Nonlinear inhomogeneous Fokker-Planck equation within a generalized Stratonovich prescription

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We deduce a nonlinear and inhomogeneous Fokker-Planck equation within a generalized Stratonovich, or stochastic α, prescription ($\alpha = 0$, 1/2, and 1, respectively, correspond to the Itô, Stratonovich and anti-Itô prescriptions). We obtain its stationary state $p_{st}(x)$ for a class of constitutive relations between drift and diffusion and show that it has a q-exponential form, $p_{st}(x) = N_q[1-(1-q)\beta V(x)]^{1/(1-q)}$, with an index q which does *not* depend on α in the presence of any nonvanishing nonlinearity. This is in contrast with the linear case, for which the index q is α dependent.

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I. INTRODUCTION

The nonlinear Fokker-Planck (FP) equation has been largely used to study a wide class of physical systems which exhibit anomalous diffusion [1–5]. A particular feature of this equation is that its stationary solutions are probability distributions obeying *nonextensive statistical mechanics* [6,7].

From a mesoscopic point of view, nonlinear FP equations are related with a class of Langevin equations with multiplicative noise [8]. In these processes, the inhomogeneity of the diffusion function is proportional to a function of the probability density itself. Therefore, the computation of stochastic trajectories turns out to be very cumbersome. Indeed, one needs to know the complete time evolution of the probability density for each noise realization, making the problem a self-consistent one, very hard to deal with. Moreover, it is well known that to correctly define the stochastic multiplicative process it is necessary to fix a prescription to perform the Wiener integrals. The stochastic evolution depends on this prescription, and the final stationary state, if it exists, might also be prescription dependent. The most popular conventions are the methods of Itô and Stratonovich. However, it is possible to work within a more general scheme, usually referred to as a *generalized Stratonovich prescription* [9] or *α-prescription* [10]. This convention is parametrized by a parameter α ($0 \leqslant \alpha \leqslant 1$) and recovers the Itô and Stratonovich prescriptions as the $\alpha = 0$ and $\alpha = 1/2$ particular cases. The concept of equilibrium in these systems should be carefully defined, since the forward and backward stochastic evolutions are generally performed with different dual prescriptions and, as a consequence, the usual detailed balance relations should be properly generalized [11].

Recently, a class of inhomogeneous and nonlinear FP equations [12] was considered within the Itô prescription. It was shown that stationary solutions for the probability density are of the q-exponential form, namely,

$$p_{\rm st}(x) = P_q(V) \equiv N_q\, e_q^{-\beta V(x)} \equiv N_q\,[1-(1-q)\beta V(x)]^{\frac{1}{1-q}}, \qquad (1)$$

where $V(x)$ is any confining potential, N_q a normalization constant, β and q are two real numbers characterizing the distribution, and $e_1^z = e^z$; β is an inverse effective "temperature" [13]. In particular, $\lim_{q\to 1} P_q(V) = N_1\, e^{-\beta V(x)}$, the usual Boltzmann-Gibbs (BG) distribution.

Nonextensive statistical mechanics, also known as q-generalized statistical mechanics, is based on the nonadditive entropy $S_q = k(1-\sum_{i=1}^{W} p_i^q)/(q-1)$ [6]. This entropic functional was introduced with the aim of studying the thermodynamic properties of a class of systems with strongly correlated elements. Many different entropic forms have been defined throughout the years, in physics, cybernetics, and information theory, for a variety of specific purposes. In particular, a commonly used (for example, in multifractal systems) additive entropy is the Rényi's entropy $S_q^R = \ln \sum_{i=1}^{n} p_i^q/(1-q)$ [14]. Both S_q and S_q^R recover the Boltzmann-Gibbs-Shannon entropy, as $q \to 1$, and are monotonic functions of each other, $S_q^R = \ln[1+(1-q)S_q]/(1-q)$. So the stationary distributions associated with these entropies (or any monotonic function of them), for identical constraints, are the same, for instance, the q-exponential distributions [Eq. (1)]. In this way, for applications involving merely a probabilistic description based in distributions, both entropic forms can be indistinguishably used. However, for a full thermostatistical approach, S_q and S_q^R certainly are different. Among other differences, S_q is concave for all positive values of q, whereas S_q^R is concave only for q within the interval (0,1] [15]. Similarly, S_q is always Lesche stable, which is not the case of S_q^R [7].

As mentioned above, in [12], the FP equation was deduced within the Itô convention. In the present paper we analyze how the stationary distributions depend on the particular prescription used to define the stochastic process. To do this, we first deduce the inhomogeneous and nonlinear FP equation within the generalized Stratonovich convention, parametrized by α, and then we look for stationary solutions of a family of processes, defined by a particular class of constitutive relations between drift and dissipation. As we shall see, the model is parametrized with two real numbers, η and θ, defined hereafter. The first one measures the nonlinearity of the system, while the second one is related with inhomogeneity; the point $(\eta,\theta) = (0,0)$ corresponds to the linear homogeneous particular case and represents a normal diffusion process. The stationary-state solutions depend on the values of these parameters. In particular, it will become clear that the solutions are, in the linear limit $\eta \to 0$, nonanalytic in the space (η,θ). We have found that, for the general case in which $(\eta,\theta) \neq (0,0)$, the stationary probability distributions are q exponentials with an index q which is *independent of the stochastic prescription*. The different conventions, characterized by α, do modify the temperature parameter β but *not* q.

The paper is organized as follows. In Sec. II we present the nonlinear inhomogeneous FP equation and define our model. In Sec. III we study the linear limit ($\eta = 0, \forall\theta$), and in Sec. IV we address the general case. Finally, we discuss our results in Sec. V.

II. THE NONLINEAR FOKKER-PLANCK EQUATION FOR MULTIPLICATIVE MARKOV PROCESSES

Consider a Markovian multiplicative stochastic process described by the Langevin equation,

$$\frac{dx}{dt} = F(x,t) + [\phi(x,t)]^{1/2}\xi(t), \tag{2}$$

where $\langle \xi(t)\rangle = 0$ and $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$. $F(x,t)$ is the drift force, and $\phi(x,t)$ is in principle an arbitrary function that models the state-dependent diffusion process. As it is well known, this equation should be complemented with a prescription to integrate the Wiener integral. In this paper, we use the generalized Stratonovich prescription [9] or α prescription [10]. Briefly speaking, it is necessary to give sense to the ill-defined product $[\phi(x(t),t)]^{1/2}\xi(t)$, since $\xi(t)$ is δ correlated. By definition, the Riemann-Stieltjes integral of a Wiener process $W(t)$ with $\xi(t) = dW(t)/dt$ is

$$\int [\phi(x(t),t)]^{1/2}\, dW(t) = \lim_{n\to\infty} \sum_{j=1}^{n} [\phi(x(\tau_j),\tau_j)]^{1/2}[W(t_{j+1}) - W(t_j)], \tag{3}$$

where τ_j is taken in the interval $[t_j,t_{j+1}]$ and the limit is taken in the sense of *mean-square limit* [16]. For a smooth measure $W(t)$, the limit converges to a unique value, regardless the value of τ_j. However, $W(t)$ is not smooth; in fact, it is integrable nowhere. In any interval, white noise fluctuates an infinite number of times with infinite variance. Therefore, the value of the integral depends on the prescription for the choice of τ_j. In the generalized Stratonovich prescription we choose

$$x(\tau_j) = (1-\alpha)x(t_j) + \alpha x(t_{j+1}) \text{ with } 0 \leqslant \alpha \leqslant 1. \tag{4}$$

In this way, $\alpha = 0$ corresponds with the prepoint Itô interpretation and $\alpha = 1/2$ coincides with that of Stratonovich (midpoint). Moreover, the postpoint prescription, $\alpha = 1$, is also known as the kinetic or anti-Itô interpretation. In principle, each particular choice of α fixes a different stochastic evolution.

In many physical applications, a weakly colored Gaussian-Markov noise with a finite variance [17] is considered. In this case, there is no problem with the interpretation of equation (2) and the limit of infinite variance can be taken at the end of the calculations. This regularization procedure is equivalent to the Stratonovich interpretation, $\alpha = 1/2$ [18,19]. However, in other applications, like chemical Langevin equations [18] or econometric problems [20,21], the noise can be considered principally white, since it could be a reduction of jumplike or Poisson-like processes. In such cases, the Itô interpretation ($\alpha = 0$) should be more suitable. Hence, the interpretation of Eq. (2) depends on the physics behind a particular application. Once the interpretation is fixed, the stochastic dynamics is unambiguously defined.

From the stochastic equation (2) it is possible to derive a Fokker-Planck equation, given by [11,22–24]

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left\{\left[F(x) + \frac{\alpha}{2}\frac{\partial \phi(x,t)}{\partial x}\right]p(x,t)\right\} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\{\phi(x,t)p(x,t)\}, \tag{5}$$

where $p(x,t)$ is the time-dependent probability distribution and $\alpha \in [0,1]$ parametrize the stochastic prescription.

If the function $\phi(x,t)$ is an "external" fixed function, modeling a simple state diffusion process, then Eq. (5) is linear. However, as discussed in Ref. [8], the diffusion function could depend on the probability distribution itself, for instance,

$$\phi(x,t) = D[g(x)]^{\theta}[p(x,t)]^{\eta}, \tag{6}$$

where D is a constant diffusion coefficient, $g(x)$ is an arbitrary well-behaved function, and $p(x,t)$ is a solution of the FP equation. With this choice, Eq. (5) is a nonlinear equation describing a state-dependent diffusion process with nontrivial particle-bath couplings [8]. The real constants θ and η control the relative strength of these effects. For instance, the point $\eta = \theta = 0$ represents a normal diffusion process driven by a stochastic additive Langevin equation. On the other hand, the line $\eta = 0, \theta \neq 0$ represents a usual state-dependent diffusion process, described by a multiplicative Langevin equation. Moreover, the general case $\eta \neq 0$ is a multiplicative process whose diffusion functions should be self-consistently computed by solving the related nonlinear FP equation.

Equation (5) can be written as a continuity equation,

$$\frac{\partial p(x,t)}{\partial t} = \frac{\partial J(x,t)}{\partial x}, \tag{7}$$

where the current of probability is given by

$$J(x,t) = \left[-F(x) + (1-\alpha)\frac{1}{2}D\frac{\partial \phi(x,p)}{\partial x} + \frac{1}{2}D\phi(x,p)\frac{\partial}{\partial x}\right]p(x,t). \tag{8}$$

Here we have indicated that $\phi(x,p(x,t))$ could be a function of $p(x,t)$ given by Eq. (6).

The equilibrium distribution is defined as the stationary solution with zero current of probability, i.e.,

$$P_{\rm eq}(x) = \lim_{t\to\infty} p(x,t), \tag{9}$$

supplemented with $\lim_{x\to\pm\infty} J(x,t) = 0$. In the following sections we will find the equilibrium probability distribution in the whole parameter range $\{\eta,\theta\}$.

III. THE LINEAR CASE, $\eta = 0$

Let us begin by analyzing stationary states of the simpler case $\eta = 0$. For this case,

$$\phi(x) = D[g(x)]^{\theta}, \tag{10}$$

and the Fokker-Planck equation (5) is linear. The stationary states have been studied in Refs. [11,23,24] for different particular cases. In this section we summarize the main results and procedures in order to present them in a unified scheme and to compare them with the nonlinear case.

The equilibrium solution takes the form [11]

$$P_{\rm eq}(x) = N\, e^{-U_{\rm eq}(x)}, \tag{11}$$

where N is a normalization constant and the effective potential is given by

$$U_{\rm eq}(x) = -2\int^{x} \frac{F(\bar{x})}{D[g(\bar{x})]^{\theta}} d\bar{x} + (1-\alpha)\theta \ln g(x). \tag{12}$$

Thus, as already mentioned, the equilibrium distribution depends on the particular stochastic prescription used to define the Langevin equation. For general functions $F(x)$ and $g(x)$, the probability density distribution is given by Eqs. (11) and (12). The only constraint is a condition of integrability in order to compute the normalization factor N. To go further, we need to impose constitutive relations between drift and dissipation. For instance, suppose that the system is submitted to a conservative force, with energy potential $V(x)$. For $\theta = 0$, the resulting process is additive and

$$U_{\rm eq}(x) = \left(\frac{2}{D}\right) V(x), \tag{13}$$

up to an unimportant constant term that is absorbed in the normalization. Then the Einstein relation imposes for the inverse temperature $\beta = 2/D$, leading to the Boltzmann distribution. On the other hand, we could impose, for $\theta \neq 0$, a local generalization of the Einstein relation,

$$F(x) = -\left(\frac{\beta}{2D[g(x)]^{\theta}}\right) V'(x), \tag{14}$$

where V' indicates a differentiation with respect to x, ending with the solution [11,22]

$$U_{\rm eq}(x) = \beta V(x) + (1-\alpha)\theta \ln g(x). \tag{15}$$

We see that for multiplicative noise, the final distribution is generally not of the Boltzmann type, even for the usual prescriptions of Itô ($\alpha = 0$) or Stratonovich ($\alpha = 1/2$). The exception is the anti-Itô interpretation ($\alpha = 1$), which, together with the local Einstein relation, leads to the usual thermodynamical equilibrium distribution. For this reason, this convention is also called the kinetic prescription. An interesting particular case is to consider a “free” particle in an inhomogeneous dissipative medium, where $V = 0$, and the probability distribution is a power law of the form

$$P_{\rm eq} = N \frac{1}{[g(x)]^{\theta(1-\alpha)}}, \tag{16}$$

assuming it is normalizable.

Moreover, we could impose constitutive relations different from the local Einstein relation, such as the one used in Ref. [23]. We can choose, for instance, $F(x) = -V'(x)$, $Dg(x) = A + BV(x)$, and $\theta = 1$; for simplicity we shall assume $A > 0$ and $B > 0$. Substituting these expressions in Eq. (12) we immediately find a q-exponential form [Eq. (1)] with

$$q = \frac{2(B+1) - \alpha B}{B + 2 - \alpha B} \quad \text{and} \quad \beta = \frac{B(1-\alpha)+2}{A}. \tag{17}$$

Therefore, we have shown that using the general solution Eqs. (11) and (12), in the linear case $\eta = 0$, we can find different types of equilibrium distributions, such as the Boltzmann or the q-exponential distribution, depending on the constitutive relation imposed between drift and dissipation and on the particular stochastic prescription used to derived the FP equation. Let us also notice that whenever A and B have the same sign, the inverse temperature β is positive, as normally expected; if both are positive (negative), then $q > 1$ ($q < 1$), which corresponds to long-tailed distributions (compact support distributions).

IV. THE NONLINEAR CASE, $\eta \neq 0$

The solution of the nonlinear and inhomogeneous FP equation (5) with (6) for general values of $g(x)$ and $F(x)$ is quite involved. We will look for solutions imposing the constitutive relations [12]

$$F(x) = -V'(x) \text{ and } g(x) = A + BV(x), \tag{18}$$

where, as already mentioned, A, B are real positive constants.

Looking for stationary solutions $\partial p(x,t)/\partial t = 0$ and assuming appropriate boundary conditions which guarantee a null net flux, we have

$$\frac{\partial F(x)p(x,\infty)}{\partial x} = \frac{D}{2}\frac{\partial}{\partial x}\left[(1-\alpha)\frac{\partial \phi(x,p)}{\partial x} + \phi(x,p)\frac{\partial}{\partial x}\right]p(x,\infty). \tag{19}$$

The choices made in Eq. (18) allow us to write the differential equation (19) in terms of the variable V, obtaining

$$(1-\alpha)\frac{\partial[g(V)^{\theta}p(V)^{\eta}]}{\partial V}p(V) + g(V)^{\theta}p(V)^{\eta}\frac{\partial p(V)}{\partial V} = -\frac{2p(V)}{D}, \tag{20}$$

where $p(V) \equiv p(V(x),\infty)$. This equation can be rewritten in the form of a *Bernoulli equation* [25] (see also [26]),

$$\frac{dp(V)}{dV} + \frac{(1-\alpha)\theta B g(V)^{-1}}{[(1-\alpha)\eta + 1]}p(V) = -\frac{2g(V)^{-\theta}}{D[(1-\alpha)\eta+1]}[p(V)]^{1-\eta}, \tag{21}$$

a class of nonlinear differential equations that can be linearized by a suitable change of variables.

For $\eta = 0$, we recover the linear equation we have treated in the last section. For $\eta \neq 0$ we can perform the nonlinear change of variables,

$$Z(V) = C_N \; p(V)^\eta \;, \tag{22}$$

where C_N is a normalization constant. With this, Eq. (21) becomes a first-order linear ordinary differential equation,

$$\frac{dZ}{dV} + \frac{(1-\alpha)\eta\theta B g(V)^{-1}}{[(1-\alpha)\eta+1]} Z = -\frac{2g(V)^{-\theta} C_N \eta}{D[(1-\alpha)\eta+1]}, \tag{23}$$

with the general solution

$$Z = (A+BV)^{-\frac{(1-\alpha)\eta\theta}{(1-\alpha)\eta+1}} \left\{ C_I - \frac{2C_N\eta}{BD[(1-\alpha)\eta+1-\theta]} \times \left[(A+BV)^{\frac{(1-\alpha)\eta+1-\theta}{(1-\alpha)\eta+1}} - A^{\frac{(1-\alpha)\eta+1-\theta}{(1-\alpha)\eta+1}} \right] \right\}, \tag{24}$$

where C_I is an integration constant which depends on the "initial" condition. Thus, Eqs. (22) and (24) provide a family of explicit solutions of the nonlinear FP equation in terms of two constants, C_N and C_I, which should be adjusted by means of an initial condition and the probability distribution normalization.

A. $\theta = 0$

In the particular case $\theta = 0$, the inhomogeneity of the dissipation function $\phi(x)$ comes only from the probability density. The stationary solution can be read from Eq. (24),

$$Z = C_N p(V)^\eta = C_I \left(1 - \frac{2C_N\eta}{DC_I[1+\eta-\alpha\eta]} V \right), \tag{25}$$

which can be rewritten in terms of a q-exponential [Eq. (1)] with $N_q = (C_I/C_N)^{1/\eta}$,

$$q = 1-\eta \quad \text{and} \quad N_q^{1-q}\beta = \frac{2}{D(1+\eta-\alpha\eta)}\,. \tag{26}$$

Interestingly enough, the index q is α independent. We will show that this is a general feature of nonlinearity ($\eta \neq 0$). In contrast, the inverse temperature β depends on the prescription that has been used. In particular, Eq. (26), in the Itô prescription, i.e., $\alpha = 0$, coincides with the result for the homogeneous nonlinear model obtained in Refs. [1] and [2].

B. $\theta \neq 0$

For the inhomogeneous and nonlinear case, i.e. $\theta \neq 0$, we notice that a sensible simplification occurs for a specific choice of constants. More precisely, we shall assume

$$\frac{C_I}{C_N} = -\frac{2\eta A^{\frac{(1-\alpha)\eta+1-\theta}{(1-\alpha)\eta+1}}}{BD[(1-\alpha)\eta+1-\theta]}\,. \tag{27}$$

Consequently, using Eq. (24), we obtain

$$Z = -\frac{2C_N\eta}{BD[(1-\alpha)\eta+1-\theta]}(A+BV)^{1-\theta}\,. \tag{28}$$

Hence, from Eq. (22),

$$p(V) = \left\{ -\frac{2\eta}{BD[(1-\alpha)\eta+1-\theta]} \right\}^{\frac{1}{\eta}} (A+BV)^{\frac{1-\theta}{\eta}}\,. \tag{29}$$

This expression can be written in the form of a q-exponential [Eq. (1)] with

$$q = 1 - \frac{\eta}{1-\theta}, \tag{30}$$

and the parameter β

$$\beta = -\frac{B}{A}\frac{1-\theta}{\eta} \tag{31}$$

$$= \frac{C_N}{C_I}\frac{2(1-\theta)}{D[(1-\alpha)\eta+1-\theta]} A^{-\theta/(1-\alpha)\eta+1}. \tag{32}$$

We see that the q-exponential distribution is a solution of the nonlinear and inhomogeneous FP equation *for any value of the stochastic prescription* α. Moreover, the value of q itself is *universal* in the sense that it does not depend on the prescription α, as can be seen in Eq. (30). The index q depends only on the parameters η and θ, which measure the relative importance of nonlinearity and inhomogeneity. Of course, this value of q coincides with the one computed in Ref. [12] using the Itô prescription. Different stochastic prescriptions affect only the temperature parameter β.

From Eq. (30), it could be wrongly concluded that in the linear limit $\eta \to 0$, the probability distribution is of the Boltzmann type, since $q \to 1$. However, as we showed in the previous section, in the linear case, we find power-law solutions with nonuniversal q (α dependent). In other words, the solutions are *not* analytic in the linear limit $\eta \to 0$. In variance with this fact, the limit $\theta \to 0$ is perfectly well defined, as can be seen from Eqs. (30) and (26).

V. CONCLUSIONS

We have presented an inhomogeneous and nonlinear Fokker-Planck equation describing a generalized Markov stochastic process in the "generalized Stratonovich prescription." This prescription is parametrized by a real parameter, $0 \leqslant \alpha \leqslant 1$, and contains the usual Stratonovich ($\alpha = 1/2$), Itô ($\alpha = 0$), and kinetic ($\alpha = 1$) prescriptions as particular cases. We have also parametrized nonlinearity and inhomogeneity by means of two parameters, η and θ, in such a way that the point $\eta = \theta = 0$ represents a normal diffusion process. In this way, we have unified and generalized several results already obtained in the literature [1,2,11,12,23,24].

We have solved the stationary FP equation for all values of the parameters η,θ,α.For the linear case, $\eta = 0$, we have found a general solution depending on the drift, dissipation, and the prescription α. The Boltzmann distribution is obtained when a generalization of the Einstein relation and the kinetic prescription, $\alpha = 1$, are imposed. In all other cases, the solution is more involved.

There exists a link between the stationary solutions of the linear or nonlinear FP equation and the distribution obtained by extremizing a particular entropy under simple specific constraints. Indeed, the connection between the linear FP equation and Boltzmann-Gibbs entropy has long been well

TABLE I. Stationary solutions of the nonlinear inhomogeneous FP equation. The q-exponential distribution has been obtained for the family of constitutive relations given by (18). Notice that, in the nonlinear case, the exponent q does not depend on the stochastic prescription, while this is not the case for the linear inhomogeneous FP equation. (The q value for this case recovers the results in [23] for the particular instances $\alpha = 0$ and $\alpha = 1/2$ by doing $B \to 2M/\tau C$; it also recovers, for $\alpha = 0$, the results in [12] by doing $B \to BD$.) Let us emphasize that q cannot be arbitrarily large for a given $V(x)$; otherwise the normalizability property will be lost. For example, if we are dealing with q Gaussians, then it must be $q < 3$.

Fokker-Planck equation	Linear ($\eta = 0$)	Nonlinear ($\eta \neq 0$)
Homogeneous ($\theta = 0$)	Additive noise $q = 1$	Multiplicative noise $q = 1 - \eta$
Inhomogeneous ($\theta \neq 0$)	Multiplicative noise $q = \frac{2(B+1)-\alpha B}{B+2-\alpha B}$	Multiplicative noise $q = 1 - \frac{\eta}{1-\theta}$

known. Analogously, for nonlinear FP equations yielding specific classes of anomalous diffusion, nonadditive entropic functionals have been analyzed in detail [1,2]. In addition, this remarkable link has also been found for even more general nonlinear FP equations and entropic forms [5,27].

We analyzed a family of constitutive relations between drift and dissipation that results in a q-exponential distribution, thus exhibiting a possible mechanism compatible with nonextensive statistical mechanics. In the nonlinear case, $\eta \neq 0$, the value of the exponent q is, remarkably enough, α independent. The different prescriptions that define the stochastic process *only* affect the inverse temperature β. In the linear case, in contrast, the exponent of the power law does depend on the stochastic prescription α. This clearly shows that the linear limit of the solutions is not analytic, namely, $\lim_{\eta\to 0} p_{\rm eq}[\eta,\theta] \neq p_{\rm eq}[0,\theta]$.

Table I summarizes the values of the entropic index q for the q-exponential distributions obtained as the stationary-state distributions for nonlinear inhomogeneous Fokker-Planck equations when using the particular relations given by (18). A variety of possible physical applications of the present results can be found in Ref. [7] and references therein.

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