-Kutta scheme for time integration. We use a time step $\delta t = 10^{-4}$, the number of collocation points $N = 16\,384$, $\nu = 10^{-20}$, and $k_d = 100$. Crucially, we use $2 \leqslant \alpha \leqslant 500$ to study this transition from dissipative dynamics to conservative, thermalized states. Our initial condition $u_0(x) = \sin(x + 1.5)$ leads to $x_{\rm s} = 2\pi - 1.5 \approx 4.8$ and, in the absence of viscosity, shock formation at time $t_* = 1.0$.

We begin our simulations from $\alpha = 2$ and observe [16] that with increasing α, oscillations in a thin layer around the shock become pronounced. However, near $x_{\rm s}$, no oscillations are seen for $\alpha \lesssim 40$ (Fig. 1, inset). However, as α increases, finite, but small, oscillations start to reach $x_{\rm s}$ from the boundary layer around the shock (Fig. 1). Furthermore, for values of $\alpha \gtrsim 80$,

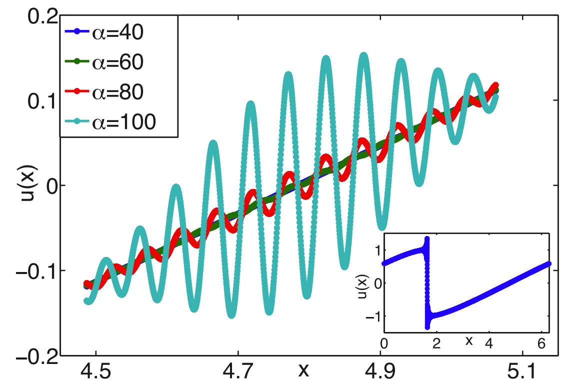

FIG. 1. (Color online) Solutions of the HBE, zoomed around $x_{\rm s}$, for various values of α (see legend) at $t = 1.5$ showing oscillations at $x_{\rm s}$ with increasing amplitude as α increases (see text). Inset: Solution of the HBE with no oscillations at $x_{\rm s}$ for small $\alpha = 20$.

a distinct bulge, reminiscent of the tygers found in solutions of the GTB [9], is clearly seen at $x_{\rm s}$. This, then, is the first evidence of what triggers thermalization in a dissipative system and whose dramatic consequences were studied in Ref. [14] for the special case of $\alpha \to \infty$.

How similar is this bulge at $x_{\rm s}$ for $\alpha \gtrsim 80$, to that seen at t_* for the Hamiltonian system of the GTB? In order to answer this question, it is useful to examine the bulge, via $u_{\rm subtracted} = u - U$, where U is the (nonoscillatory) solution of the inviscid Burgers equation. In Fig. 2 we show this subtracted bulge for $\alpha = 100$ and find that this bulge has the same symmetric shape as tygers [9]. The wavelength of these oscillations, as is expected from a resonance build-up argument, is the same as the wavelength of the oscillations emanating from the boundary layer (2). A significant difference between the bulge observed for moderate values of α (Figs. 2 and 1), and that of a tyger [9], is its large width and the small number of oscillations inside it. In the truncated system, the bulge width is proportional to $K_{\rm G}^{-1/3}$ and the wavelength of the oscillations proportional to $1/K_{\rm G}$; this yields the number of oscillations in the bulge to be proportional to $K_{\rm G}^{2/3}$. In the present problem, the

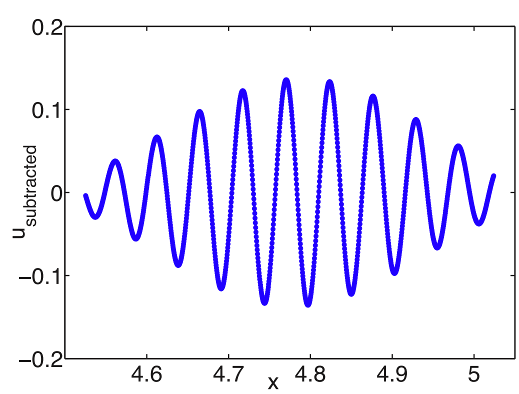

FIG. 2. (Color online) The solution of the HBE for $\alpha = 100$, with the solution for the ordinary Burgers subtracted out, zoomed around $x_{\rm s}$. A clear symmetric bulge, similar to those seen in inviscid, conservative truncated systems [9], is seen.

width of the bulge is explained as follows: At time t, such that $\tau = t - t_* \sim O(1)$ (when the bulge is still symmetric around x_s), resonant interactions are confined to particles such that $\tau \Delta u \equiv \tau |u - u_{\rm shock}| \lesssim \lambda_\alpha^{\rm th}$, where $u_{\rm shock}$ is the velocity of the shock. Since, $\tau \sim 1$, this leads to $\Delta u \sim \lambda_\alpha^{\rm th}$. Given that around x_s the velocity is proportional to $x - x_s$, this yields a bulge width $\sim \lambda_\alpha^{\rm th}$ with a few oscillations inside.

In the case of the GTB, the early bulge become asymmetric in time, leading to an eventual collapse and thermalized states. In the present dissipative problem, although a bulge is guaranteed to form at x_s, its eventual dynamics—and indeed whether the system actually thermalizes—depends on the interplay between the local dissipation around x_s, the fraction of oscillations reaching x_s from the boundary layer, and the effect of the nonlinearity. For smaller values of α, when the dissipation is strong and the amplitude of oscillations is small, this bulge at large times remains stationary in time. However, as α increases, the amplitude of oscillations reaching the stagnation point is significant: Consequently for values of α higher than a threshold $\alpha_{\rm crossover}$, the local dissipation can no longer compensate for the resonant pileup at x_s leading to the emergence of thermalized states in a manner exactly similar to that of the GTB. Heuristically, an estimate of $\alpha_{\rm crossover}$ can be obtained as follows: The fraction of amplitude at the boundary layer that reaches x_s is given by $e^{-K_\alpha^{\rm th}\pi}$. We assume that a significant level of oscillations is present at x_s when at least a fraction $1/e$ of the oscillations produced near the shock reaches x_s, i.e., $K_\alpha^{\rm th}\pi = 1$. For moderately large values of α, numerical simulations demand $k_d \gg 1$ and hence values of ν such that $\nu^{-\beta} \to 1$. Thus (3) yields $\alpha_{\rm crossover} = \frac{1}{4}(2 + k_d\pi^2)$. Although this result is obtained heuristically it predicts that for hyperviscous systems a finite crossover value of α exists which leads to a transition from conservative to dissipative dynamics.

We now examine the accuracy of our estimate of $\alpha_{\rm crossover}$ through detailed simulations with increasing values of α and for $k_d = 100$. As we increase α, our simulations show that the bulge at the x_s reaches a stationary state without collapsing. However at around $\alpha \gtrsim 220$ we observe that the bulge which forms, due to resonance, collapses in a finite time and then the system thermalizes. This is best seen in Fig. 3 where we show the solution of the HBE for $\alpha = 250$ at time $t = 1.5$. We note that, just as in the GTB [9], the bulge at the resonance point becomes very large, asymmetric, and nonmonochromatic with secondary structures on either side of it. This is exactly similar to the onset of thermalization in conservative systems [9]. Indeed at larger times the solution completely thermalizes (inset of Fig. 3, at $t = 5.0$). Our simulations illustrate quite clearly that (a) the heuristic estimate of $\alpha_{\rm crossover}$ is correct (by using the value of $k_d = 100$, we obtain $\alpha_{\rm crossover} \simeq 230$ which is consistent with the results from the numerical simulations) and, more importantly, (b) dissipative systems, such as the HBE, can thermalize at finite values of the order of dissipativity in a manner similar to that of conserved, truncated systems. (We have performed several other simulations with different values of ν and k_d and found our results consistent.) The fact that dissipative systems can start to mimic a truncated, Hamiltonian system through the tuning of a single parameter (α) is a striking result and resolves a long-standing paradox in the area of turbulence and statistical mechanics. It is not hard to conjecture that this crossover should be possible for $\alpha \to \infty$ [14]; however, remarkably, we now show that the onset to thermalization actually occurs at a finite value $\alpha_{\rm crossover}$.

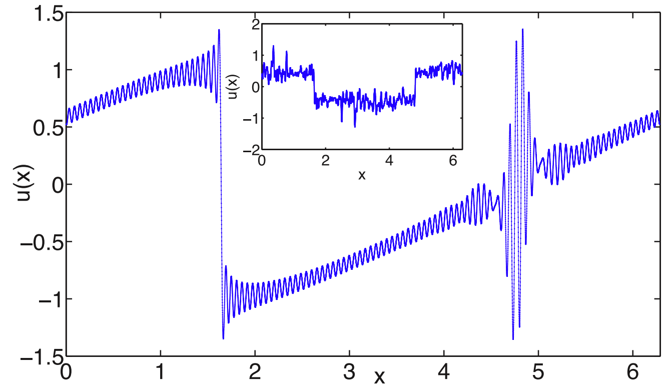

FIG. 3. (Color online) Solution of the HBE, for $\alpha = 250$ at $t = 1.5$, showing the presence of a *tyger* [9] at x_s. Inset: $u(x)$ for $\alpha = 250$ at a later time ($t = 5.0$) confirming that for $\alpha \gtrsim \alpha_{\rm crossover}$ the system eventually thermalizes.

Let us finally address the question of whether this phenomenon can be captured within a systematic theoretical framework. Rewriting Eq. (1) in terms of the solution U of the inviscid Burgers equation and the discrepancy $\tilde{u} \equiv u - U$, and using $\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial U^2}{\partial x} = 0$, we obtain

$$\frac{\partial \tilde{u}}{\partial t} + \frac{\partial}{\partial x}(U\tilde{u}) + \frac{1}{2}\frac{\partial \tilde{u}^2}{\partial x} = -\nu\left(-\frac{1}{k_d^2}\frac{\partial^2}{\partial x^2}\right)^\alpha (\tilde{u} + U). \quad (4)$$

At times close to t_*, and away from the shock, we can linearize (1) since $\tilde{u}/U \ll 1$. Next, we note that U is linear in x away from the shock which implies that higher derivatives of U vanish around x_s. By using these two approximations, we obtain

$$\frac{\partial \tilde{u}}{\partial t} + \frac{\partial}{\partial x}(U\tilde{u}) = -\nu\left(-\frac{1}{k_d^2}\frac{\partial^2}{\partial x^2}\right)^\alpha \tilde{u}. \quad (5)$$

We first validate our linear theory by numerically solving (5) for $\tilde{u}$ with a further approximation that U is the solution at t_* of the inviscid Burgers equation with the initial condition $\sin(x + 1.5)$. We choose two kinds of initial conditions $\tilde{u}_0 = \tilde{u}(t = 0)$: (I1) $\tilde{u}_0$ is a low amplitude sinusoidal function with a wave number equal to 10; and (I2) $\tilde{u}_0 = e^{-K_\alpha^{\rm th}|x - x_{\rm shock}|}\sin\frac{2\pi(x - x_{\rm shock})}{\lambda_\alpha^{\rm th}}$, where $x_{\rm shock}$ is the position of the shock. Our numerical integration of Eq. (5) for both initial conditions yield similar results as illustrated in Fig. 4 where we present a representative plot of $\tilde{u}$, solved for Eq. (5), at time $t = 2$ (blue curve) and $t = 2.5$ (red curve) for $\alpha = 100$ by using I2; the inset shows the solution of Eq. (5) for I1 at time $t = 10.0$. A symmetric bulge at the stagnation point, just like in the solutions Eq. (1) for large α, is clearly seen. The essential features of the bulge are reproduced by our linear model. Having established the validity of the linear model to predict the location and the nature of the bulge, we can now solve Eq. (5) by various standard analytical means such as by using the method of separation of variables or through a Fourier transform of Eq. (5), to obtain solutions (up to constants) which

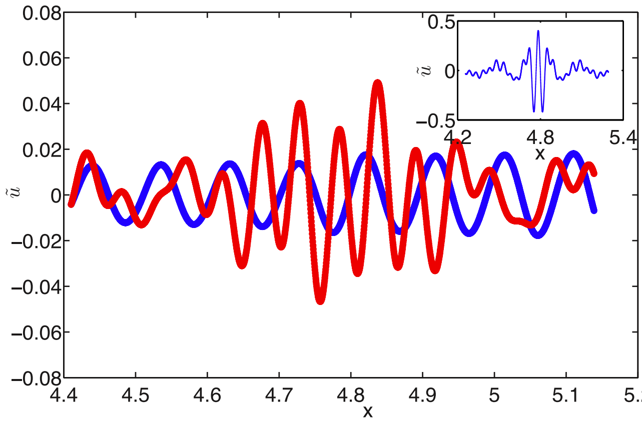

FIG. 4. (Color online) The solution $\tilde{u}$ of the linearized equation (5) with initial conditions (I2) at time $t = 2.0$ (blue or darker curve) and $t = 2.5$ (red or lighter curve) for $\alpha = 100$ (see text). Inset: $\tilde{u}$ with initial conditions (I1) at time $t = 10.0$.

show the existence of symmetric bulges at x_s which decay on either side of the stagnation point. We note, in passing, that although the linear model predicts the early stages of the formation of the bulge at x_s, our simulations of the linear model, for various large values of α, not surprisingly, fails to capture the collapse of the bulge and eventual thermalization [9]. A plausible conjecture for this is that the nonlinearity, however weak, is responsible for the stretching of the bulge and generating an associated Reynolds stress which makes the symmetric bulge collapse and trigger thermalization.

For the past many decades, a vexing and open question in the areas of turbulence and statistical mechanics is, how meaningful are thermalized states in such problems? In this Rapid Communication we answer this question via detailed numerical simulations and linear models. Our results show that just as in the case of Hamiltonian systems of the Galerkin-truncated equation, where monochromatic truncation waves can reach x_s, leading to an accumulation, via resonance, and eventual thermalization, similarly, for dissipative systems such as the HBE, for moderately large α, monochromatic boundary-layer oscillations reach and accumulate, via the same resonant effect, at x_s. These bulges are the seeds of an eventual thermalized regime and for $\alpha \gtrsim \alpha_{\text{crossover}}$ the dissipative system does thermalize at large times. Our work thus connects the apparently disconnected worlds of conservative and dissipative systems. Although we have confined ourselves to the one-dimensional Burgers equation, the central result obtained in this Rapid Communication should be valid in the multidimensional Navier-Stokes equation for the reasons outlined in Refs. [9,14,16]. A detailed study of this is beyond the scope of this Rapid Communication and is left for the future.

We are grateful to M.-E. Brachet, U. Frisch, T. Matsumoto, S. Nazarenko, and R. Pandit for many useful discussions. We thank CSIR, UGC, DAE, and DST (India) for support, and SERC (IISc) for computational resources. We acknowledge the support of the Indo-French Center for Applied Mathematics (IFCAM). S.S.R. acknowledges support from the AIRBUS Group Corporate Foundation Chair in Mathematics of Complex Systems established in ICTS.

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