Exponential Fermi acceleration in general time-dependent billiards

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We show, that under very general conditions, a generic time-dependent billiard, for which a phase space of corresponding static (frozen) billiards is of the mixed type, exhibits the exponential Fermi acceleration in the adiabatic limit. The velocity dynamics in the adiabatic regime is represented as an integral over a path through the abstract space of invariant components of corresponding static billiards, where the paths are generated probabilistically in terms of transition-probability matrices. We study the statistical properties of possible paths and deduce the conditions for the exponential Fermi acceleration. The exponential Fermi acceleration and theoretical concepts presented in the paper are demonstrated numerically in four different time-dependent billiards.

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I. INTRODUCTION

An unbounded energy growth of particles in a time-dependent potential is known as Fermi acceleration (FA), which was first proposed by Fermi [1] to explain the high energies of cosmic particles as a consequence of repeated collisions with moving interstellar magnetic domains. Nowadays different models of FA are investigated in many areas of physics, such as astrophysics [2], plasma physics [3], atom optics [4], and time-dependent billiards, which are the subject of this paper.

Billiards are very simple and generic dynamical systems of a fundamental importance for theoretical as well as numerical investigations in classical [5–7] and quantum mechanics [8,9]. Billiards have been realized also experimentally as microwave cavities [10], acoustic resonators, optical laser resonators [11], and quantum dots [12]. The first time-dependent billiard investigated in the context of FA was the one-dimensional Fermi-Ulam model (a particle between the moving walls) [13], for which it is nowadays known that it does not permit FA if a motion of the walls is sufficiently smooth, due to the existence of invariant tori which suppress the global energy transport [14]. The presence of chaos in two- (or higher) dimensional billiards make such an unbounded energy transport possible [15].

Two-dimensional periodic time-dependent billiards have been the subject of intense investigations for almost two decades. Numerical studies suggest that, asymptotically, the average velocity of an ensemble of particles obeys the power law,

$$\langle v \rangle \propto n^{\beta}, \tag{1}$$

with respect to the number of collisions n, where several different values of the *acceleration exponent* β were observed [15–20]. The velocity dynamics is strongly related to the dynamical properties of a continuous set of *corresponding static billiards* which coincide with different shapes of a time-dependent billiard. If all corresponding static billiards of a time-dependent billiard are ergodic then, in general, $\beta = 1/2$ [21], except if the billiard motion is shape-preserving: In this case, β depends only on the rotational properties of the billiard and can have only one of the three possible values $\{0,1/6,1/4\}$ [22–24]. However, if the dynamics is not ergodic then β could be greater than $1/2$ [18]; moreover, it can even reach a theoretical maximum asymptotic value $\beta = 1$, which corresponds to the exponential acceleration in the continuous time [17].

Recently a lot of attention has been given to the possibility of a very efficient unbounded exponential acceleration of particles in time-dependent billiards. It was shown theoretically, under very general conditions, that in time-dependent billiards possessing the chaotic component, there exist trajectories of zero measure which accelerate exponentially fast [21]. However, under some circumstances the exponential acceleration can take place for most initial conditions. This was first demonstrated in the rectangular billiard with the oscillating bar [25–27] and then in a class of chaotic billiards which undergo a separation of ergodic components by physically splitting the billiard in several unconnected parts [28,29]. Recently, it was shown in the study of an oscillating mushroom billiard how, in this particular example, the presence of the regular component results in the exponential acceleration [30]. Although the phenomenon of exponential acceleration is understood in particular special examples, general insights have been lacking.

In this paper we consider the velocity dynamics in the adiabatic regime and deduce general conditions for the exponential acceleration in time-dependent billiards. The basic idea is to represent the motion of fast variables as a Markov model of a transport between the invariant components of corresponding static billiards. It is shown that the exponential acceleration arises if in the strict adiabatic limit, the number of possible paths through the space of invariant components proliferate exponentially in time. This condition is expected to be fulfilled if a corresponding static billiards of a time-dependent billiard have more than one invariant component, e.g., mixed-type billiards. We also clearly demonstrate the exponential acceleration numerically in several different time-dependent billiards.

II. THEORY

The state of the particle in a time-dependent billiard is described by the set $\{\mathbf{r},\theta,v,t\}$, where $\mathbf{r}$ is a two-component position vector, θ is a direction (angle) of a particle velocity vector $\mathbf{v} = v\,(\cos\theta, \sin\theta)$, $v = \|\mathbf{v}\|$ is a particle velocity, and

t is time. The motion of the particle is restricted to the billiard domain which is periodically changing with time.

The only force that acts on a particle in a billiard is that of a boundary at collisions. Between collisions the particle velocity vector is preserved. Collisions are elastic, which means that at a collision, in a reference frame in which the collision point is at rest, the normal component of the velocity vector changes sign, while the tangential component remains unchanged.

A velocity vector after the n-th collision at the collision point $\mathbf{r}_n$ equals

$$\mathbf{v}_n = \mathbf{v}_{n-1} - 2\, P_n\, (\mathbf{v}_{n-1} - \mathbf{u}_n), \tag{2}$$

where P is a projection matrix onto the normal to the boundary at $\mathbf{r}_n$, and $\mathbf{u}_n$ is the velocity vector of the boundary at $\mathbf{r}_n$. Squaring Eq. (2) gives

$$v_n^2 = v_{n-1}^2 - 4\, \mathbf{u}_n \cdot P_n\, (\mathbf{v}_{n-1} - \mathbf{u}_n), \tag{3}$$

where we have used the properties of the projection matrix, $P^T = P$ and $P^T P = P$. Now, using

$$v_n^2 - v_{n-1}^2 = (v_n - v_{n-1})(v_n + v_{n-1}) \tag{4}$$

and Eq. (2) in Eq. (3), we find the change of the velocity at the n-th collision,

$$v_n - v_{n-1} = 2 \left(\frac{\mathbf{v}_n - \mathbf{v}_{n-1}}{v_n + v_{n-1}} \right) \cdot \mathbf{u}_n. \tag{5}$$

If the velocity of the boundary is zero, the particle velocity is preserved.

The objective of the paper is to understand under what conditions does the sequence v_n on average increase exponentially in time. In the limit of large particle velocities, the exponential acceleration in time corresponds to the linear acceleration with respect to the number of collisions n, which corresponds to the acceleration exponent $\beta = 1$, as defined in (1). This is because the number of collisions on the fixed time interval is proportional to the particle velocity,

$$\Delta n \propto \frac{v}{\bar{\ell}}\, \Delta t, \tag{6}$$

where $\bar{\ell}$ is an average distance between two collisions on the time interval Δt, which is large compared to the period of billiard oscillations but small enough to neglect the variations of v. It has to be assumed that in the adiabatic limit $\bar{\ell}$ is positive and independent of v, which is a reasonable assumption in general time-dependent billiards. For the extended discussion see Ref. [28]. We shall show in numerical examples in next section that $\beta = 1$ indeed corresponds to the exponential acceleration.

In the adiabatic regime the velocity of the particle v is much bigger than any velocity of the boundary $u = \|\mathbf{u}\|$ and the time between two collisions is much smaller than a period of a billiard motion. For a small but finite time interval δt at time t, on which the billiard changes only very slightly, the following inequality is satisfied in the adiabatic regime:

$$u\, \delta t \ll \langle \ell \rangle \ll v\, \delta t, \tag{7}$$

where $\langle \ell \rangle$ is an average distance between two collisions on a time interval δt at time t. If (7) is satisfied then a trajectory on the time interval δt around some time t is approximately the same as if the particle would be in the corresponding static billiard at time t, where a corresponding static billiard at time t is a static billiard ($u \equiv 0$) which has the same boundary as a time-dependent billiard at time t. In the adiabatic limit, the geometry of trajectories in a time-dependent billiard becomes independent of the particle velocity, the same as in a static billiard.

In the adiabatic regime, the change of the particle velocity at a collision can be considered to depend only on $\{\mathbf{r},\theta,t\}$, as, for example,

$$v_n - v_{n-1} \approx \left(\frac{\mathbf{v}_n}{v_n} - \frac{\mathbf{v}_{n-1}}{v_{n-1}} \right) \cdot \mathbf{u}_n, \tag{8}$$

which follows from (5) and the approximation $v_n \approx v_{n-1}$. Thus, formally, in the adiabatic regime, the particle velocity can be approximately written as a path integral,

$$v(t_1) = v(t_0) + \int_{s(t_0)}^{s(t_1)} ds\; f(\mathbf{r}(s),\theta(s),t(s)), \tag{9}$$

over a trajectory parametrized with s, where s is a geometrical length of the path in the configuration space, and $f(\mathbf{r},\theta,t)$ is a field independent of v. By parametrizing the trajectory in terms of time t and using $ds = v\, dt$, the integral equation (9) can be equivalently written in the form of the differential equation,

$$\dot{v} = v\; f(\mathbf{r}(t),\theta(t),t), \tag{10}$$

where the dot denotes the time derivative. A possible definition of the field $f(\mathbf{r},\theta,t)$ is presented in Appendix A. However, the concrete construction of the field $f(\mathbf{r},\theta,t)$ is not important for general conclusions of the theory that follows.

The integration of Eq. (10) gives

$$v(t) = v(t_0)\, e^{F(\mathcal{S})}, \tag{11}$$

where we have introduced

$$F(\mathcal{S}) = \int_{t_0}^{t} f(\mathbf{r}(t),\theta(t),t)\, dt, \tag{12}$$

which is the integral of $f(\mathbf{r},\theta,t)$ along a trajectory $\mathcal{S}$ on the time interval from t_0 to t.

Our goal is to describe the statistical properties of F and deduce conditions for the exponential acceleration. We are going to introduce a discrete time and represent the dynamics of the fast variables $\{\mathbf{r},\theta\}$ as a stochastic hopping between the invariant components of corresponding static billiards at discrete instances of time, exploiting the fact that on a sufficiently small time interval δt and for a sufficiently big particle velocity v, the motion of the fast variables is restricted to (and ergodic on) a single invariant component of a corresponding static billiard.

We divide the time interval of one period T into N small intervals of length $\delta t = T/N$ on which the billiard can be considered static and introduce a discrete time $j \in \{1,2\ldots\}$. In the adiabatic regime, on a time interval δt at time j the motion of the fast variables $\{\mathbf{r},\theta\}$ is restricted to only one of the invariant components $\{\zeta_n^j\}$ of the corresponding static billiard at time j, where $n \in \{1,2\ldots\}$ labels invariant components.

We have assumed that there are countable many invariant components in a static billiard. This is not exactly true, because there is a continuum of invariant tori in regular domains if these are present in a billiard. However, conceptually we can always partition a regular domain into a countable many invariant components which are very thin layers of invariant tori. On the other hand, we consider a connected chaotic domain as a single invariant component, neglecting the zero measure set of isolated periodic orbits.

In the adiabatic regime, almost every trajectory on any invariant component ζ_n^j uniformly covers ζ_n^j within the time interval δt. Thus, the integral over f, along almost any trajectory segment on the interval δt, that lives in the invariant component ζ_n^j at time j, approximately equals $\delta F \approx \delta t\, \bar{f}_{\zeta_n^j}$, where $\bar{f}_\zeta$ denotes the average of f on the invariant component ζ.

We shall call a chronologically ordered sequence of invariant components $\{\zeta_{n_j}^j\}$ a ζ trajectory. In the adiabatic regime, every trajectory can be represented as a ζ trajectory. We shall not distinguish between trajectories which are represented with the same ζ trajectory. In other words, the ζ trajectory represents a maximal resolution of our theory. In terms of a ζ trajectory, F is an "integral" over the path through the space of invariant components of corresponding static billiards,

$$F \approx \sum_j \delta t\, \bar{f}_{\zeta_{n_j}^j}. \tag{13}$$

ζ trajectories are generated probabilistically in terms of transition matrices $\{P^j\}$, where a matrix element $P_{n,m}^j$ is a probability for the transition $\zeta_m^j \to \zeta_n^{j+1}$ between two invariant components of two successive corresponding static billiards at times j and $j+1$, respectively. A transition probability $P_{n,m}^j$ is bounded between 0 and 1 and can be only a monotonic function of the particle velocity v, thus, in the adiabatic limit, it either vanishes or it converges to a positive constant independent of v. In the adiabatic regime, we consider $\{P^j\}$ to be constant matrices independent of v.

If at least some of transition matrices $\{P^j\}$ are stochastic matrices, which means that at lest some matrix elements differ from 0 or 1, then a number of possible ζ trajectories increases exponentially with increasing j.

A transition matrix $M = P^N \dots P^2 P^1$ determines transition probabilities between invariant components of an initial corresponding static billiard after one cycle of a billiard motion. If not all invariant components of an initial corresponding static billiard are connected, then the transition matrix M is a block matrix. In this case we can consider each block separately as an independent system. In the following, let the matrix M correspond to a single block which represents a subset of invariant components which are connected. Then, by the Perron-Frobenius theorem, there exists a unique invariant probability vector $\boldsymbol{\pi}$, such that $\boldsymbol{\pi} = M\boldsymbol{\pi}$, and the sequence of the powers of M converges to a stationary matrix M^∞ which has all columns equal to $\boldsymbol{\pi}$. A vector $\boldsymbol{\pi}$ is an invariant discrete probability distribution on a discrete set of invariant components of an initial corresponding static billiard.

Let F_m denote a value of F after m cycles of a billiard motion and let $\rho_{\boldsymbol{\pi}}(F_m)$ be a probability distribution for F_m with respect to an invariant probability distribution $\boldsymbol{\pi}$. All possible values of F_m are determined by the transition matrix M and by a pool of all possible values of F_1 corresponding to all possible ζ trajectories within one cycle of a billiard motion. Thus, for a particular billiard, it is essential to understand what are possible values of F_1. We shall show, using the Liouville theorem, that if for some ζ trajectories the corresponding values of F_1 do not vanish in the adiabatic limit, then this leads to the exponential acceleration on average, which is nontrivial, since F_1 can be negative as well, resulting in the exponential deceleration according to Eq. (11).

There are three types of time-dependent billiards in which F_1 vanishes in the adiabatic limit for almost all initial conditions and, consequently, the acceleration is slower than exponential:

(1) A time-dependent billiard in which all corresponding static billiards have only one invariant component which is necessarily ergodic, excluding a zero measure set of isolated periodic orbits. In this case there is only one ζ trajectory, for which $F_1 \to 0$ according to the adiabatic law $v_1 \sqrt{\mathcal{A}_1} = v_0 \sqrt{\mathcal{A}_0}$, where $\mathcal{A}$ is the area of a billiard [31]. In the adiabatic regime, the fluctuations of F_1, denoted by δF_1, scale with the velocity as $\delta F_1 \propto 1/\sqrt{v_0}$, which follows from the following consideration. The difference $\delta v = v_1 - v_0$ between the initial velocity v_0 and the final velocity v_1 is a sum of $n \propto v_0$ terms from each collision within one cycle of the billiard motion. If these terms are uncorrelated, then $\langle \delta v^2 \rangle \approx \kappa^2 n$ and

$$\langle \delta F_1{}^2 \rangle = \left\langle \left(\log \frac{v_0 + \delta v}{v_0} \right)^2 \right\rangle \approx \frac{\kappa^2 n}{v_0^2} \propto \frac{\kappa^2}{v_0}, \tag{14}$$

where κ^2 is some number independent of the velocity. So the distribution $\rho_{\boldsymbol{\pi}}(F_1)$ depends on the velocity of the initial ensemble. We expect that $\rho_{\boldsymbol{\pi}}(F_1)$ satisfies the following scaling property:

$$\frac{1}{\sqrt{v_a}}\, \rho_{\boldsymbol{\pi}}(\sqrt{v_a}\, F_1) = \frac{1}{\sqrt{v_b}}\, \rho_{\boldsymbol{\pi}}(\sqrt{v_b}\, F_1), \tag{15}$$

for every pair of sufficiently large initial velocities v_a and v_b of an ensemble of particles.

(2) A time-dependent billiard in which all corresponding static billiards are integrable [32]. In this case the adiabatic invariance of actions [33] ensures that $F_1 \to 0$ for every ζ trajectory. Moreover, a matrix M is an identity matrix for every initial corresponding static billiard, thus, a number of ζ trajectories is constant, equal to a number of invariant components. In the case of a time-dependent ellipse, as demonstrated in [32,34], the motion of the boundary induces a chaotic layer around the separatrices of corresponding static billiards, which plays a crucial role in the acceleration. The width of the chaotic layer depends on the particle velocity and vanishes in the strict limit $v \to \infty$. The chaotic layer cannot be considered as an invariant component of a corresponding static billiard.

(3) A billiard which undergoes shape-preserving transformations [24], such that a distance ℓ between each pair of points on a boundary changes by the same proportion, which means that $\dot{\ell}/\ell$ is constant, where $\dot{\ell}$ is a time derivative of ℓ. If a billiard driving is periodic this implies $F_1 \to 0$ as shown in Appendix B. In the adiabatic limit, in a reference frame in which a shape preserving billiard is at rest, the particle dynamics quickly converges to the dynamics of a static billiard [24]. This implies that in the adiabatic limit the transport between invariant components of a static billiard is

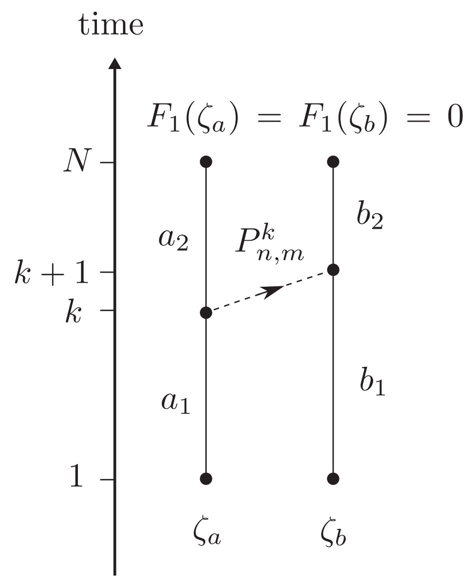

FIG. 1. Two ζ trajectories, namely ζ_a and ζ_b, which are represented as straight vertical lines, for which $F_1(\zeta_a) = a_1 + a_2 = 0$ and $F_1(\zeta_b) = b_1 + b_2 = 0$, are mixing with the probability $P^k_{n,m}$ from the invariant component n, which is the state on the trajectory ζ_a at time $j = k$, to the invariant component m, which is the state of the trajectory ζ_b at time $j = k + 1$. This allows an additional ζ trajectory for which $F_1 = a_1 + b_2 \neq 0$ if $a_1 \neq b_1$.

suppressed and, consequently, the number of ζ trajectories is constant in time, equal to the number of invariant components.

If some corresponding static billiards of a time-dependent billiard host more than one invariant component, then in such a time-dependent billiard there are many possible ζ trajectories. If transitions among invariant components are stochastic, then ζ trajectories are mixing in a sense that they can cross on a single invariant component. In this case a number of possible ζ trajectories increases exponentially in time.

In a generic case the mixing of ζ trajectories implies the existence of ζ trajectories for which $F_1 \neq 0$. This can be proved by contradiction. Assume that for all ζ trajectories $F_1 = 0$, then, because some ζ trajectories are mixing, a pair of mixing ζ trajectories can be found for which $F_1 = 0$. Let ζ_a and ζ_b be two such ζ trajectories. See Fig. 1. Let the mixing be such that at time k there is a nonzero probability $P^k_{n,m}$ to switch from ζ_a to ζ_b. Thus, there exists a ζ trajectory, denoted by ζ_{ab}, which follows ζ_a from time $j = 1$ up to time $j = k$ and then switches with the probability $P^k_{n,m}$ to ζ_b at time $j = k + 1$ and stick to it until the cycle of the billiard motion completes. Let a_1 and b_1 denote the changes of F on the time interval from $j = 1$ to $j = k$ and let a_2 and b_2 denote the changes of F on the time interval from $j = k + 1$ to $j = N$ on the trajectories ζ_a and ζ_b, respectively. By the assumption

$$F_1(\zeta_a) = a_1 + a_2 = 0, \quad F_1(\zeta_b) = b_1 + b_2 = 0, \tag{16}$$

and thus

$$F_1(\zeta_{ab}) = a_1 + b_2. \tag{17}$$

If $a_1 = b_1$ and $a_2 = b_2$, then $F_1(\zeta_{ab}) = 0$ as well. However, we expect that in a generic case different ζ trajectories are not correlated, which means that, in general, $a_1 \neq b_1$ and $a_2 \neq b_2$, and then, according to Eq. (17), $F_1(\zeta_{ab}) \neq 0$. Thus, we have found a ζ trajectory for which $F_1 \neq 0$, which contradicts the original assumption that for all ζ trajectories $F_1 = 0$.

Now, knowing that that for some ζ trajectories F_1 is nonvanishing, we shall show that this must result in the exponential acceleration, which is nontrivial since F_1 can be negative as well.

Let γ be a finite number of cycles of a billiard motion after which M^γ (M raised to the power of γ) can be considered sufficiently close to M^∞, which is effectively a number of cycles after which correlations between initial and final states of ζ trajectories are lost. Note that if at least one corresponding static billiard is ergodic (has only one invariant component), then $\gamma = 1$. By the definition of γ, a probability distribution $\rho_{\boldsymbol{\pi}}(F_{\gamma\, k})$ for F after $m = \gamma\, k$ cycles of a billiard motion, where k is some positive integer, equals the k-fold convolution power of $\rho_{\boldsymbol{\pi}}(F_\gamma)$. Using this fact and the distribution of the velocity in terms of F,

$$\rho(v) = \int dF\, \delta\big(v - v_0\, e^F\big)\, \rho_{\boldsymbol{\pi}}(F), \tag{18}$$

we find that a corresponding average velocity after $m = \gamma\, k$ cycles equals

$$\langle_{\gamma\, k}\rangle = v_0 \big\langle e^{F_\gamma}\big\rangle^k, \tag{19}$$

where v_0 is an initial velocity.

Now we show that the incompressibility of the phase-space flow (Liouville theorem) implies $\langle^{F_\gamma}\rangle > 1$ and thus the exponential acceleration. We have constructed the velocity dynamics in terms of the repeated convolutions of the distribution $\rho_\pi(F_\gamma)$ and now we want to deduce its properties. Because the dynamics of time-dependent billiards is Hamiltonian, we have to insist that $\rho_\gamma(F)$ is such that the velocity dynamics does not violate the Liouville theorem. For the arguments sake, suppose γ is big enough for $\rho_{\boldsymbol{\pi}}(F_\gamma)$ to be approximately Gaussian with the mean μ and the width $\sigma > 0$,

$$\rho_{\boldsymbol{\pi}}(F_\gamma) = \frac{1}{2\,\pi\,\sigma^2}\, e^{-\frac{(F_\gamma - \mu)^2}{2\sigma^2}}. \tag{20}$$

Consider some finite velocity v_c and denote with Ω_c the volume of the phase space below v_c. Take some large part of the phase space above v_c having the volume $\Omega \gg \Omega_c$ and the initial velocity distribution $\rho(v_0)$. A phase-space volume $\Omega_{v<v_c}$ that leaks below v_c after $m = \gamma\, k$ cycles equals

$$\begin{aligned}\Omega_{v<v_c} &= \Omega \int_0^{v_c} dv\, \rho(v)\\ &= \frac{\Omega}{2} \int \left[1 - \operatorname{erf}\left(\frac{\mu\, k + \log(v_0/v_c)}{\sqrt{2\, k\, \sigma^2}}\right)\right] \rho(v_0)\, dv_0,\end{aligned} \tag{21}$$

where we have used the fact that the phase-space volume is proportional to the probability. From Eq. (21) we see that if $\mu < 0$ or $\mu = 0$, then in the limit $k \to \infty$ the phase-space volume $\Omega_{v<v_c}$ converges to Ω or $\Omega/2$, respectively. But the amount of the phase-space volume that can be occupied below v_c is limited, $\Omega_{v<v_c} \leqslant \Omega_c$, thus $\Omega_{v<v_c} \to \Omega$ or $\Omega_{v<v_c} \to \Omega/2$ contradicts either the initial assumption $\Omega \gg \Omega_c$ or the Liouville theorem. Therefore, if $\sigma > 0$, then $\mu > 0$, which implies

$$\langle^{F_\gamma}\rangle = \int_{-\infty}^{\infty} \frac{dF_\gamma}{2\pi\sigma^2}\, e^{F_\gamma - \frac{(F_\gamma-\mu)^2}{2\sigma^2}} = e^{\mu+\sigma^2/2} > 1 \qquad (22)$$

and thus Eq. (19) implies the exponential acceleration. This is the central result of the paper.

III. NUMERICAL RESULTS

The exponential acceleration was already demonstrated in time-dependent billiards for which a number of physically connected parts of a billiard domain vary with time [29]. The exponential acceleration in such a time-dependent billiard is easily explained with the theory we have developed in Sec. II. Recently, the exponential acceleration was also demonstrated in a time-dependent mushroom billiard, which is a nonsmooth billiard with sharply separated regular and chaotic domains [30]. However, there has been no clear demonstration of the exponential acceleration in a smooth time-dependent billiard of the mixed type.

Demonstrating the exponential acceleration numerically could be a demanding task [17]. The problem is that only a finite number of collisions can be computed, which might not be enough to demonstrate exponential acceleration in an affordable amount of time. The exponential acceleration cannot be demonstrated numerically if the asymptotic regime occurs at such high velocities that we cannot afford to compute all the collisions of a reasonably large ensemble within one cycle of the billiard motion and show at least that $\rho_\pi(F_1)$ is asymptotically independent of the particle velocity.

The asymptotic regime of the exponential acceleration arises when the distribution $\rho_\pi(F_1)$ becomes effectively independent of the particle velocity. A regime below the asymptotic regime is called a transient regime.

Consider a time-dependent billiard which is predominantly chaotic with relatively small regular domains. In a transient regime the presence of regular domains in the phase space of corresponding static billiards is negligible and the billiard behaves as a fully chaotic. So, if σ is the asymptotic width of $\rho_\pi(F_1)$, then, according to Eq. (14), the asymptotic regime is reached when the particle velocity v satisfies

$$\sigma \gg \frac{\kappa}{\sqrt{v}}. \qquad (23)$$

The mechanism of exponential acceleration prevails when the above condition is satisfied.

Long transient regimes are expected also if deformations of a billiard shape are small, such that a structure of the phase space of corresponding static billiards varies very little and the billiard is close to the shape-preserving regime. In this case, the transport between regular and chaotic domains is weak, which renders the mechanism of exponential acceleration weak as well.

In fully chaotic billiards, we can observe in a transient regime the more efficient acceleration with $\beta > 1/2$ rather than $\beta = 1/2$, which is expected asymptotically. This can be due to the presence partial barriers and sticky objects in chaotic domains which act as quasi-invariant components for sufficiently small but large particle velocities.

In the following we present numerical analysis of four different time-dependent billiards. Two of them were already studied before: oval billiards [18] and annular billiards [20,35]. Although these billiards are of the mixed type, the authors did not observe the exponential acceleration, because they did not reach the asymptotic regime in their numerical simulations. We consider these two billiards again and show that they indeed exhibit the exponential acceleration. We have adjusted the parameters of the billiards in order to make the mechanism of exponential acceleration stronger and the transient regimes shorter.

We have also considered two time-dependent billiards which have not been studied before: a time-dependent Robnik billiard and a billiard which is a nonconvex deformation of the elliptical billiard. The last billiard is studied in depth and is presented here as a main example.

A. Main example

In this subsection we consider a time-dependent billiard with a boundary which satisfies a time-dependent implicit equation,

$$x^2 + \frac{2\,y^2}{1 + a\,(1+\cos t)\,(x^2-1)} = 1, \qquad (24)$$

where a is the deformation parameter. The period of the billiard motion is 2π. The motion of the billiard alternates between expanding and contracting phases in which the billiard passes over the same sequence of corresponding static billiards but in the reversed order.

In the following we shall closely consider the case $a = 0.3$, shown schematically in Fig. 2(a). In this case, the corresponding static billiards change from the almost completely chaotic at $t = 0$ to the completely regular (ellipse) at the half period $t = \pi$. In between the structure of the phase space is mixed and the invariant phase-space structures are rapidly changing with time, thus a very clear exponential acceleration is expected to be observed.

In Figs. 2(b)–2(d) different projections of the phase space are presented, in which a chaotic domain is colored gray and a regular domain is colored white. Together with the structures of the phase space, we plot the contours of constant $|f'|$, where f' is defined in Appendix B in Eq. (B2) and is just one of the possible approximations of the field $f(\mathbf{r},\theta,t)$ introduced in Eq. (9). The contours help to demonstrate that the acceleration of the particle, Eq. (10), differs in different parts of the phase space and that the average of f differs on different invariant components of the corresponding static billiards. Thus, different ζ trajectories have different associated values of F_1, where not all of them can be zero. Therefore the distribution of F_1 must have a finite variance, as shown in Fig. 2(b), which, according to the theory, implies the exponential acceleration, as shown in Fig. 2(a).

The theory predicts that $\rho_\pi(F_1)$ has asymptotically a nonvanishing width and a velocity independent shape. As shown in Fig. 3(b), the velocity dependence of $\rho_\pi(F_1)$ is already barely visible for velocities $v > 10^3$, except for the central peak. This peak is a consequence of the symmetry of the driving and converges to the Dirac δ distribution in the limit $v \to \infty$. As already mentioned, the motion of the

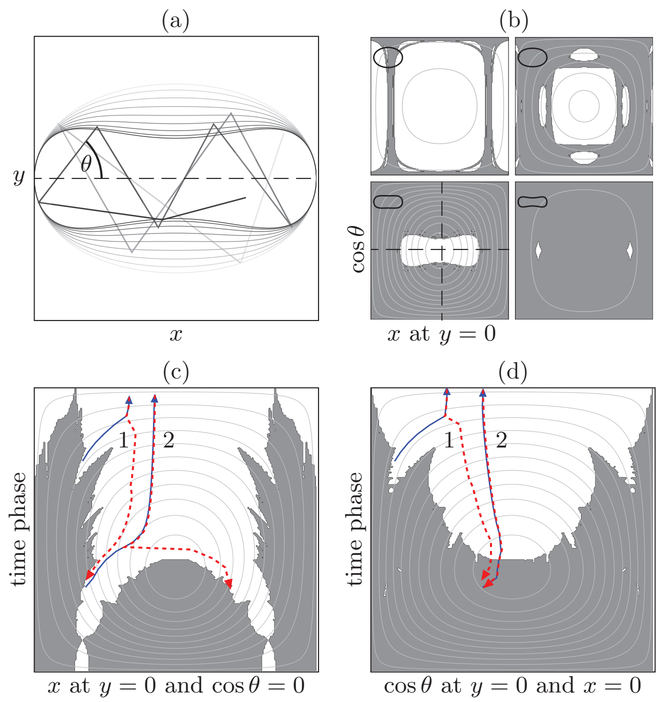

FIG. 2. (Color online) Numerical study of the phase space of a time-dependent billiard defined in (24) with the deformation parameter $a = 0.3$ (a). [(b), (c), and (d)] Projections on the Poincare line of section $y = 0$ [the dashed line in (a)]; gray denotes chaotic and white regular regions of the corresponding static billiards; in light gray, contours of constant $|f'|$, Eq. (B2), on 11 equidistant levels between 0 and $f'_{\max}$. (b) Phase-space structures of four corresponding static billiards (upper-left). [(c) and (d)] Time evolution of two slices [dashed lines in (b)]. Expanding and contracting phases are symmetric: expanding phase = direction up, contracting phase = direction down. Lines with arrows are fractions of two trajectories in two different projections: a time of one period was divided into 200 subintervals on which the value of local minimum of x (and $\cos\theta$) of a trajectory was determined and used in the plot instead of all intersections with the surface of section. Parts of trajectories in the chaotic region are not plotted. Solid blue and dashed red represent the expanding and contracting phases, respectively. The velocities of considered trajectories are $\sim 10^5$. Both trajectories start in the chaotic component at the beginning of the expanding phase, which is at the bottom of the diagrams. The trajectory 1 is a typical example for which $F_1 > 0$, as can be seen from the path through the contours of constant $|f'|$, while the trajectory 2 is symmetric and thus $F_1 \to 0$.

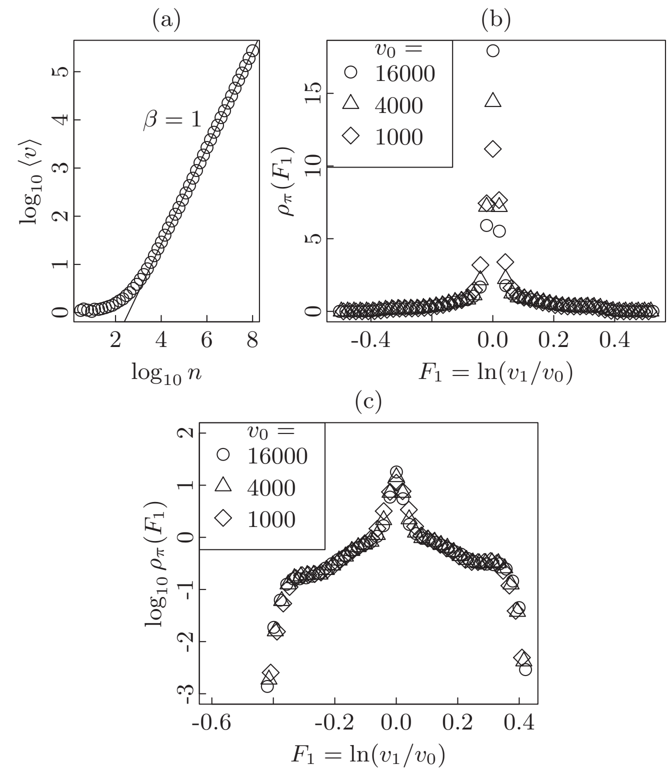

FIG. 3. (a) A linear increase of the average velocity with respect to the number of collisions n (exponential acceleration in the continuous time t); 10^3 initial conditions used. (b) The distribution of F_1 for different v_0 and for 10^6 initial conditions uniformly distributed in $\{\mathbf{r},\theta\}$ at $t = 0$, when the billiard is almost completely chaotic. The central peak is growing as $\sqrt{v_0}$ and converges to the Dirac δ distribution. The peak corresponds to symmetric ζ trajectories which pass over the same sequence of invariant components in both the expanding and contracting phases. The distribution $\rho_\pi(F_1)$ is effectively independent of the velocity and has a positive mean and a finite width, which implies the exponential acceleration. (c) The same as (b) but with $\log_{10}\rho_\pi(F_1)$ on the ordinate instead.

billiard alternates between the expanding and the contracting phases in which the billiard passes over the same sequence of corresponding static billiards, but in the reversed order. This symmetry implies that $F_1 \to 0$ for a symmetric ζ trajectory which passes over the same sequence of invariant components in both the expanding and contracting phases. Let us decompose the distribution $\rho_{\pi}(F_1)$ into a sum,

$$\rho_{\pi}(F_1) = p_a\,\rho_a(F_1) + p_b\,\rho_b(F_1), \qquad (25)$$

where $\rho_b(F_1)$ is the distribution of F_1 for symmetric ζ trajectories and corresponds to the peak at $F_1 = 0$, in Fig. 3(b). According to Eq. (15), the width of the distribution $\rho_b(F_1)$ scales as $1/\sqrt{v_0}$ and its height scales as $\sqrt{v_0}$. Now, since the fraction of symmetric ζ trajectories is constant, and thus p_b is constant, the height of the central peak of $\rho_{\pi}(F_1)$ should scale as $\sqrt{v_0}$, in agreement with the numerical results.

In Fig. 4 we show how the length of the transient regime depends on the deformation parameter a, Eq. (24). If deformations of the billiard are small ($a = 0.1$), then the structure of the phase space of corresponding static billiards does not change that much. This results in a reduced transport between invariant domains and, consequently, in the weak mechanism of the exponential acceleration, which reveals itself only after a long transient regime.

B. Oval billiard

In this subsection we shall consider a time-dependent oval billiard which was already studied in Ref. [18]. The shape of the billiard is defined in polar coordinates as

$$R(\theta) = (1 + \eta_1) + \epsilon\,(1 + \eta_2\,\cos t)\,\cos 2\,\theta, \qquad (26)$$

where η_1 and η_2 are deformation parameters. For convenience we have chosen the same symbols for the deformation

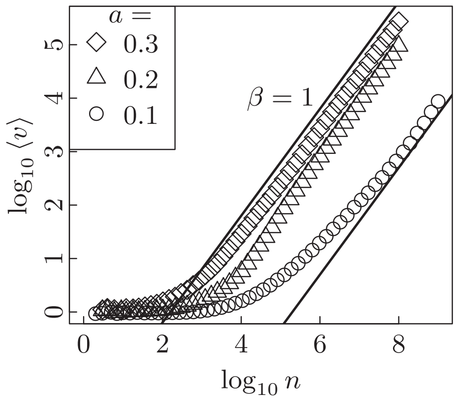

FIG. 4. The acceleration for different deformation parameters a in a billiard with the boundary as defined in (24). We see that smaller deformations (smaller a) correspond to a slower transition to the exponential acceleration (solid lines with slopes 1).

parameters as in Ref. [18]. In Ref. [18] only small deformations of a boundary were considered ($\eta_1 \leqslant 0.1$ and $\eta_2 \leqslant 0.1$) at $\epsilon = 0.4$ for which the whole phase space is almost entirely chaotic, except for the relatively small islands of regular motion. While the phase space is of the mixed type, though predominantly chaotic, the exponential acceleration is still expected in the deep adiabatic limit. In Ref. [18], however, the regime of exponential acceleration was not jet reached and thus not observed, although the numerical simulations had been evolved up to 10^9 collisions and the velocities of the order of 10^3 were reached.

We have considered again the case $\epsilon = 0.4$, $\eta_1 = 0$, and $\eta_2 = 0.1$, which was considered in Ref. [18]. As shown in Fig. 5(a), the exponential acceleration is not observed, even after 10^9 collisions, where the local acceleration exponent is $\beta \approx 0.575$. This is consistent with the distribution of F_1,

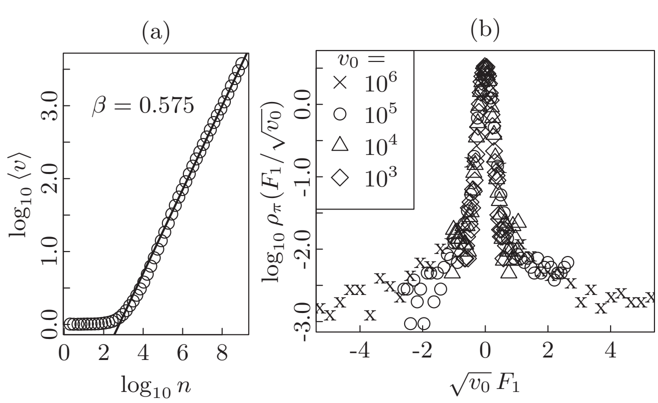

FIG. 5. The acceleration of particles (10^3 initial conditions at $v_0 = 1$) in time-dependent oval billiard with $\epsilon = 0.4$ and deformation parameters $\eta_1 = 0$ and $\eta_2 = 0.1$ (a). The acceleration exponent β is far from the expected asymptotic value $\beta = 1$. The distribution of F_1 steel strongly depends on the velocity (b), satisfying the scaling property (15) for fully chaotic systems. However, we see that tails are already independent of the velocity, thus in the deep adiabatic limit the mechanism of exponential acceleration should prevail. The distributions in (b) are calculated with ensembles of 10^4 initial conditions.

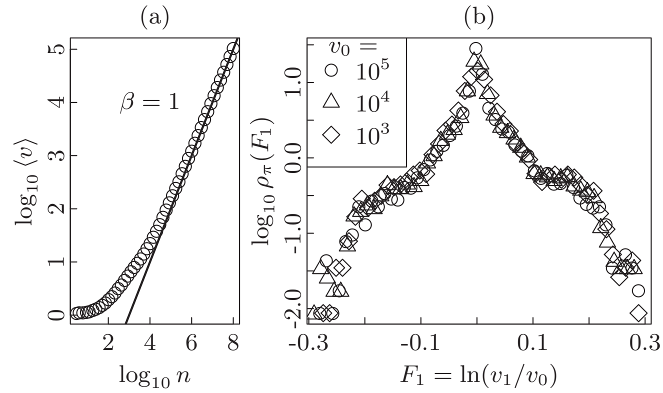

FIG. 6. The acceleration of particles (10^3 initial conditions at $v_0 = 1$) in a time-dependent oval billiard with $\epsilon = 0.2$ and deformation parameters $\eta_1 = 0$ and $\eta_2 = 0.9$ (a). The acceleration exponent β is approaching $\beta = 1$. The distribution of F_1 is (effectively) independent of the velocity in the regime $v > 10^3$ (b), which is the indicator of the exponential acceleration ($\beta = 1$). The distributions in (b) are calculated with ensembles of 10^4 initial conditions.

Fig. 5(b), where we can see that in the range of velocities 10^3–10^6 the distribution of F_1 effectively satisfies the scaling relation (15), which is valid for fully chaotic systems. However, the tails of the distribution are independent of the velocity, as expected, but in the concrete velocity range they are too weak to dominate the acceleration.

In order to demonstrate the exponential acceleration in the time-dependent oval billiard we chose a set of parameters $\epsilon = 0.2$, $\eta_1 = 0$, and $\eta_2 = 0.9$, for which the phase space of corresponding static billiards is more diverse and rapidly changes with time. This set of parameters was not studied in Ref. [18]. As expected, we see on Fig. 6 that this billiard exhibits a clear exponential acceleration.

It is worth mentioning that in Ref. [18] the authors studied the case $\epsilon = 0.4$ and $\eta_1 = \eta_2 = 0.1$, which is a time-dependent scaling transformation of the billiard. As already mentioned, in this case the exponential acceleration is not possible and the acceleration exponent equals $\beta = 1/6$, which is theoretically explained in Ref. [24].

C. Robnik billiard

In this subsection we consider a billiard with the boundary given in a parametric form as

$$x(s) = \cos(s) + \lambda(t)\cos(2s),$$
$$y(s) = \sin(s) + \lambda(t)\sin(2s), \qquad (27)$$

where s is a parameter that runs from 0 to 2π and where

$$\lambda(t) = \frac{1 - \cos t}{8} \qquad (28)$$

is the time-dependent deformation parameter, Fig. 7.

Static billiards, Eq. (27), for fixed values of λ, are known as Robnik billiards, introduced by Robnik [36]. For $\lambda = 0$ at $t = 0$, the billiard boundary is a circle and the corresponding static billiard is integrable. With the increasing λ the boundary deforms and the phase space of corresponding static billiards becomes of the mixed type, with the increasing chaotic

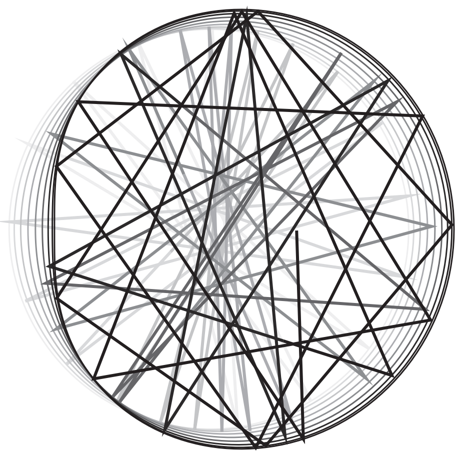

FIG. 7. A trajectory in the time-dependent Robnik billiard, Eq. (27).

component. Finally, for $\lambda = 1/4$ at $t = \pi$, the billiard is almost ergodic [37].

Again we expect the exponential acceleration, as confirmed in Fig. 8. Note a relatively slow transition to $\beta = 1$, which could be easily missed if the simulation would be terminated after 10^7 collisions.

D. Annular billiard

In this subsection we consider a time-dependent annular billiard, which was already studied in Refs. [20,35]. The domain of the annular billiard is the interior of the circle

$$x^2 + y^2 = R^2, \tag{29}$$

with the circular hole

$$(x - d)^2 + y^2 = r^2, \quad r + d < R, \tag{30}$$

which is shifted from the center of the big circle by d. An example is shown in Fig. 9. Trajectories which do not hit the internal boundary are regular as in a circle billiard, while the phase-space structure of trajectories which do hit the internal boundary is of the mixed type and depends on the parameters d and r.

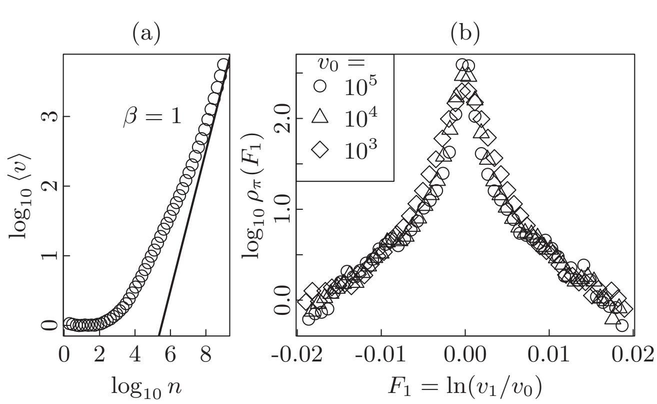

FIG. 8. The acceleration of particles (10^3 initial conditions at $v_0 = 1$) in a time-dependent Robnik billiard, Eq. (27). The distribution of F_1 is (effectively) independent of the velocity in the regime $v > 10^4$, while at $v = 10^3$ the distribution is still noticeably wider (b). The exponential acceleration is clearly visible only after 10^8 collisions (a).

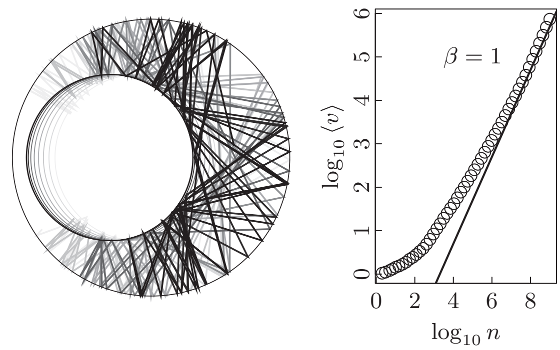

FIG. 9. A trajectory in the time-dependent annular billiard and the evident exponential acceleration.

In Refs. [20,35] authors studied a time-dependent annular billiard where $d = 0.5$ is fixed while $R = 1 + \epsilon \cos t$ and $r = 0.3 + \epsilon \cos t$ depend on time, where $\epsilon = 0.05$. This particular example is almost completely chaotic, with relatively small islands of regular motion if the trajectories which do not hit the internal boundary are excluded. Additionally, a relatively small variations of the billiard (small deformation parameter ϵ) renders the mechanism of exponential acceleration even weaker. It is thus no surprise that in Refs. [20,35] the exponential acceleration was not observed even after 10^8 collisions, although a rather large acceleration exponent $\beta \approx 0.62$ was observed.

We have considered a different set of parameters for which a phase space of corresponding static billiards is more diverse. We took fixed $R = 1$ and $r = 0.6$ and time-dependent $d = 0.3 \sin t$, Fig. 9. As expected, we observe a clear exponential acceleration, Fig. 9.

IV. CONCLUSIONS

The central result of the paper is the following statement: If a phase-space structure of some corresponding static billiards of a generic time-dependent billiard is of the mixed type, with coexisting regular and chaotic domains, then, in the adiabatic regime, such a time-dependent billiard exhibits exponential Fermi acceleration. A nongeneric example, for which the exponential acceleration is not possible, is a shape preserving time-dependent billiard [24]. Since a phase-space structure of a typical billiard is of a mixed type, we can conclude that the exponential acceleration is a most common mode of acceleration in time-dependent billiards in the adiabatic regime.

We have shown in this paper that in a time-dependent billiard a relevant part of the dynamics of fast variables in the adiabatic regime can be represented as a stochastic hopping between invariant components of corresponding static billiards where the hopping probabilities are represented as a Markovian transition matrices. The velocity dynamics is then described as an integral over a path through the space of invariant components of corresponding static billiards. We have shown that if a number of possible paths through the space of invariant components grows exponential with time, then this in general implies the exponential Fermi acceleration. This should be typically observed in a mixed type billiards such as those we have studied numerically in this paper.

Future studies should aim at a general understanding of transition probabilities among invariant components of corresponding static billiards. These were already calculated for a time-dependent mushroom billiard [30]. It is also important to understand the quantum-mechanical aspects of time-dependent billiards in the semiclassical limit [38], for which the formalism presented in this paper could prove relevant.

ACKNOWLEDGMENTS

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APPENDIX A

Here we derive the field $f(\mathbf{r},\theta,t)$ introduced in Eq. (9). First, let us introduce a boundary function $h(\mathbf{r},t)$, which is differentiable, zero on a billiard boundary, positive in a billiard domain, and negative outside a billiard domain. Then

$$\mathbf{n} = \frac{\nabla h(\mathbf{r},t)}{\|\nabla h(\mathbf{r},t)\|} \tag{A1}$$

is an inward normal unit vector at the point $(\mathbf{r},t)$ on the boundary. A total derivative of $h(\mathbf{r},t)$ on a boundary is zero by definition,

$$\nabla h(\mathbf{r},t)\cdot\mathbf{u} + \frac{\partial\, h(\mathbf{r},t)}{\partial t} = 0, \tag{A2}$$

where $\mathbf{u}$ is the velocity of the boundary. From Eqs. (A1) and (A2) we find a normal component of a boundary velocity,

$$\mathbf{n}\cdot\mathbf{u} = -\frac{1}{\|\nabla h(\mathbf{r},t)\|}\,\frac{\partial\, h(\mathbf{r},t)}{\partial t}. \tag{A3}$$

Let $\mathbf{k} = (\cos\theta, \sin\theta)$ be the unit vector in the direction of the particle velocity. By the definition,

$$\mathbf{k} = \frac{d\mathbf{r}}{ds}, \tag{A4}$$

where s is a geometrical length of the trajectory. Now, according to Eq. (8), the change of the velocity at a collision can be written as

$$\begin{aligned}\Delta v &= (\mathbf{k}_+ - \mathbf{k}_-)\cdot\mathbf{u} = (\mathbf{n}\cdot\mathbf{k}_+ - \mathbf{n}\cdot\mathbf{k}_-)\,\mathbf{n}\cdot\mathbf{u}\\ &= (|\mathbf{n}\cdot\mathbf{k}_+| + |\mathbf{n}\cdot\mathbf{k}_-|)\,\mathbf{n}\cdot\mathbf{u},\end{aligned} \tag{A5}$$

where the subscripts $-$ and $+$ denote a value right before and after a collision, respectively, and where we have taken into account that $\mathbf{n}$ is always pointing inside the billiard domain.

A change of the velocity appears only when the particle hits the boundary, thus the field $f(\mathbf{r},t)$ can be defined in terms of the Dirac δ function,

$$\begin{aligned}f(\mathbf{r},\theta,t) &= \delta(h(\mathbf{r},\theta,t))\left|\frac{d\,h(\mathbf{r},t)}{ds}\right|\ldots\\ &= \delta(h(\mathbf{r},\theta,t))\left|\nabla h(\mathbf{r},\theta,t)\cdot\mathbf{k} + \frac{1}{v}\frac{\partial\, h(\mathbf{r},t)}{\partial t}\right|\ldots\\ &\approx \delta(h(\mathbf{r},t))\Big|\nabla h(\mathbf{r},\theta,t)\cdot\mathbf{k}\Big|\ldots\end{aligned} \tag{A6}$$

where we have taken into account Eq. (A4) and $dt/ds = 1/v$, and in the last line neglected the term which is vanishing in the adiabatic limit. Combining everything together gives us, finally,

$$f(\mathbf{r},\theta,t) = -2\,\delta(h(\mathbf{r},t))\,\frac{|\nabla h(\mathbf{r},t)\cdot\mathbf{k}|^2}{\|\nabla h(\mathbf{r},t)\|^2}\,\frac{\partial\, h(\mathbf{r},t)}{\partial t}. \tag{A7}$$

This is a possible form of the field $f(\mathbf{r},\theta,t)$; however, in practice it is more convenient to work with a smooth field if we can neglect the stepwise structure of the velocity dynamics.

APPENDIX B

The velocity of the particle in a time-dependent billiard is a stepwise function of time, with jumps at collisions of the particle with the boundary. While the jumps are of the order of the velocity of the boundary, the stepwise structure of the velocity dynamics becomes unimportant in the adiabatic regime and can be as well represented with some continuous curve.

We define a continuous velocity v' of a trajectory as

$$v'(t) = \int_0^t dt\; v\, f'(\mathbf{r},\theta,t), \tag{B1}$$

where the field $f'(\mathbf{r},\theta,t)$ is defined in every phase-space point as

$$f'(\mathbf{r},\theta,t) = -\frac{\mathbf{r}_b - \mathbf{r}_a}{\|\mathbf{r}_b - \mathbf{r}_a\|^2}\cdot(\mathbf{u}_b - \mathbf{u}_a) = -\dot{\ell}/\ell, \tag{B2}$$

where $\mathbf{r}_b$ and $\mathbf{r}_a$ are two intersections between the straight line passing through $(\mathbf{r},\theta)$ and the boundary of the corresponding static billiard at time t, while $\mathbf{u}_b$ and $\mathbf{u}_a$ are their velocities and $\ell = \|\mathbf{r}_b - \mathbf{r}_a\|$ is their distance. In other words, points $\mathbf{r}_b$ and $\mathbf{r}_a$ are two successive collision points of a particle passing though $(\mathbf{r},\theta)$ in a corresponding static billiard at time t.

That Eq. (B1) approximates the true velocity can be demonstrated as follows. In every point $(\mathbf{r},\theta,t)$ on a trajectory between two successive collisions at points $\mathbf{r}_n$ and $\mathbf{r}_{n-1}$, we can approximate $\mathbf{r}_b \approx \mathbf{r}_n$ and $\mathbf{r}_a \approx \mathbf{r}_{n-1}$ up to corrections of the order of $1/v$, from which it follows

$$\begin{aligned}v'_n - v'_{n-1} &= \int_{t_{n-1}}^{t_n} dt\; v_{n-1}\, f'(\mathbf{r},\theta,t)\\ &\approx -\frac{\mathbf{r}_n - \mathbf{r}_{n-1}}{\|\mathbf{r}_n - \mathbf{r}_{n-1}\|}\cdot(\mathbf{u}_n - \mathbf{u}_{n-1})\\ &= -\frac{\mathbf{v}_{n-1}}{v_{n-1}}\cdot(\mathbf{u}_n - \mathbf{u}_{n-1}).\end{aligned} \tag{B3}$$

It is easy to see that the sum over the sequence of Eq. (8) can be rearranged into the sum over the sequence of Eq. (B3), such that

$$v_n - v_0 \approx \frac{\mathbf{v}_n}{v_n} \cdot \mathbf{u}_n - \frac{\mathbf{v}_0}{v_0} \cdot \mathbf{u}_0 + \int_{t_0}^{t_n} dt\, v\, f'(\mathbf{r},\theta,t). \quad \text{(B4)}$$

Therefore, in the adiabatic limit, the continuous velocity v' differs from the true velocity v by a term proportional to the velocity of the boundary $\|\mathbf{u}\|$ plus an error from the adiabatic approximation, which is vanishing. Thus, neglecting the structures of the velocity dynamics on the resolution $\|\mathbf{u}\|$, the velocity approximately satisfies the differential equation

$$\dot{v} = v\, f'(\mathbf{r},\theta,t), \quad \text{(B5)}$$

and, accordingly,

$$F = \int dt\, f'(\mathbf{r},\theta,t). \quad \text{(B6)}$$

Consider now a shape-preserving time-dependent billiard in which $f'(\mathbf{r},\theta,t) = f'(t) = -\dot{\ell}/\ell$ depends only on time. If a driving is periodic, then ℓ is also periodic. Thus, for every possible trajectory,

$$F_1 = \int_0^T dt\, f'(\mathbf{r},\theta,t) = \int_0^T dt\, f'(t) = 0. \quad \text{(B7)}$$

Therefore, in a shape preserving time-dependent billiard the exponential acceleration is not possible.

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