Finite-size scaling analysis of pseudocritical region in two-dimensional continuous-spin systems

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At low temperatures, the two-dimensional continuous-spin systems exhibit large correlation lengths. Some of them show the Berezinskii–Kosterlitz–Thouless-like transitions, and some others show pseudocritical behaviors for which correlation lengths are extremely large but finite. To distinguish pseudo and genuine critical behaviors, it is important to understand the nature of spin-spin correlations and topological defects at low temperatures in continuous-spin systems. In this paper, I develop a finite-size scaling analysis which is suitable for distinguishing the critical behavior and its applications to the two-dimensional XY, Heisenberg, and RP^2 models.

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I. INTRODUCTION

The finite-size-scaling (FSS) theory [1] has long been an indispensable paradigm for accurate analyses in computational physics. Using only a few data sets, the FSS theory enables us to infer the dependence of physical quantities on system size. In the field of phase transitions and critical phenomena in particular, estimates of critical exponents and central charges by FSS analyses provide approaches to grasp outstanding issues. While the FSS theory has provided powerful tools for investigating phase transitions and critical phenomena, there are intricate issues that give ambivalent results when we apply FSS analyses. One of the intricate issues is to distinguish pseudo phase transitions in systems that show extremely large but finite correlation lengths at finite temperatures. In such systems, we could erroneously recognize disordered phases as critical phases because conventional FSS analyses indicate divergence of the correlation length and scale-invariant behavior below their pseudocritical temperatures. In fact, such pseudocritical behavior would be found as fictitious were meticulous FSS analyses executed. But these signs that the system is in pseudocritical region are usually very subtle even when we execute extensive simulations.

In view of the pseudocritical behavior, continuous-spin models in a two-dimensional (2D) system give suitable test grounds. While the Mermin–Wagner–Hohenberg theorem [2] proved there is no true long-range order in the 2D continuous-spin systems with short-range bilinear interactions, the 2D XY model shows the Berezinskii–Kosterlitz–Thouless (BKT) transition [3,4] in which the correlation length diverges. Whereas the scenario of the BKT transition, the Z-vortex binding-unbinding mechanism, is well established, the existence of the criticality in the 2D Heisenberg model is not conclusive. Although the 2D Heisenberg model does not exhibit a transition at a finite temperature in the majority view, there are also reasons for skepticism about the absence of the transition [5–7]. On the other hand, a transition in the 2D RP^2 model (also known as the Lebwohl–Lasher model [8]) is widely accepted [9–12]. Since the topological point defects caused by the Z_2 vortices are stable in the 2D RP^2 model [13,14], it is probable that the topological transition could take place at a finite temperature as in the 2D XY model. However, the transition in the 2D RP^2 model is recently questioned from the point of view of the scaling theory and behavior of the order parameter [15,16]. The origin of the above contradicting results in the 2D Heisenberg and RP^2 model is their extremely large correlation length at low temperatures. Therefore, the development of a new FSS analysis that works properly in the pseudocritical region is highly desired.

The present paper gives a remedy for the above difficulty in FSS analyses. Since the remedy is a simple combination of an FSS analysis proposed by Caracciolo *et al.* [17] and an analytical form of the two-point correlation function, it is applicable to a wide range of problems. This paper is organized as follows: In Sec. II, the FSS analysis is explained, and a scaling function for systems that show a large but finite correlation length is given. The applications of the FSS analysis to the 2D continuous-spin systems, the 2D XY, Heisenberg, and RP^2 models, are given in Sec. III. Section IV is devoted to the summary and discussion.

II. FIXED-SCALE-FACTOR FINITE-SIZE-SCALING ANALYSIS

In this section, we explain a FSS analysis proposed by Caracciolo *et al.* [17] [hereafter we call this FSS analysis the fixed-scale-factor FSS (FSF-FSS) analysis] and give an asymptotic scaling function by combining an analytic form of the two-point correlation function with the FSS analysis. The asymptotic scaling function gives a criterion for judging a system as whether the system is in the critical region or in the pseudocritical region.

The essential point of the FSF-FSS analysis is to calculate the ratio of an observable $\mathcal{O}$ for different system sizes at the same temperature. According to the FSS theory, the ratio can be written as follows:

$$\frac{\mathcal{O}(\beta,sL)}{\mathcal{O}(\beta,L)} = F_{\mathcal{O}}(\xi(\beta,L)/L) + O(\xi^{-\omega},L^{-\omega}), \qquad (1)$$

where L, ξ, ω, and β denote the linear size of the system, the correlation length, the correction-to-scaling exponent, and the inverse temperature T (the Boltzmann constant k_B is set to unity). The parameter s (>1) in Eq. (1) is a fixed scale factor, and $F_{\mathcal{O}}$ is a scaling function. We call the function $F_{\mathcal{O}}$ the FSF-FSS function. The correlation length, $\xi(\beta,L)$, is estimated

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by the second-moment correlation length [18],

$$\xi(\beta,L) = \frac{1}{2\sin(\Delta k/2)}\sqrt{\frac{K(0)}{K(\Delta \boldsymbol{k})} - 1}, \tag{2}$$

where the function

$$K(\boldsymbol{k}) = \frac{1}{N}\sum_{i,j}\langle \boldsymbol{S}(\boldsymbol{r}_i)\cdot\boldsymbol{S}(\boldsymbol{r}_j)\rangle e^{i\boldsymbol{k}\cdot\boldsymbol{r}_{ij}} \tag{3}$$

is the structure factor for an order parameter $\boldsymbol{S}$. Here, $\Delta k = 2\pi/L$, and N is the number of spins. The angle brackets $\langle\cdots\rangle$ denote a thermal average. The behavior of the FSF-FSS function is categorized by the critical inverse temperature β_c and β as follows:

$$F_{\mathcal{O}}(\xi(\beta,L)/L) = \begin{cases} s^{\Delta_-(\beta,L)} & (\text{for } \beta > \beta_c) \\ s^{\Delta_{\mathcal{O}}} & (\text{for } \beta = \beta_c) \\ s^{\Delta_+(\beta,L)} & (\text{for } \beta < \beta_c), \end{cases} \tag{4}$$

where $\Delta_{\mathcal{O}}$ is the scaling dimension of the observable $\mathcal{O}$, and $\Delta_-(\beta,L)$ and $\Delta_+(\beta,L)$ are monotonic functions of the system size L around the critical temperature. The relation above is especially powerful when we choose an observable $\mathcal{O}$ whose $\Delta_{\mathcal{O}}$ is self-evident. For example, a critical point is easily identified by a point where $F_{\mathcal{O}}(\xi(\beta_c,L)/L) = 1$ ($\Delta_{\mathcal{O}} = 0$), when we choose the Binder parameter [19] as the observable $\mathcal{O}$. Or it can be identified by a point where $F_{\mathcal{O}}(\xi(\beta_c,L)/L) = s$ ($\Delta_{\mathcal{O}} = 1$) when the correlation length is chosen as the observable $\mathcal{O}$. This fact would be underestimated when the phase transition is clearly observed. However, the explicit criterion, e.g., $F_{\mathcal{O}}(\xi(\beta_c,L)/L) = 1$, is an indispensable advantage when a severe crossover behavior is observed around a phase transition.

Although the FSF-FSS analysis is quite useful, the analysis of marginal critical phenomena like the BKT transition is still difficult. To clarify the difficulty in the FSF-FSS analysis, we show the asymptotic form of $F_{\mathcal{O}}$ below. For calculating convenience, we choose the correlation ratio $\mathcal{C}$ [20] as an observable $\mathcal{O}$:

$$\mathcal{C}(\beta,L) = \frac{G(\beta,L/2;L)}{G(\beta,L/4;L)}, \tag{5}$$

where $G(\beta,r;L)$ is the two-point correlation function of the distance r in the system size L. When the system is in a *genuine* critical region, we obtain $F_{\mathcal{C}} = 1$, i.e., $\Delta_{\mathcal{C}} = 0$, by the power-law form of the two-point correlation function, $G(\beta_c,r) \approx r^{-d+2-\eta}$ for $r \ll L$. Here, d is the dimensionality of the system, and η is the critical exponent of the correlation function. However, when the system is in a *pseudo*critical region, the asymptotic formula of the two-point correlation function [21] is written by

$$G(\beta,r) \approx \frac{e^{-\kappa r}}{r^{d-2+\eta}}, \tag{6}$$

where κ is a parameter that is proportional to the inverse of the correlation length, $1/\xi(\beta)$. While this two-point correlation function behaves like a power function ($\approx r^{-d+2-\eta}$) in the $r \ll \kappa^{-1}$ region, it decays exponentially in the $r \geqslant \kappa^{-1}$ region. Using the asymptotic formula [22], the FSF-FSS scaling function is given by

$$F_{\mathcal{C}}(\xi(\beta,L)/L) = \exp\left\{-\frac{(s-1)\kappa(\beta)L}{4}\right\}. \tag{7}$$

Therefore, it becomes difficult to distinguish the between a genuine ($\kappa = 0$) and a pseudocritical region ($\kappa > 0$) from the behavior of $F_{\mathcal{C}}$ when a system approaches a "critical" region ($\kappa \simeq 0$). However, Eq. (7) also says that $F_{\mathcal{C}}$ never equals unity when the correlation length does not diverge. For the convenience of observing $F_{\mathcal{C}}$ approaching unity, we define a function

$$U_{\mathcal{C}}(\xi(\beta,L)/L) = 1 - F_{\mathcal{C}}(\xi(\beta,L)/L). \tag{8}$$

In the asymptotic region, $U_{\mathcal{C}}$ behaves like

$$U_{\mathcal{C}}(\xi(\beta,L)/L) \approx \frac{(s-1)\kappa(\beta)L}{4}. \tag{9}$$

Since the parameter κ is proportional to the inverse of the correlation length, we can read from the curve of $U_{\mathcal{C}}$ whether the correlation length diverges or not. That is, $U_{\mathcal{C}}$ reaches zero at a genuine critical point, so that $\ln U_{\mathcal{C}}$ is recognized as a concave function; $\kappa(\beta)$ in a pseudocritical region never diverges at finite temperature, so that $\ln U_{\mathcal{C}}$ is recognized as a convex function. In the next section, we show the usefulness of this simple criterion.

III. RESULTS

In this section, we show the FSF-FSS functions $F_{\mathcal{C}}$ and $U_{\mathcal{C}}$ and analyze the criticality of the 2D continuous-spin systems. The 2D XY model is employed as an example which possesses the genuine critical region, and the 2D Heisenberg model is employed as an example which shows the pseudocritical region. The Hamiltonian of these models is written as

$$\mathcal{H} = -J\sum_{\langle i,j\rangle}\boldsymbol{S}_i\cdot\boldsymbol{S}_j, \tag{10}$$

where J is the ferromagnetic exchange-coupling constant ($J > 0$) and $\boldsymbol{S}_i$ is a spin variable at site i. In this paper, we fix the constant J to unity. The length of the vector $\boldsymbol{S}$ is also fixed to unity, and the vector has two-dimensional (three-dimensional) degrees of freedom for the XY (Heisenberg) model. The summation in Eq. (10) runs over all nearest-neighbor sites. To examine the usefulness of the FSF-FSS function $U_{\mathcal{C}}$, we choose the 2D RP^2 model as a system which shows a nontrivial "critical" behavior. The RP^2 model resembles the Heisenberg model but the interaction is biquadratic,

$$\mathcal{H} = -J\sum_{\langle i,j\rangle}(\boldsymbol{S}_i\cdot\boldsymbol{S}_j)^2. \tag{11}$$

In the quite-low-temperature region, the RP^2 and the Heisenberg models are described by the same spin-wave formalism. Therefore, in the view of the point, the 2D RP^2 model would not exhibit the genuine critical region. But, in the view of topological theory [13,14], the 2D RP^2 model possesses the nonremovable line singularities, there might be a BKT-like critical point. In the BKT-like critical region, if it exists, the 2D RP^2 model exhibits a (quasi) long-range order in the low temperatures. To study the nematic order in the RP^2 model, we measure the largest eigen value $\lambda_{\max}$ of the nematic tensor order parameter Q,

$$Q = \frac{1}{N}\sum_{i=1}^{N}\begin{pmatrix} S_i^x S_i^x & S_i^x S_i^y & S_i^x S_i^z \\ S_i^y S_i^x & S_i^y S_i^y & S_i^y S_i^z \\ S_i^z S_i^x & S_i^z S_i^y & S_i^z S_i^z \end{pmatrix}. \qquad (12)$$

The thermal average of the nematic order parameter $\langle m\rangle$ is defined by

$$\langle m\rangle = \tfrac{3}{2}\left(\langle\lambda_{\max}\rangle - \tfrac{1}{3}\right). \qquad (13)$$

By this definition, $\langle m\rangle$ is zero in a disordered phase and unity in a fully ordered phase.

Physical quantities are calculated by Monte Carlo (MC) simulations. We executed MC simulations on the square lattice with the periodic boundary condition for all the three spin systems. The Swendsen–Wang multicluster updates [10,23,24] are employed for spin updates. For thermalization of the systems, at most initial 3.2×10^5 MC steps are discarded depending on the system size. The following 8×10^5 MC steps are allotted to measure physical quantities. In order to drive a system to its thermal equilibrium state as quickly as possible, we chained MC simulations from the highest to the lowest temperature. Ten independent samples are simulated to obtain statistical errors. The fixed scale factor s is set to two for the entire calculation of FSF-FSS functions.

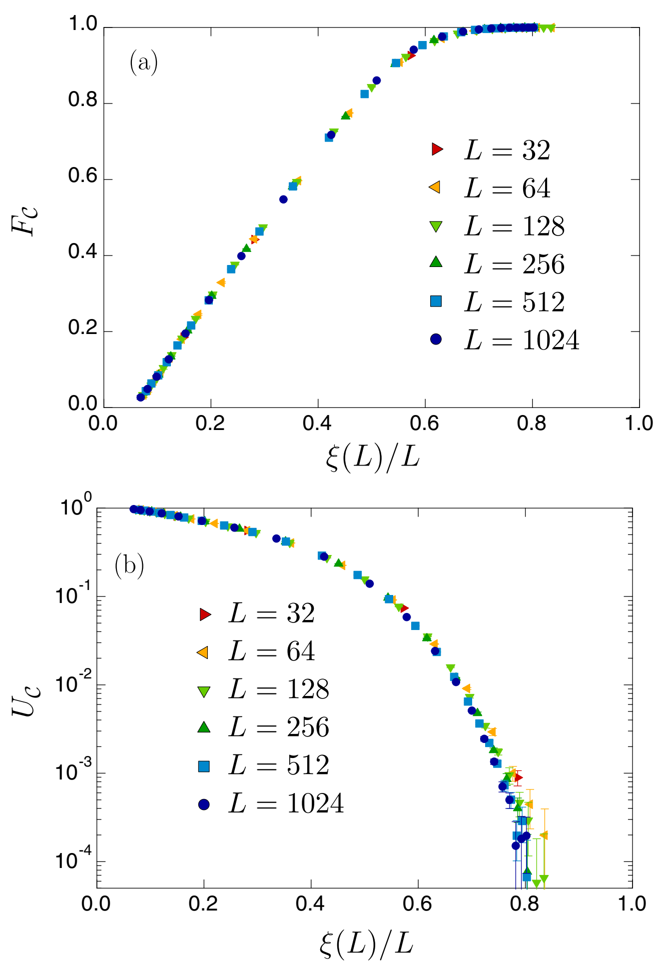

FIG. 1. (Color online) (a) FSF-FSS plot of correlation ratio for 2D XY model. Error bars are smaller than size of marks. (b) A semilogarithmic plot of an FSF-FSS function U_C for the 2D XY model. Since $\ln U_C$ is a concave function, U_C reaches zero at the critical temperature.

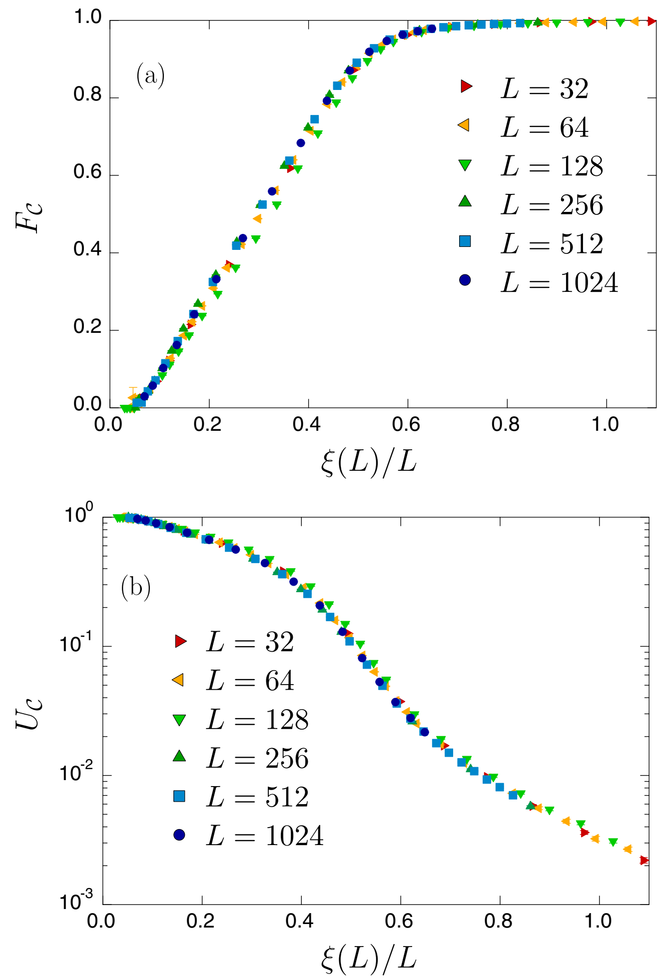

FIG. 2. (Color online) (a) FSF-FSS plot of correlation ratio for 2D Heisenberg model. Error bars are smaller than size of marks. (b) A semilogarithmic plot of an FSF-FSS function U_C for the 2D Heisenberg model. Since $\ln U_C$ is a convex function in a sufficiently large $\xi(L)/L$ region, U_C never reaches zero at a finite temperature.

Figure 1(a) shows the FSF-FSS function of the correlation ratio for the 2D XY model. Converging to unity of the function in $\xi/L > 0.7$ indicates that the 2D XY model has the genuine critical region. To confirm the existence of the critical region, we plot the FSF-FSS function U_C in Fig. 1(b). The function $\ln U_C$ of the 2D XY model is a concave function, and the function F_C converges to unity in the range of $0.8 \leqslant \xi/L < 0.9$.

Figure 2(a) shows the FSF-FSS function of the correlation ratio for the 2D Heisenberg model. The FSF-FSS function seems to converge to unity at sufficiently large ξ/L, but the logarithmic plot of U_C denies the possibility of the convergence. The function $\ln U_C$ of the 2D Heisenberg model is a convex function in a sufficiently large ξ/L region, so that we conclude the function U_C never reaches zero at a finite temperature.

The application to the 2D RP^2 model demonstrates that the FSF-FSS analysis is quite useful for distinguishing the genuine

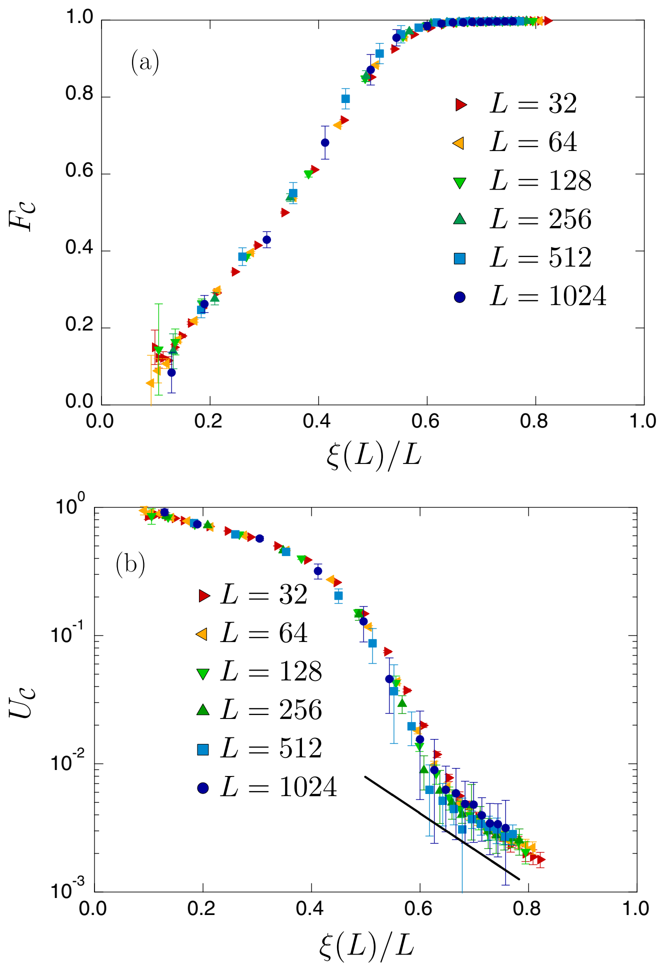

FIG. 3. (Color online) (a) FSF-FSS plot of correlation ratio for 2D RP^2 model. (b) A semilogarithmic plot of a FSF-FSS function $U_{\mathcal{C}}$ for the 2D RP^2 model. Since $\ln U_{\mathcal{C}}$ is a convex function in a sufficiently large $\xi(L)/L$ region, $U_{\mathcal{C}}$ never reaches zero at a finite temperature. The slope abruptly changes at the inflection point $\xi_\times/L$ (∼0.6). The solid line is a guide to the eye.

critical behavior from that of the pseudocritical behavior. To estimate the correlation length $\xi(\beta,L)$ in Eq. (2) and the correlation ratio $\mathcal{C}$ in Eq. (5), spin-spin correlations in biquadratic form [i.e., $(\boldsymbol{S}_i \cdot \boldsymbol{S}_j)^2$] are used, while they are in bilinear form [i.e., $(\boldsymbol{S}_i \cdot \boldsymbol{S}_j)$] for the XY and Heisenberg models. Since Fig. 3(a) exhibits a smooth approach toward unity, one might deduce that the 2D RP^2 model possesses a genuine critical region. But the logarithmic plot of $U_{\mathcal{C}}$ clearly denies the existence of the genuine critical region. The rather large error bars in Fig. 3 compared with Figs. 1 and 2 originate from the difference in the computational method. While we utilize improved estimators [25,26] constructed by Swendsen–Wang clusters for the XY and Heisenberg models, bare observables are calculated for the RP^2 model. Because Swendsen–Wang clusters in the RP^2 model do not directly relate to the spin-spin correlation, utilization of improved estimators is not sufficiently effective. Therefore, bare observables are chosen in the RP^2 model.

Whereas both of the 2D Heisenberg and the 2D RP^2 models possess the pseudocritical region, the behavior of $\ln U_{\mathcal{C}}$ is

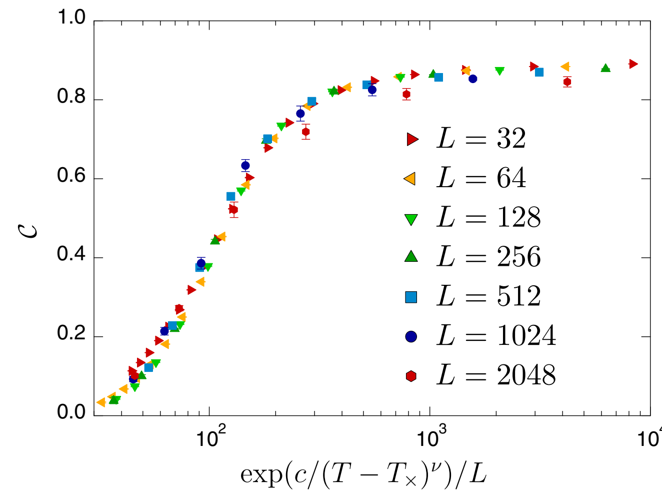

FIG. 4. (Color online) FSS plot of correlation ratio $\mathcal{C}$ for 2D RP^2 model.

not the same. The slope of the $\ln U_{\mathcal{C}}$ for the 2D Heisenberg model is smooth in all the range of ξ/L, but the slope of $\ln U_{\mathcal{C}}$ for the 2D RP^2 model abruptly changes at the inflection point $\xi_\times/L$ (∼0.6). This abrupt change indicates that a mode of development of the correlation length changes at $\xi_\times/L$. This behavior has also been observed in the triangular-lattice Heisenberg antiferromagnet [27–29] in which the topological point defects caused by the Z_2 vortices are stable as in the 2D RP^2 model. Assuming the correlation length develops as

$$\xi \approx \exp\left[\frac{c}{(T - T_\times)^\nu}\right] \quad (14)$$

in the range of $\xi < \xi_\times$, the crossover temperature $T_\times$, the exponent ν, and a nonuniversal constant c are estimated by the FSS analysis combined with Bayesian inference [30]. It should be mentioned that the correlation length ξ defined in Eq. (14) is not equal to $\xi(L)$ defined in Eq. (2). While $\xi(L)$ measures a correlation length in a finite system of size L, ξ is an extrapolated correlation length in the thermodynamic limit. In Fig. 4, we plot the scaled correlation ratio $\mathcal{C}$. The estimated values are $T_\times = 0.3431\,(2)$, $\nu = 0.247\,(2)$, and $c = 4.18\,(4)$. The results are consistent with preceding studies of the 2D RP^2 model [9–12]. It must be stressed that the FSS plot in Fig. 4 does not mean the existence of the transition in the 2D RP^2 model, but the plot results from the subtle and persistent crossover in the model. The FSS plot shows the data are well fit except the data point $L = 2048$. Indeed, the fact erroneously deduces results that claim the existence of the transition. This difficulty in distinguishing the pseudocritical behavior from that of genuine behavior can be overcome by using the FSF-FSS analysis. That is, as shown in Figs. 1, 2, and 3, plotting the FSF-FSS function $U_{\mathcal{C}}$, it can be seen whether the correlation length diverges at a finite temperature or not.

The above FSF-FSS analysis shows that the critical behavior in the 2D RP^2 model is pseudocritical, while some studies mention that the 2D RP^2 model possesses the genuine critical region. This discrepancy comes from a subtle and persistent crossover in the model, which disguises the pseudocritical region from that of the genuine. To see the crossover, we plot the distribution function of the nematic order parameter [see Eq. (13)] at a couple of temperatures. The distribution function $P(L;m)$ of the order parameter m for the system size L is scaled by the scaling function $\tilde{P}$ as

$$P(L;m) = L^{\psi} \tilde{P}((m - \langle m\rangle_{\infty}) L^{\psi}), \quad (15)$$

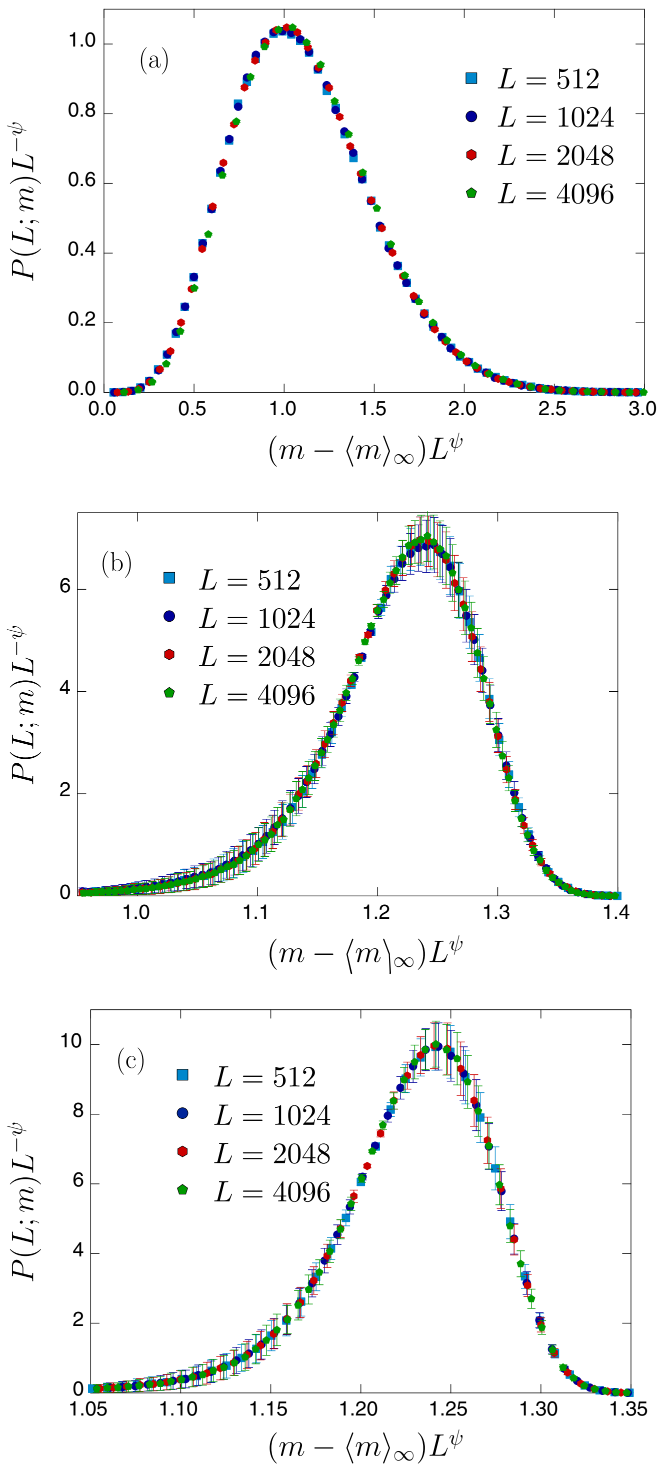

FIG. 5. (Color online) FSS plot of distribution functions of the nematic order parameter m. (a) FSS plot at a high temperature ($T = 1.0$). (b) FSS plot at the crossover temperature ($T = 0.3431$). (c) FSS plot at a low temperature ($T = 0.3$).

where $\langle m\rangle_{\infty}$ is the value of the order parameter at the thermodynamic limit. In a critical region, the expected value of $\langle m\rangle_{\infty}$ is zero, since $\langle m\rangle$ is proportional to $L^{-\psi}$. At a high-enough temperature, the distribution function $P(L;m)$ should be scaled by the Gaussian exponent ($\psi = 1$). Figure 5(a) shows the scaled distribution function at a high temperature ($T = 1.0$) for several system sizes. The estimated ψ and $\langle m\rangle_{\infty}$ are 0.994 (2) and $-8.0\,(9) \times 10^{-6}$, respectively. The number in parenthesis represents one standard error in the last digit. Since both the temperature and the system sizes are finite, the exponent ψ is a little smaller than unity. In a genuine critical region, the exponent ψ is equal to β/ν, where β is the critical exponent of the order parameter and ν is that of the correlation length. When a system is in a pseudocritical region, one might observe a pseudocritical exponent ψ. However, the value of ψ converges to the Gaussian value when the system size is sufficiently larger than the correlation length $[L \geqslant \kappa(\beta)^{-1}]$. Figure 5(b) shows scaled data at the crossover temperature ($T = T_{\times}$), and Fig. 5(c) shows scaled data at a low temperature ($T = 0.3$). Using the Bayesian inference, pseudocritical exponents are estimated as $\psi(T_{\times}) = 0.115\,(6)$ and $\psi(0.3) = 0.08\,(4)$, and the estimated order parameters are $\langle m(T_{\times})\rangle_{\infty} = -0.211\,(30)$ and $\langle m(0.3)\rangle_{\infty} = -0.249\,(37)$. In both temperatures, the distribution functions are scaled by a single effective exponent $\psi(T)$ at a temperature T. In the conventional FSS analysis, this fact means that the system is in the critical region. However, because $\langle m\rangle$ is non-negative by definition, observing negative $\langle m\rangle_{\infty}$ implies that the obtained scaling plot of the distribution function is pathological. In fact, as demonstrated by the FSF-FSS analysis above, the system is in a disordered phase at finite temperatures, and therefore the true value of ψ must be unity. The observation of the pseudocritical exponents without any crossover indicates that the crossover from the pseudocritical to the disordered phase is quite subtle and persistent, at least in the range of $L \sim O(10^3)$.

IV. SUMMARY AND DISCUSSION

In this paper, we applied the FSF-FSS analysis [17] to three two-dimensional continuous-spin systems and examined their criticalities.

A merit of the FSF-FSS analysis is that we know the value of the scaling function with a self-evident scaling dimension $\Delta_{\mathcal{O}}$ at the critical point, $F_{\mathcal{O}}(\xi(\beta_c,L)/L)$, in advance [see Eq. (4)]. In the conventional FSS analysis, we have to identify the crossing point of dimensionless quantities (e.g., Binder cumulant [19]) in the two-dimensional parameter space, temperature T and an observable $\mathcal{O}$, but the identification is executed in the one-dimensional parameter space (temperature T) in the FSF-FSS analysis. This merit reduces uncertainties in the FSS analysis and improves reliability of the FSS analysis. In particular, in the case of the correlation length is finite but considerably long: we obtained asymptotic formulas of the scaling functions [Eqs. (7) and (9)] by assuming the correlation function in the pseudocritical region [21]. The analysis of the 2D RP^2 model is consistent with the obtained formulae (see Fig. 3), so that we conclude the correlation length of the model is finite at finite temperatures.

We also executed an FSS analysis of the distribution function of the order parameter, and quite slow and persistent crossover from the pseudocritical to the disordered phase was shown in pseudocritical exponents ψ and the value of the order parameter at thermodynamic limit $\langle m\rangle_\infty$. If the correction form is given by a multiplicative logarithmic form as in the 2D XY model, the system size dependence of the order parameter is written by $\langle m\rangle \sim (\ln aL)^\phi L^{-\psi}$. Here a and ϕ are constants. Then the effective pseudocritical exponent is obtained as a function of L, $\psi(L) = \psi_\infty - \phi/(\ln aL)$, where ψ_∞ is the Gaussian exponent ($\psi_\infty = 1$). In this study, we cannot recognize a correction in the pseudocritical exponent. To reveal the correction form could be helpful to understand the roles of correlations of topological defects in the pseudocritical region.

While the correlation length was finite at finite temperature, we observed notable change in the development of the correlation length at the crossover temperature $T_\times$. This notable change, which is not observed in the 2D Heisenberg model, would be caused by the binding unbinding of the Z_2 vortices. The binding unbinding of the Z_2 vortices is extensively discussed in the antiferromagnetic Heisenberg model on the triangular lattice [27–29]. Kawamura and coworkers insist that there is a thermodynamic phase transition at a finite temperature driven by the binding unbinding of the Z_2 vortices [28], but clear evidence of the existence of the phase transition is not obtained in this study. Besides the study of the binding unbinding of the Z_2 vortices, Hasenbusch pointed out that FSS functions of $O(N)$ and RP^{N-1} should be the same if the boundary condition is properly controlled [31]. In this study, although we employed the periodic boundary condition for both the XY and Heisenberg models and the RP^2 model, a systematic study of the influence of the boundary condition on FSS functions would shed a light on the discussion about the universality class of the $O(N)$ and RP^{N-1} models. The effect of the nonlinear interaction in the RP^2 model is also an interesting issue in view of the pseudocritical behavior. The effect of nonlinearity in the interaction is studied by a nonlinear model [32–35], $\mathcal{H} = \sum_{\langle i,j\rangle}[(1 + \boldsymbol{S}_i \cdot \boldsymbol{S}_j)/2]^p$, and the critical value of p at which the first-order transition emerges is estimated as $p_c \gtrsim 20$ [33,34]. Considering that the parameter p in the RP^2 model ($p = 2$) largely deviates from p_c, the pseudocritical behavior in the RP^2 model could hardly be affected by the first-order transition point. However, it is interesting to study the relation between the Z_2 vortex and the first-order transition when p is sufficiently large. I leave these issues open for future study.

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