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Unusual eigenvalue spectrum and relaxation in the Lévy–Ornstein-Uhlenbeck process

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We consider the rates of relaxation of a particle in a harmonic well, subject to Lévy noise characterized by its Lévy index μ. Using the propagator for this Lévy–Ornstein-Uhlenbeck process (LOUP), we show that the eigenvalue spectrum of the associated Fokker-Planck operator has the form $(n+m\mu)\nu$ where ν is the force constant characterizing the well, and $n,m \in \mathbb{N}$. If μ is irrational, the eigenvalues are all nondegenerate, but rational μ can lead to degeneracy. The maximum degeneracy is shown to be 2. The left eigenfunctions of the fractional Fokker-Planck operator are very simple while the right eigenfunctions may be obtained from the lowest eigenfunction by a combination of two different step-up operators. Further, we find that the acceptable eigenfunctions should have the asymptotic behavior $|x|^{-n_1-n_2\mu}$ as $|x| \to \infty$, with n_1 and n_2 being positive integers, though this condition alone is not enough to identify them uniquely. We also assert that the rates of relaxation of LOUP are determined by the eigenvalues of the associated fractional Fokker-Planck operator and do not depend on the initial state if the moments of the initial distribution are all finite. If the initial distribution has fat tails, for which the higher moments diverge, one can have nonspectral relaxation, as pointed out by Toenjes *et al*. [Phys. Rev. Lett. **110**, 150602 (2013)].

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In the recent past, diffusion processes that are anomalous have been observed in a wide variety of areas ranging from financial markets, movement of active particles in biological systems, diffusion in turbulent media, etc. [1–3]. In general, the relaxation of a system undergoing normal diffusion is multiexponential. The exponents are determined by the eigenvalues of the Fokker-Planck (FP) operator associated with the diffusion. The time scales involved are thus an intrinsic property of the system, independent of the initial condition. For normal diffusion of a particle in a potential $V(x)$, the associated FP operator is non-Hermitian [4]. However, one can make a similarity transformation to get a Hermitian operator, analogous to the Hamiltonian operator for a quantum system. This shows that the eigenvalues of the operator are all real and further, by imposing physically motivated boundary conditions (vanishing at infinity and square integrability), one finds the eigenvalue spectrum of the operator. This spectrum determines the relaxation characteristics of the system completely. It has been recently argued by Toenjes *et al.* [5] that this traditional wisdom may not hold in general. They suggested [5] that "initial distributions which are not mapped to square integrable functions by the similarity transformation, cannot be expanded in terms of the eigenfunctions of the corresponding Hamiltonian operator and will therefore relax at rates that may not be given by the Hermitian spectrum." This has been referred to as nonspectral relaxation. Further, it was also suggested that "the smallest nonspectral rate can be smaller than the smallest spectral relaxation rate and thus, it will dominate the relaxation behavior over the whole time range." This has been argued [5] to happen even for the simplest of processes, viz., the Ornstein-Uhlenbeck process (OUP).

In the recent past, a number of investigations have focused on processes driven by Lévy noise. Several physical problems in which they appear have been discussed in excellent reviews [3,6,7]. The result of such driving is anomalous diffusion, having the displacement scaling like $(\text{time})^{1/\mu}$ with $0 < \mu \leqslant 2$. The process is governed by a fractional Fokker-Planck equation [8–10], which is much more difficult to analyze. The properties of such operators are not well understood, as there are very few results on them. Laskin [11] introduced a generalization of quantum mechanics referred to as fractional quantum mechanics which has similar operators and the eigenvalue spectra and eigenfunctions of some operators have been investigated [12,13]. Toenjes *et al.* [5] discuss the case of a particle in a harmonic well, subject to Lévy noise which is governed by the equation

$$\frac{dx}{dt} = -\frac{V'(x)}{m\gamma} + \eta(t),$$

where $V(x)$ is the potential that the particle is subjected to, and m and γ are the mass and the friction coefficient, respectively. $\eta(t)$ is the Lévy noise, best described by its characteristic functional $\langle e^{i\int_0^T dt\,\eta(t)p(t)}\rangle$ which is equal to $e^{-D\int_0^T dt|p(t)|^\mu}$ [14–16]. The resultant Lévy–Ornstein-Uhlenbeck process (LOUP) is one of the simplest of such processes. Though LOUP appears to be quite simple, it has found applications in econophysics and statistics, to analyze stochastic volatility models for financial assets [1,2]. Also, the underdamped limit of LOUP has found application in biology in describing the anomalous dynamics of cell migration [17], and stochastic models for active particles [18]. LOUP is governed by the corresponding fractional Fokker-Planck equation, given by [8,19]

$$\frac{\partial P(x,t)}{\partial t} = \left\{-D\left(-\frac{\partial^2}{\partial x^2}\right)^{\mu/2} + \frac{\partial}{\partial x}\frac{V'(x)}{m\gamma}\right\} P(x,t). \quad (1)$$

The usual Smoluchowski equation is a special case of this and is obtained when one puts $\mu = 2$. Taking the potential to be of the form $V(x) = kx^2/2$, and changing over to a new variable $D^{-1/\mu}\,x \to x$ and putting $k/(m\gamma) = \nu$, we can write this as

$$\frac{\partial P(x,t)}{\partial t} = -\hat{H}_\mu P(x,t), \quad (2)$$

with

$$\hat{H}_\mu = -\left\{-\left(-\frac{\partial^2}{\partial x^2}\right)^{\mu/2} + \nu\frac{\partial}{\partial x}x\right\}. \quad (3)$$

The relaxation of the system governed by Eqs. (2) and (3) was studied by Toenjes *et al.* [5]. We briefly summarize their main arguments. The relaxation is governed by the eigenvalues λ of the operator determined by $\hat{H}_\mu \psi = \lambda \psi$, if the initial state satisfies the acceptability conditions (see below). Interestingly, one can find solutions of $\hat{H}_\mu \psi = \lambda \psi$ for any λ, but when one imposes acceptability conditions on the x dependence of ψ, only certain discrete values λ_n and the associated eigenfunctions ψ_n are allowed. In particular, when $\mu = 2$, the acceptability condition imposed is $\psi(x) e^{\nu x^2/4} \to 0$ as $x \to \pm\infty$, which leads to $\lambda_n = \nu n$, with $n \in \mathbb{N}$. According to them, the relaxation of the system, from any initial state $P_0(x_i)$, which can be expanded in terms of these eigenfunctions as

$$P_0(x_i) = \sum_n c_n \psi_n(x_i) e^{\nu x_i^2/4}, \tag{4}$$

is determined by these eigenvalues. In the above, c_n are the expansion coefficients. This means that the slowest relaxation would correspond to ν (i.e., $n = 1$). According to the paper, if the initial condition cannot be expanded as in Eq. (4), then the relaxation can be much slower.

On going over to Fourier space, with $\mathcal{P}(p,t) = \int dx\, P(x,t) e^{ipx}$, Eq. (2) becomes

$$\frac{\partial \mathcal{P}(p,t)}{\partial t} = -\hat{\mathcal{H}}_\mu\, \mathcal{P}(p,t), \tag{5}$$

with

$$\hat{\mathcal{H}}_\mu = |p|^\mu + \nu p \frac{\partial}{\partial p}. \tag{6}$$

In the case with $\mu \neq 2$, Toenjes *et al.* [5] perform a similarity transformation given by $p = |\kappa|^{2/\mu} \mathrm{sgn}(\kappa)$, under which the operator $\hat{\mathcal{H}}_\mu$ gets transformed to $\kappa^2 + \frac{\mu\nu}{2}\kappa \frac{\partial}{\partial \kappa}$. As this operator is similar to the one for OUP, they impose boundary conditions appropriate for OUP and get the eigenvalues $n\mu\nu/2$, with $n \in \mathbb{N}$, similar to that of OUP. For cases where the initial distribution has a long tail behaving like $|x|^{-\alpha-1}$, Toenjes *et al.* claim to solve the time evolution exactly and find that it contains time scales determined not by $n\mu\nu/2$, but by the numbers $\nu(n + m\mu + l\alpha)$, where α is determined by the characteristics of the initial distribution and $l,m,n \in \mathbb{N}$. This does not coincide with the eigenvalue spectrum of the operator found by the similarity transformation and hence, the authors argue that nonspectral relaxation is the rule rather than the exception for such processes. Note that according to their paper, the long term relaxation is not necessarily determined by the lowest nonzero eigenvalue of the operator $\hat{H}_\mu$.

In this Rapid Communication, we point out that the Green's function for the operator in Eq. (2) is enough to propagate any arbitrary initial condition for any value of μ, including $\mu = 2$, which is the usual OUP. We use the expression for the propagator that is already known [6,8,15,20] and find the exponents that are involved in the time evolution. They are of the form $\nu(n + m\mu)$, thus showing that the eigenvalue spectrum of the FP operator for the LOUP is characterized, in general, by two "quantum numbers" n and $m \in \mathbb{N}$ (and not one, as one would normally expect). If $\mu = 1$ or 2, then the spectrum coincides with that for the OUP, but the degeneracies are different for $\mu = 1$. Also, we identify the left and right eigenfunctions of the operator and arrive at a generalization of the expansion of the propagator for OUP which is given in terms of the Hermite polynomials [4]. Further, we give operators that can be used to generate the right eigenfunctions from the lowest possible eigenfunction, similar to the step-up operators of quantum mechanics. We also discuss the boundary conditions that when imposed on the solutions would lead naturally to the correct identification of these eigenfunctions and eigenvalues.

We now give our analysis of the problem. Equation (5) can be solved by the method of characteristics, for any initial condition $\mathcal{P}(p,0) = \mathcal{P}_0(p)$ to get the solution at a final time T as

$$\mathcal{P}(p,T) = \mathcal{P}_0(pe^{-\nu T}) e^{-|p|^\mu (1-e^{-\mu\nu T})/(\mu\nu)}. \tag{7}$$

Writing $\mathcal{P}_0(p) = \int dx_i\, e^{ipx_i} P_0(x_i)$, we can express the position space probability distribution at the final time T as

$$P(x_f,T) = \int dx_i\; G(x_f,T|x_i,0) P_0(x_i) \tag{8}$$

with

$$\begin{aligned} &G(x_f,T|x_i,0) \\ &= \int \frac{dp}{2\pi} e^{-|p|^\mu (1-e^{-\mu\nu T})/(\mu\nu) + ip(x_f - x_i e^{-\nu T})} \qquad (9) \\ &= \left(\frac{(\mu\nu)^{1/\mu}}{(1-e^{-\mu\nu T})^{1/\mu}}\right) L_\mu \left(\frac{(\mu\nu)^{1/\mu}(x_f - x_i e^{-\nu T})}{(1-e^{-\mu\nu T})^{1/\mu}}\right), \qquad (10) \end{aligned}$$

where $L_\mu(x)$ is the Lévy stable distribution defined by

$$L_\mu(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp\, e^{ipx - |p|^\mu}. \tag{11}$$

If $\mu > 1$, this may be evaluated as the series

$$L_\mu(x) = \frac{1}{\pi\mu} \sum_{n=0}^{\infty} (-1)^n x^{2n} \frac{\Gamma\left(\frac{2n+1}{\mu}\right)}{\Gamma(2n+1)}. \tag{12}$$

We have recently developed a path integral approach to Lévy flights which leads to exactly this result [15]. The above analysis shows that irrespective of what the initial distribution is, the time development of the system is determined only by the propagator. Note that this conclusion is valid for all values of μ and is therefore applicable to the usual Brownian motion too. It is possible to expand the propagator $G(x_f,T|x_i,0)$ in terms of the left eigenfunctions $\tilde{\psi}_n(x)$ and the right eigenfunctions $\psi_n(x)$ of the operator $\hat{H}_\mu$ as

$$G(x_f,T|x_i,0) = \langle x_f | e^{-T\hat{H}_\mu} | x_i \rangle = \sum_n \tilde{\psi}_n(x_i) \psi_n(x_f) e^{-\lambda_n T}. \tag{13}$$

As $\hat{H}_\mu$ is not a Hermitian operator, the eigenfunctions are not necessarily orthogonal. It is obvious that if we can expand the right-hand side of Eq. (9) as a series in exponentials involving T (note that there are two exponentials involving T leading to a double summation), then we would be able to find the eigenvalues and eigenfunctions of the operator $\hat{H}_\mu$.

$$\begin{aligned} G(x_f,T|x_i,0) = &\sum_{n,m=0}^{\infty} \frac{(-x_i)^n}{\Gamma(n+1)\,\Gamma(m+1)} \\ &\times \psi_{n,m}(x_f)\, e^{-(n+m\mu)\nu T}, \end{aligned} \tag{14}$$

where

$$\psi_{n,m}(x_f) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp\; e^{-|p|^\mu/\mu\nu}\,(ip)^n |p|^{m\mu} e^{ipx_f}. \quad (15)$$

We can now expand e^{ipx_f} as a series and perform the integral over p to obtain the right eigenfunction in the position space as

$$\psi_{n,m}(x_f) = \frac{1}{\mu\pi}\sum_{k=0}^{\infty}\frac{(-1)^{\{(n+[n])/2\}+k}}{([n]+2k)!} \times \Gamma\left(\frac{n+[n]+2k+1}{\mu}+m\right) x_f^{(2k+[n])}, \quad (16)$$

where $[n] = n$ modulo 2. Note that the above expansion is convergent for all x_f only if $\mu > 1$. From Eq. (14), it is clear that the eigenvalues of $\hat{H}_\mu(x)$ have the form $(n+m\mu)\nu$ with n and m belonging to $\mathbb{N}$. Thus we have a very interesting situation that if μ is irrational, the eigenvalues are characterized by two "quantum" numbers n and m, unlike the usual situation where there is only one quantum number for a 1-D problem such as this. Further, for such values of μ, the left eigenfunction is simply $\tilde{\psi}_n = (-x_i)^n/\Gamma(n+1)$ while the right eigenfunction is given by $\psi_{n,m}(x_f)$. If μ is a rational number, the left eigenfunction can become a little bit more complex. This point will be elaborated in the later part of this Rapid Commnication. Equation (14) is the generalization of the classic expansion of the propagator for the Ornstein-Uhlenbeck process [4], to the LOUP. The eigenvalue spectrum for LOUP is shown in Fig. 1 as a function of μ. We also note that to use the expansion (14) and to calculate $P(x_f,T)$ using Eq. (8), it is necessary that *all the moments of the initial distribution $P(x_i)$ should exist*. If one chooses an initial distribution for which the moments diverge [for example, $P(x_i) = \frac{1}{w_0}L_\alpha(x_i/w_0)$], then one cannot use Eq. (14) and the relaxation can contain other time scales [21] leading to nonspectral relaxation as pointed out in [5].

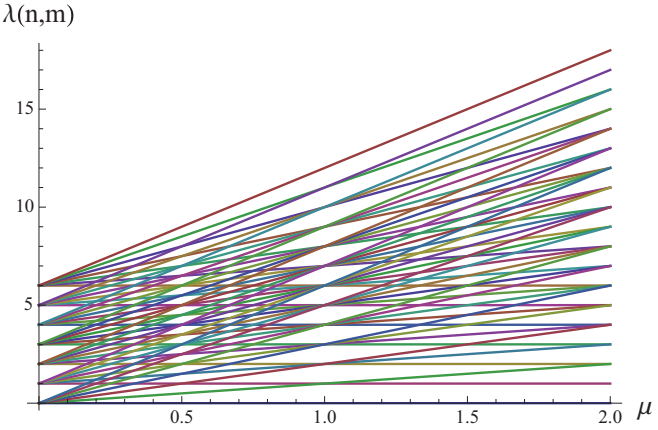

FIG. 1. (Color online) Eigenvalues $n+m\mu$ plotted as functions of μ for $\mu \in [0,2]$, for $n = 0,1,2\ldots6$ and $m = 0,1,2\ldots6$. The spectrum has only nondegenerate eigenvalues if μ is irrational. Intersections of two or more lines occur when μ is rational and can lead to points where there is degeneracy. However, the maximum degeneracy is only 2, even though more than two lines may intersect at the same point. At $\mu = 2$, all eigenvalues are nondegenerate, and when $\mu = 1$, all eigenvalues except the lowest are doubly degenerate.

However, if the initial distribution has $\alpha = \mu$ or is a truncated Lévy distribution, then the relaxation is purely spectral.

It is possible to express the eigenfunctions in an interesting way. If one allows $T \to \infty$, $G(x,T|x_i,0)$ approaches the steady equilibrium state and becomes

$$\psi_{0,0}(x) = (\mu\nu)^{1/\mu} L_\mu[(\mu\nu)^{1/\mu}\, x].$$

This is the lowest eigenfunction of the operator $\hat{H}_\mu$, having the eigenvalue zero. It is easy to prove the following commutation relations:

$$\left[\frac{\partial}{\partial x}, \hat{H}_\mu\right] = -\nu\frac{\partial}{\partial x} \quad (17)$$

and

$$\left[\left(-\frac{\partial^2}{\partial x^2}\right)^{\mu/2}, \hat{H}_\mu\right] = -\mu\nu\left(-\frac{\partial^2}{\partial x^2}\right)^{\mu/2}. \quad (18)$$

The above imply that if ψ is an eigenfunction of $\hat{H}_\mu$ with an eigenvalue ε, then $\frac{\partial\psi}{\partial x}$ and $(-\frac{\partial^2}{\partial x^2})^{\mu/2}\psi$, too are eigenfunctions with eigenvalues $\varepsilon+\nu$ and $\varepsilon+\mu\nu$, respectively. It follows that the right eigenfunction $\psi_{n,m}(x) \propto (-\frac{\partial^2}{\partial x^2})^{\mu m/2}\frac{\partial^n}{\partial x^n}\psi_{0,0}(x)$. The asymptotic behavior as $x \to \pm\infty$ is

$$\psi_{n,m}(x) \sim \frac{1}{|x|^{n+1+\mu}} \qquad \text{when } m = 0 \quad (19)$$

and

$$\psi_{n,m}(x) \sim \frac{1}{|x|^{n+1+m\mu}} \qquad \text{when } m \neq 0. \quad (20)$$

However, one can find eigenfunctions other than these, as shown below. This is easily done in the momentum space.

The eigenvalue equation in the momentum space

$$\left(|p|^\mu + \nu p\frac{\partial}{\partial p}\right)\overline{\psi}_\lambda(p) = \nu\lambda\overline{\psi}_\lambda(p), \quad (21)$$

has the solution

$$\overline{\psi}_\lambda(p) = e^{-|p|^\mu/(\mu\nu)}\, p^\rho\, |p|^\sigma\,, \quad (22)$$

for any real positive numbers ρ and σ such that $\lambda = \rho+\sigma$. Its position space representation is given by

$$\psi_\lambda(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp\, e^{-|p|^\mu/(\mu\nu)+ipx}\, p^\rho\, |p|^\sigma\,. \quad (23)$$

For an arbitrary $\lambda = \rho+\sigma$, we may rearrange Eq. (23) to get

$$\psi_\lambda(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp\, e^{-|p|^\mu/(\mu\nu)+ipx}\, |p|^\lambda\, (\text{sgn } p)^\rho$$

$$= \frac{(1+e^{i\pi\rho})}{2\pi}\, C_\lambda(x) + i\frac{(1-e^{i\pi\rho})}{2\pi}\, S_\lambda(x), \quad (24)$$

with

$$C_\lambda(x) = \int_0^\infty dp\, e^{-p^\mu/(\mu\nu)}\, p^\lambda \cos(px), \quad (25)$$

$$S_\lambda(x) = \int_0^\infty dp\, e^{-p^\mu/(\mu\nu)}\, p^\lambda \sin(px). \quad (26)$$

From the momentum space eigenfunction in Eq. (22), it appears that a continuous infinity of λ will satisfy the differential

equation in Eq. (21). However, we find that $\lambda = n + m\mu$ are the only eigenvalues we find in the spectrum. The question that we ask is, what is the sanctity of these $\lambda = n + m\mu$ and why are the other values of λ unacceptable? In order to answer this question, we now look at the asymptotic behavior of the two linearly independent solutions in the position space for the eigenvalue equation (21), viz., $C_\lambda(x)$ and $S_\lambda(x)$ which have the same eigenvalue $\nu\lambda$. For this we use the identity

$$\cos(px) = \frac{1}{2i}\int_{c-i\infty}^{c+i\infty} ds\, \frac{\Gamma(s)}{\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)}(px)^{-s},$$

where $c > 0$ (note the right-hand side of the equation is a Mellin-Barnes integral) in the expression for $C_\lambda(x)$ and then perform the integral over p, to get

$$C_\lambda(x) = \frac{(\mu\nu)^{(\lambda+1)/\mu}}{2i\mu}\int_{c-i\infty}^{c+i\infty} ds\, \frac{\Gamma(s)\Gamma\left(\frac{1+\lambda-s}{\mu}\right)}{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}[(\mu\nu)^{1/\mu}x]^{-s}. \tag{27}$$

If one closes the contour on the left-hand side using a semicircle of radius $R \to \infty$, one gets a series expansion in terms of x^n, while if one closes the contour on the right-hand side, one gets an asymptotic expansion in inverse powers of x. This asymptotic expansion is

$$C_\lambda(x) = \frac{(\mu\nu)^{(\lambda+1)/\mu}}{\mu}\sum_{k=0}^{\infty}(-1)^k\,\frac{\Gamma(\lambda+k\mu+1)}{\Gamma(k+1)} \times \cos\left(\frac{\pi}{2}(\lambda+k\mu+1)\right)[(\mu\nu)^{1/\mu}x]^{-(\lambda+k\mu+1)}. \tag{28}$$

$S_\lambda(x)$ can be evaluated in an exactly similar fashion to be

$$S_\lambda(x) = \frac{(\mu\nu)^{(\lambda+1)/\mu}}{\mu}\sum_{k=0}^{\infty}(-1)^k\,\frac{\Gamma(\lambda+k\mu+1)}{\Gamma(k+1)} \times \sin\left(\frac{\pi}{2}(\lambda+k\mu+1)\right)[(\mu\nu)^{1/\mu}x]^{-(\lambda+k\mu+1)}. \tag{29}$$

From these expressions, we can see that unless $\lambda = n + m\mu$, the functions will not have the correct asymptotic behavior as prescribed by Eqs. (19) and (20). Interestingly even imposing this condition is not enough to restrict eigenfunctions and eigenvalues to those found from the Green's function, because for any λ we still seem to be having two solutions, viz., $C_\lambda(x)$ and $S_\lambda(x)$. We now discuss how to identify the correct eigenfunctions which appear in the expansion of Eq. (14).

For any value of λ, we can write down the different possible ways in which the eigenvalue can have the form $n + m\mu$. We now consider two separate possibilities:

(1) *μ is irrational:* For any $\lambda = n + m\mu$, the values (n,m) are unique. Further, from Eqs. (25) and (26), it is clear that if n is even, then $C_n(x)$ behaves like $|x|^{-n-\mu-1}$, and $S_n(x) \sim |x|^{-n-1}$ and hence, only $C_n(x)$ is acceptable. The function $C_{n+m\mu}(x)$ is obtained from $C_n(x)$ by the application of $(-\frac{\partial^2}{\partial x^2})^{\mu/2}$ m times, an operation which does not change its symmetry. Therefore, $C_{n+m\mu}$ becomes the acceptable eigenfunction for $\lambda = n + m\mu$ where n is even, as $C_n(x)$ itself is acceptable. $S_{n+m\mu}(x)$ is an unacceptable solution when n is even since $S_n(x)$ itself is not. One can make a similar argument when n is odd.

(2) *μ is rational:* μ can be written as p/q, where p and q are integers having no common factors. Then one can have degeneracy if m is an integral multiple of q equal to kq. The associated eigenvalue is $(n + kp)\nu$. Then all the states having quantum numbers $(n + kp,0)$, $(n + (k-1)p,q)$, $(n + (k-2)p,2q)\ldots(n,kq)$ will have the same eigenvalue and hence, one expects degeneracy. However, if n and p are even, then all these possibilities lead to the even eigenfunction, viz., $C_{n+kp}(x)$, and the level will be nondegenerate. On the other hand, if either n or p is odd or both are odd, then one of the states will be $S_{n+kp}(x)$ and the other will be $C_{n+kp}(x)$. Thus one would have degeneracy in this case, and the degeneracy would be 2.

Another interesting observation about the degeneracies of states has emerged out this analysis. If we consider a particular state, say, $\lambda = 4$ for $\mu = 1$ in Fig. 1, we see that five lines intersect at this point, seeming to suggest that the degeneracy of this state is 5. However, from the previous arguments the degeneracy of this state can be found to be 2. Of the five states which intersect at $\lambda = 4$, three of them become exactly identical to $C_4(x)$ and two of them to $S_4(x)$, and therefore are not independent states. When the right eigenfunctions become identical, the corresponding left eigenfunctions, $(-x_i)^n/\Gamma(n+1)$, will add up to give a single left eigenfunction. This sudden reduction in the number of eigenstates as one goes from $\mu = 1 - \epsilon$ to $\mu = 1$ to where ϵ is infinitesimally small is not a problem as they are not constrained to be orthogonal, as in the case in quantum mechanics. This can also be treated as evidence for the lack of a similarity transformation for the LOUP operator converting it to a Hermitian operator, as proposed by Toenjes *et al.* If such a transformation existed, it seems impossible for the eigenfunctions to undergo this sudden reduction in their number.

In conclusion, we have shown that the eigenvalue spectrum of the Fokker-Planck operator for LOUP to be of the form $n + m\mu$, characterized by the two "quantum numbers" $n,m \in \mathbb{N}$. Using the spectral expansion of the propagator, we have found the left and right eigenfunctions of the operator. For irrational values of μ the spectrum is nondegenerate, while for rational μ there could be degeneracies with the maximum degeneracy being 2. We also find that any acceptable eigenfunction of the operator should satisfy the condition that as $|x| \to \infty$, the functions should behave like $|x|^{-(n_1+\mu\, n_2)}$, where n_1 and n_2 are positive integers, though this condition alone is not enough to uniquely identify the eigenstates. If the moments of the initial distribution are all finite, then the relaxation is governed only by these eigenvalues, while for initial distributions having long tails, one can have nonspectral relaxation in agreement with [5].

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