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Exploring chaos in the Dicke model using ground-state fidelity and Loschmidt echo

Utso Bhattacharya, Sayak Dasgupta, and Amit Dutta
Department of Physics, Indian Institute of Technology, 208016, Kanpur
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We study the quantum critical behavior of the Dicke Hamiltonian with finite number of atoms and explore the signature of quantum chaos using measures like the ground-state fidelity and the Loschmidt echo and the time-averaged Loschmidt echo. We show that these quantities clearly point to the classically chaotic nature of the system in the superradiant (SR) phase. While the ground-state fidelity shows aperiodic oscillations as a function of the coupling strength, the echo shows aperiodic oscillations in time and decays rapidly when the system is in the SR phase. We clearly demonstrate how the time-averaged value of the echo already incorporates the information about the ground-state fidelity and stays much less than unity, indicating the classically chaotic nature of the model in the SR phase.

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I. INTRODUCTION

A classical system is said to be integrable if the number of independent conserved quantities in the system equals the number of degrees of freedom. The motion of a particle then takes place on a d-dimensional tori, whereas the absence of symmetries in the system causes the particle trajectory to get delocalized over the whole of the energy surface within a bounded region of the phase space. Such trajectories may have hypersensitivity to initial conditions resulting in chaotic dynamics. Such chaotic dynamics in classical systems are generally characterized by a nonzero positive Lyapunov exponent, which quantifies the exponential divergence of "nearby" trajectories [1].

There are two different types of motions in classical Hamiltonian mechanics: regular motion of integrable systems or regular and chaotic motion produced by nonintegrable systems. To understand whether a system is chaotic we look at a cluster of trajectories of a Hamiltonian H originating from nearly same initial conditions in the phase space. In chaotic systems any two trajectories separate exponentially fast with time, while for a regular system the separation varies with a power law involving time (t). The linearity of quantum mechanics disallows the phenomenon of chaos in quantum systems [2]. Taking two eigenstates of the Hamiltonian H at slightly separate phase space points, after time t, $|\langle\phi(t)|\psi(t)\rangle|^2 = |\langle\phi(0)|\psi(0)\rangle|^2$ due to the unitary nature of the time evolution operator $U = \exp(-iHt/\hbar)$; hence this direct method of taking overlaps does not work in trying to identify the possibility of chaos for the corresponding classical Hamiltonian.

The correspondence principle, however, demands that just like their classical counterparts, exponential sensitivity to initial conditions should also manifest itself somehow in quantum dynamics. That is, signatures of chaos can be identified for quantum Hamiltonians, which will indicate their classical counterparts to be chaotic [2]. The Loschmidt echo (LE), which is the overlap of the same wave function evolved under two slightly different Hamiltonians, was proposed as a measure to identify chaos in quantum systems [3]. In order to understand the role of the LE in identifying "quantum" chaos in the Dicke Hamiltonian (DH) [4], we study the LE and another measure related to it, the ground-state quantum fidelity.

The Dicke model is a system of N interacting two-level atoms placed in a bosonic cavity with a coupling characterized by the parameter λ. This model is widely studied in quantum optics to understand collective effects. In the limit of an infinite system the model is integrable and shows a sharp quantum phase transition. For a finite-size system (characterized by a finite number of atoms proportional to j) the transition is rounded off; however, it shows a transition from a normal phase (quasi-integrable) to a superradiant (SR) (chaotic) phase. This is well understood from the studies of energy-level statistics performed on it [5,6]. We use this finite j case to investigate chaos in this present article.

Emary and Brandes [5,6] used level statistics of the energy eigenvalues of the DH in the finite j case to indicate the presence of chaos. They have used the fact that quantum systems have conserved quantities when their classical counterparts have a high degree of symmetry, which leads to degeneracy in the energy spectrum. This enables them to construct a distribution $P(S)$, of the nearest neighbor level spacing (denoted by S); $P(S)$ is given by the Poisson distribution as $S \to 0$, when such symmetries exist. Such a quantum system is "quasi-integrable." On the contrary, the classically chaotic regime is devoid of symmetries, and, hence, the quantum Hamiltonian is nondegenerate and absent energy level crossings leads to $P(S) \to 0$ as $S \to 0$ giving rise to the Wigner-Dyson distribution $[P_W(S) = \pi(S/2)\exp(-\pi S^2/4)]$.

For finite j, the appearance of Poisson distribution of $P(S)$ in the normal phase and the Wigner-Dyson distribution in the SR phase serves as a good signature for the transition to chaos. However, this correspondence between the $P(S)$ and the "chaoticity" of the classical or the quantum Hamiltonian is not general or unique, and a good number of exceptions do exist [6]. This motivates us to look for other signatures to identify chaos in a more general fashion using two quantum information theoretic measures: the ground-state fidelity and the time average of the LE. We note that there exists a different approach based on the operator fidelity metric [7] which bypasses the necessity of a perturbative expansion in the coupling strength in order to generate the eigenstates for the modified Hamiltonian with a shifted parameter value. In our case, on the other hand, we use a numerical method to obtain the eigenstates in a direct fashion. We note here that the signature of chaos is also found by studying the semiclassical equations of motion [8].

In recent years, there have been many works which have studied the connection between quantum phase transitions [9], quantum information [10,11], and quantum critical dynamics [12,13]. Two important measures which show interesting behavior close to a quantum critical point are the LE [14–17] and the ground-state quantum fidelity [18] (see review articles [12,19]). The former in particular has been studied extensively in recent years in connection with the dynamics of decoherence [20–22], work statistics [23], equilibration [24], and the dynamical phase transition [25,26]. As noted already, the concept of the LE was proposed in connection to quantum chaos [3] to describe the hypersensitivity of the time evolution of the system to the perturbations experienced by the surrounding environment; there have been many studies in this direction in recent years [27–30]. We emphasize at the outset that though contradictions exist, our result obtained using quantum information theoretic measures are in complete agreement with those of Refs. [5,6] for the DH.

The paper is organized in the following manner. In Sec. II we discuss the DH briefly, also providing an approach to numerically diagonalize the DH. We then move onto the study of ground-state fidelity in both thermodynamic and finite size limits in Sec. III. We then discuss the LE for the DH in Sec. IV followed by the numerical analysis for the time average of the LE in Sec. V and present the final conclusions in Sec. VI.

II. THE DICKE MODEL: INFINITE AND FINITE j

We look for the signatures of quantum chaos in the Dicke Hamiltoninian (DH), which describes a single-mode bosonic field interacting with an ensemble of N two-level atoms [4], given by

$$H = \omega_0 \sum_{i=1}^{N} s_z^i + \omega a^\dagger a + \sum_{i=1}^{N} \frac{\lambda}{\sqrt{N}}(a^\dagger + a)(s_+^{(i)} + s_-^{(i)}) \quad [\hbar = 1]. \tag{1}$$

Here ω_0 is the energy level splitting between the two-level systems. $a^\dagger(a)$ is the creation (annihilation) operator for the bosonic field, with $[a^\dagger,a] = 1$. In our case, we consider only a single bosonic mode which interacts with two-level atoms with the interaction strength λ. The ith atom is described by the spin-half operators ($s_k^i; k = z,\pm$), obeying the commutation rules $[s_z,s_\pm] = \pm s_\pm$ and $[s_+,s_-] = 2s_z$. The origin of the factor $1/\sqrt{N}$ in the interaction term results from the dipole interaction, which is proportional to $1/\sqrt{V}$, where V is the volume of the cavity. Taking into consideration that the density of atoms in the cavity is $\rho = N/V$, we find that the coupling strength is of the form $\lambda/\sqrt{N}$. The scaling factor $\sqrt{N}$ appearing in the interaction plays an important role for the finite "size" system.

The DH [Eq. (1)] is further simplified by using collective atomic operators,

$$J_z \equiv \sum_{i=1}^{N} s_z^{(i)}, \quad J_\pm \equiv \sum_{i=1}^{N} s_\pm^{(i)}, \tag{2}$$

which obey the usual angular momentum commutation relations. Here j is assigned its maximum value $j = N/2$, and this value is constant for a fixed value of N. Thus, the N two-level system effectively gets reduced to a $(2j+1)$ [$=(N+1)$] -level system. The final form of the single-mode DH then looks like

$$H = \omega_0 J_z + \omega a^\dagger a + \frac{\lambda}{\sqrt{2j}}(a^\dagger + a)(J_+ + J_-). \tag{3}$$

The resonance condition, $\omega = \omega_o = 1$, has been used in the rest of the paper. The parity operator (Π) can be defined here in terms of the total number of excitation quanta ($\hat{N}$) in the system as

$$\Pi = \exp\{i\pi \hat{N}\}, \quad \hat{N} = a^\dagger a + J_z + j. \tag{4}$$

Clearly, the operator Π can have only two eigenvalues (± 1), N being even or odd. Thus, the DH turns out to be parity conserving as $[H,\Pi] = 0$, and, correspondingly the Hilbert space of the total system is split into two noninteracting subspaces.

The DH shows a Quantum Phase Transition (QPT) in the thermodynamic limit (as $N \to \infty$) at a critical value of the atom-field coupling strength (λ), $\lambda_c = \sqrt{\omega\omega_o}/2$ where the symmetry associated with the parity operator (Π) is broken. The second derivative of the ground-state energy per j with respect to λ shows a sharp discontinuity at the point $\lambda = \lambda_c$ clearly marking the occurrence of a phase transition; this transition separates the normal phase (for $\lambda < \lambda_c$) from the SR phase (for $\lambda > \lambda_c$). The system in the normal phase is only microscopically excited, whereas the SR phase shows macroscopic excitations.

In the finite j limit, however, parity symmetry holds and Π continues to be a good quantum number for all values of λ, and there is no discontinuity in the ground-state energy per j ($=E_G/j$) with respect to λ, indicating the absence of a sharp phase transition. However, the finite j results tend to the infinite j (i.e., thermodynamic limit) very rapidly. The system, however, shows microscopic excitations below λ_c even for finite j and is macroscopically excited above that value, although the crossover from the microscopically excited phase to the macroscopically excited phase is not sharp. Therefore, one observes that the initially localized wave function for a small but finite j gets delocalized rapidly with a slight increase in j. Finally as $j \to \infty$, the wave function breaks into two lobes (creating a degeneracy); the parity symmetry breaks and there is a proper QPT at λ_c in this limit [6]. There is no QPT for a finite j case in the true sense of the term, because the parity symmetry remains intact, but there is a crossover at around λ_c indicating a transition from a localized (normal) phase to a delocalized (chaotic) phase.

Exact solutions of the DH at finite j do not exist except for $j = 1/2$ (though a study based on finite size scaling has been carried out in Ref. [31]). Hence, we resort to a numerical diagonalization scheme using the number states of the field $|n\rangle$ and the Dicke states $|j,m\rangle$ as our combined basis $\{|n\rangle \otimes |j,m\rangle\}$. To perform the diagolnalization the bosonic

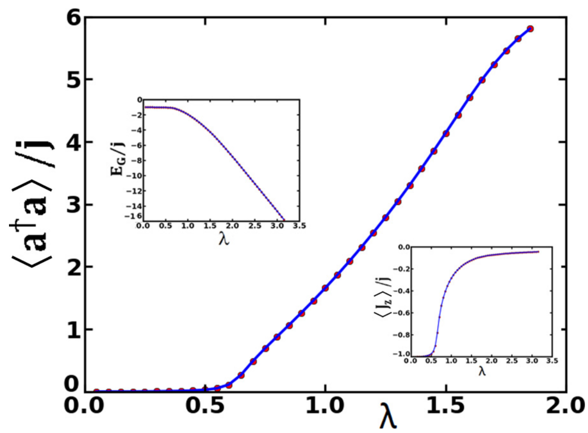

FIG. 1. (Color online) The ground-state expectation values of mean photonic number $N = \langle a^\dagger a\rangle$ (background), the ground-state energy E_g (left inset), and the expectation value of the atomic inversion J_z (right inset) per j as a function of λ for the numerically diagonalized Hamiltonian ($\lambda_c = 0.5$, $\omega = \omega_o = 1$). The plots produced here match the ones in Ref. [6] and hence establish the justification of the chosen system parameters $j = 5$ and $n_c = 40$ for the remaining simulations.

Hilbert space is truncated, while for the pseudospin the full Hilbert space is considered. We take n_c as the maximum boson number in the artificially truncated space. Finally, diagonalizing the DH for finite $j(= 5)$, we evaluate and plot the ground-state energy (E_G/j) and the ground-state expectation values of the scaled atomic inversion $\langle J_z\rangle/j$ and the photonic number $\langle a^\dagger a\rangle/j$ as a function of the coupling strength λ (see Fig. 1). We emphasize that our results match those produced in Ref. [6], in terms of both the phase transition point and behavior at high λ.

III. IDENTIFICATION OF CHAOS THROUGH GROUND-STATE FIDELITY

The ground-state quantum fidelity (F), which measures the overlap between many-body ground states at slightly different values of a parameter λ of the Hamiltonian, usually serves as an important tool for detecting quantum phase transitions. We shall discuss below that it also acts as a good indicator of transition to quantum chaos. We define the the ground-state fidelity as

$$F = |\langle\psi(\lambda+\delta)|\psi(\lambda)\rangle|^2, \tag{5}$$

where $|\psi(\lambda+\delta)\rangle$ and $|\psi(\lambda)\rangle$ are the ground states of the DH with parameters λ and $\lambda+\delta$, respectively. We present results for the ground-state fidelity defined in Eq. (5) of the DH in both limits, thermodynamic (N, i.e., $j\to\infty$) and finite j (=5), in both phases. Although results obtained in the thermodynamic limit were already reported in Ref. [18], we present them here to contrast with the features that emerge in the finite j case in the SR phase.

A. Thermodynamic limit

To exactly diagonalize the Hamiltonian in the thermodynamic limit one resorts to the Holstein-Primakoff transformation (applied to the DH as in Ref. [32]) of the angular momentum operators, given by

$$J_+ = b^\dagger\sqrt{2j - b^\dagger b}, \tag{6}$$

$$J_- = \sqrt{2j - b^\dagger b}\, b, \tag{7}$$

$$J_z = (b^\dagger b - j), \tag{8}$$

where $[b,b^\dagger] = 1$. With these substitutions we get the DH in the normal phase as

$$H = \omega_0(b^\dagger b - j) + \omega a^\dagger a + \lambda(a^\dagger + a)\left(b^\dagger\sqrt{1-\frac{b^\dagger b}{2j}} + \sqrt{1-\frac{b^\dagger b}{2j}}\, b\right). \tag{9}$$

In the SR phase to capture the macroscopic occupations of both the field and the atomic ensembles we have to displace the bosonic modes in Holstein-Primakoff representation, in either of the following ways:

$$a^\dagger \to c^\dagger + \sqrt{\alpha},\quad b^\dagger \to d^\dagger - \sqrt{\beta}, \tag{10}$$

$$a^\dagger \to c^\dagger - \sqrt{\alpha},\quad b^\dagger \to d^\dagger + \sqrt{\beta}, \tag{11}$$

and retain only the terms linear in j. Both choices of bosonic displacements give identical Hamiltonians. Hence, every state is doubly degenerate in the SR phase.

Diagonalizing the Hamiltonian in the uncoupled (q_1,q_2) basis we obtain the ground states as

$$\Psi_G(q_1,q_2) = G_-(q_1)G_+(q_2). \tag{12}$$

In this scheme the ground states in both phases have a Gaussian profile [$G_\pm$, with different (q_1,q_2) in both the phases], given in the artificial (x,y) basis by

$$g(x,y) = \left(\frac{\epsilon_+\epsilon_-}{\pi^2}\right)^{1/4}\exp\left[\frac{-\langle\mathbf{R},\mathbf{AR}\rangle}{2}\right], \tag{13}$$

$$A = U^{-1}MU, \tag{14}$$

$$M = \text{diag}[\epsilon_-,\epsilon_+]. \tag{15}$$

A is the rotation matrix parametrized with the angle γ, which is needed to transfer the basis from (q_1,q_2) to $R = (x,y)$, and U is an orthogonal matrix. $\epsilon_\pm$ are the atomic and the photonic excitations of the DH. The ground-state fidelity is given by [18,33]

$$\langle g|g'\rangle = 2\frac{[\det A \det A']^{1/4}}{[\det(A+A')]^{1/2}}, \tag{16}$$

which on simple determinant manipulation gives

$$\langle g|g'\rangle = 2\frac{[\det M \det M']^{1/4}}{[\det(M+M')]^{1/2}}. \tag{17}$$

This yields a fidelity expression for the normal phase as a function of the parameter λ. In the plot of ground-state fidelity versus the λ in the thermodynamic limit we have color coded the two phases differently as they arise from two different representations of the same Hamiltonian, on either side of the QPT. For the calculation of the ground-state fidelity in the SR phase, we take the Hamiltonan in which both the modes have acquired nonzero mean fields above λ_c, formed by using the Holstein-Primakoff representation, to calculate the atomic and photonic excitations, which are different from those obtained

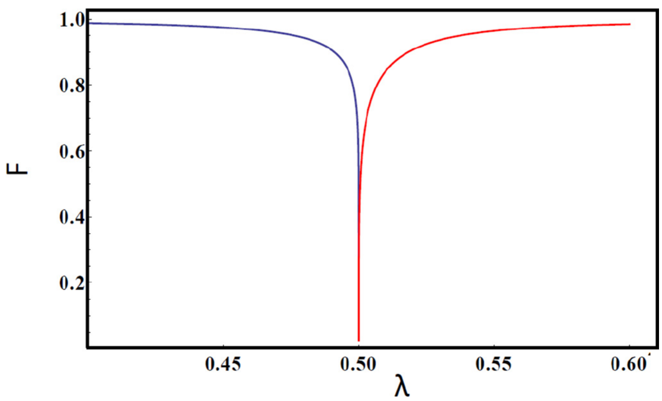

FIG. 2. (Color online) The plot for the exact fidelity expression in the thermodynamic limit; one finds a sharp dip at the quantum critical point $\lambda_c = 1/2$.

in the normal phase, which we insert in Eq. (17) to obtain the ground-state fidelity.

B. Finite *j*

To obtain the numerical value of fidelity we set the parameter $\delta = 0.1$ and diagonalize the DH on either side of the critical point. We numerically obtain the ground states $|\psi(\lambda)_g\rangle$ and $|\psi(\lambda+\delta)_g\rangle$ to calculate the fidelity as defined earlier. The fidelity when plotted against λ shows a dip near the thermodynamic Quantum Critical Point (QCP); the slight difference is due to the finite size of the system.

Immensely interesting behavior of the fidelity occurs in the SR phase at $\lambda > \lambda_c$. We see a significant number of oscillations in the fidelity, which drops from a value less than unity to near zero. It rises and falls aperiodically, staying below 1 till a value of λ when n_c is no longer a sufficient cutoff. Then the fidelity rises to one, but the aperiodic oscillations persist.

On increasing the value of n_c (i.e., the bosonic cutoff), we observe that the fidelity remains less than one up to even larger value of λ though the oscillation persists. Ideally, an infinite bosonic cutoff would see the fidelity never rise to one at any finite value of λ. In the $j \to \infty$ limit, the DH is integrable in both its phases, and we recall the absence of aperiodic oscillations in both the phases of the plot at all values of λ as shown in Fig. 2.

Remarkably, the presence of chaos in the SR phase, as indicated by the level crossing arguments and their statistics [6], manifests itself in the fidelity as aperiodic oscillations. Even for a small change in the parameter δ in the Hamiltonian, we find that the ground states are widely separated (in state space) for certain values of $\lambda(>\lambda_c)$ resulting in a nearly vanishing fidelity. For other values of $\lambda(>\lambda_c)$ also, the overlap is small (<1) and decreases further with increasing δ as seen in the insets of Figs. 3 and 4. Thus, unlike the normal phase where the fidelity remains very close to unity throughout with a dip at the critical point, one finds a significantly different behavior in the SR phase.

IV. LOSCHMIDT ECHO FOR FINITE *j*

The modulus of the overlap between the two ground states, where one is evolved with $H(\lambda)$ and the other with a shifted parameter $\lambda + \delta$ is known as the Loschmidt echo (LE) given

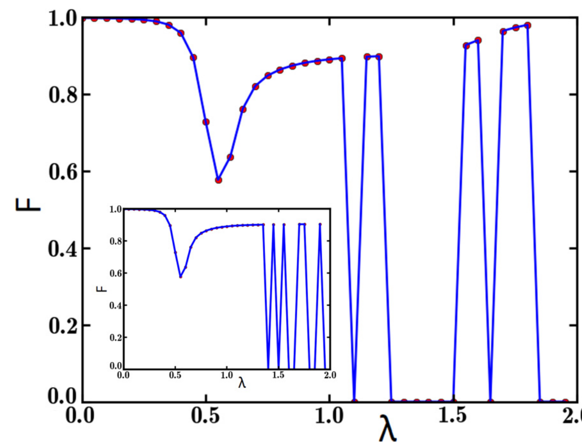

FIG. 3. (Color online) Fidelity for $\delta = 0.1$ and $\delta = 0.3$ (inset) for $n_c = 40$. As is evident the ground-state fidelity oscillates wildly as the system crosses λ_c into the SR phase. Inset: The plot in the case of a large deviation $\delta = 0.3$ clearly shows a lower recovery for the ground-state fidelity.

by the expression

$$L(t) = |\langle\psi(\lambda)|\, e^{iH(\lambda)t} e^{-iH(\lambda+\delta)t}\, |\psi(\lambda)\rangle|^2. \tag{18}$$

We study the time evolution of the LE in the normal phase and the SR phase as well as at the QCP for appropriate values of the parameter λ. We list the observations below.

In the normal phase Fig. 5 (top left), we find that the amplitude of the LE varies from a value of 1.0 to 0.55, and the peaks in the envelope have nearly the same amplitude.

Near the QCP Fig. 5 (background), the ground states at λ_c and at $\lambda_c + \delta$ are widely separated; hence, we see that the LE dips from 1 to 0 and there is no apparent periodicity marking the QCP at around $\lambda = 0.5$.

In the SR phase Fig. 5 (top right), there is an overall decay in the amplitude of the LE with time. The amplitude of the envelope revives after a long time.

We see from Fig. 6, which is a plot of the overlap between the ground state at λ and all states at $\lambda + \delta$ against the total

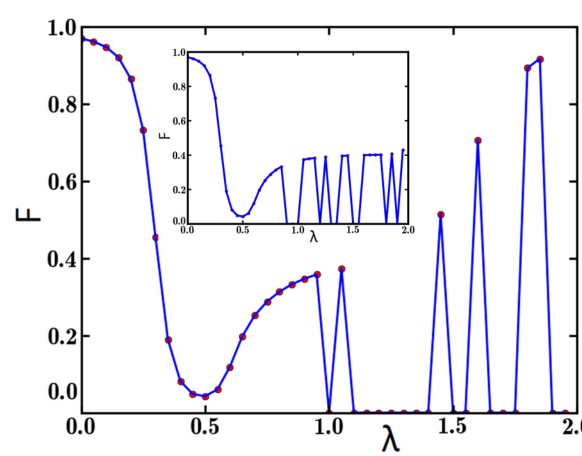

FIG. 4. (Color online) Fidelity for $\delta = 0.1$ and $\delta = 0.3$ (inset) for $n_c = 70$. The plot in the case of a larger bosonic cutoff shows that the fidelity remains less than one for larger range of λ without any change in the oscillatory behavior in the SR phase.

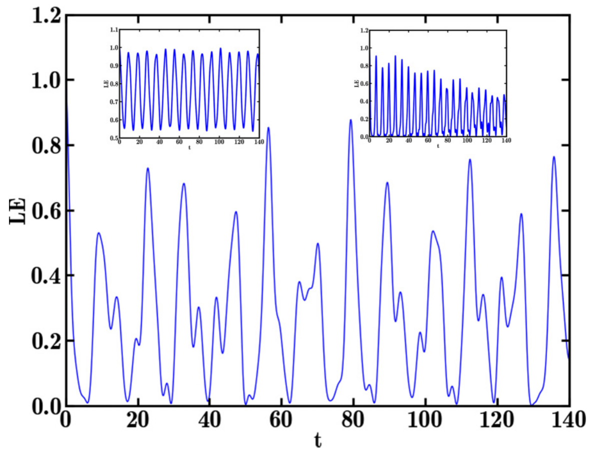

FIG. 5. (Color online) The LE for finite j and with $\delta = 0.3$ at a set time in the normal phase (top left), at the QCP ($\lambda = \lambda_c$) (background), and in the SR phase (top right). The LE in the normal phase shows sustained oscillations (periodic) as only a few states are involved; as we move into the SR phase the nature of the LE becomes aperiodic with many states contributing to the LE.

number of states, that in the normal phase the overlap between the ground state and the states of the Hamiltonian with a shifted value of λ is limited to one or two excited states. Hence there is no decay of the LE with time, the system oscillates with the superposition of two or three frequencies associated with the energy differences of the nonzero overlaps. In the SR phase, in contrary, we see a delocalization of the wave function with parameter $\lambda + \delta$. As evident from a greater number of states of $|\psi_i(\lambda + \delta)\rangle$ contributing to the overlap with the ground state with smaller amplitudes than in the normal phase. As a larger number of overlaps are involved the phases interfere destructively, leading to a decay of the LE with time. Finally at the crossover point λ_c we see a mixture of both of the above mentioned behaviors: chaotic and nonchaotic regimes of λ get involved in the LE, and so we get an aperiodic pattern.

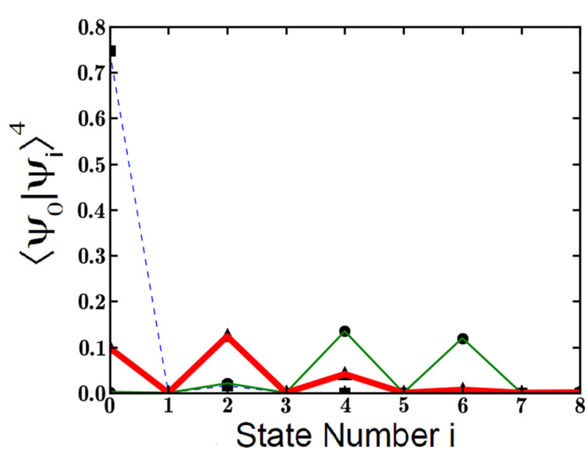

FIG. 6. (Color online) This plot shows the mod square of the overlap between $\langle\psi_o(\lambda)|$ and $|\psi_i(\lambda + \delta)\rangle$ for $i \in [0,n]$ corresponding to the LE plots (with the parameters $\delta = 0.3$, $n_c = 40$, and $j = 5$): blue (dotted) for the normal phase ($\lambda < 0.5$), green (thin line) at the QCP, and red (thick line) for the SR phase ($\lambda > 0.5$). One can clearly observe that after the crossover point $\lambda_c = 0.5$, the overlap between the initial ground state and the excited states of the perturbed Hamiltonian is significant.

V. THE TIME AVERAGE OF THE LE

Generally the LE serves as a good indicator of QCP, but to understand the transition to chaos in the SR phase of the DH, one should explore the time average of the LE as argued by Peres [3]. It has been suggested that if a quantum system has a chaotic classical analog, then the time average of the overlap between two states nearly vanishes in the chaotic phase while it remains close to unity in the regular phase. We employ the same technique in the present context using the two ground states evolved with two slightly different Hamiltonians. It is easy to show that

$$\langle L\rangle = \lim_{T\to\infty} \frac{\int_0^T L(t)\,dt}{\int_0^T dt} = \sum_i |\langle\psi_o(\lambda)|\psi_i(\lambda+\delta)\rangle|^4, \quad (19)$$

where ψ_0 is the ground state of $H(\lambda)$ and ψ_i is the ith excited state of the Hamiltonian with the modified value of λ. We emphasize that though there is an apparent similarity with the expression for fidelity, there is also a subtle difference; this expression incorporates information about all the excited states of the Hamiltonian $H(\lambda + \delta)$. Thus the time average of the LE is expected to capture the entire delocalization scheme unlike the fidelity.

A simple mathematical expression connects the time-averaged LE and the ground-state fidelity:

$$\langle L\rangle = F^2 + \sum_{i\neq 0} |\langle\psi_0(\lambda)|\psi_i(\lambda+\delta)\rangle|^4. \quad (20)$$

Figure 7 clearly shows that the first dip of the ground-state fidelity (green) in the chaotic phase occurs at the a value of λ where the LE average (red) just starts to flatten out. This implies that the terms with $i \neq 0$ in Eq. (20) oscillate complementary to that of the square of the fidelity, showing

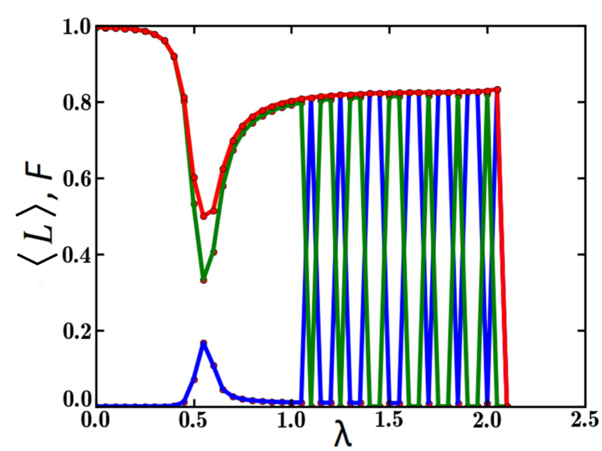

FIG. 7. (Color online) A combined plot of the time average LE (red), ground-state fidelity (green), and complementary higher state fidelity sum (blue) versus λ ($\delta = 0.3$). It can be clearly seen that the time average of the LE contains within it information about the ground-state fidelity as expected, making the picture of delocalization of the wave function in state space clear as we move into the SR phase.

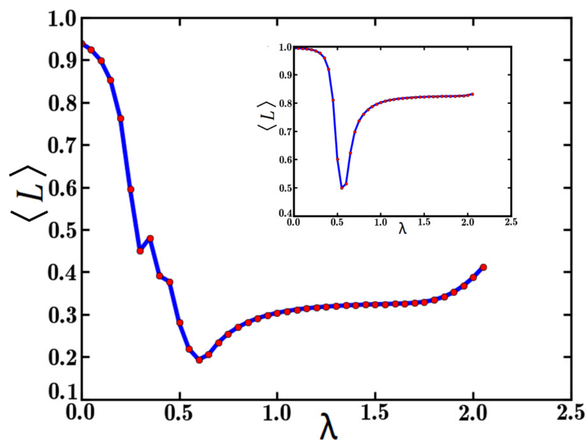

FIG. 8. (Color online) Time-averaged LE for $\delta = 0.3$ (background) and $\delta = 0.1$ (inset). In both the cases the time average of the LE dips significantly from 1 as λ goes into the chaotic SR phase. Inset: Like in the case of ground-state fidelity (Fig. 3) as δ is increased the recovery of the time average of the LE to 1 never really occurs.

a clear connection between the fidelity and the time-averaged LE. Thus we can conclude that the time average of the LE incorporates the effect of ground-state fidelity while providing a clearer picture of the delocalization in state space.

The time average of LE, as shown in Fig. 8, distinctly separates the two phases of the DH. As argued in Ref. [3], the occurrence of chaos in the SR phase is indicated by the time average dipping to a value much less than one, whereas in the normal phase, the time average remains close to unity indicating regularity. This is because in the normal phase, on a slight change of the perturbing parameter δ, only a few excited states of the perturbed Hamiltonian have significant overlap with the ground state of the unperturbed Hamiltonian $\psi_0(\lambda)$; this indicates that the initial ground state remains close to itself (in the state space) even when the Hamiltonian is shifted, which should indeed be the case for a regular behavior.

Contrary to this, in the chaotic phase, one finds a significant reduction of the overlap between the initial ground state and the ground state $\psi_0(\lambda + \delta)$ of the Hamiltonian with parameter $\lambda + \delta$; this can be understood as a signature of chaos. On a small change of the parameter λ here, there is an overlap between the initial ground state and a large number of the excited states of the perturbed Hamiltonian; these states having nonzero (but small in magnitude) overlap with the initial ground state are randomly distributed. Thus we see that on shifting the parameter λ by δ the shifted ground state $\psi_0(\lambda + \delta)$ is widely separated from the initial ground state $\psi_0(\lambda)$ in the state space. In the classical picture, chaos is understood as the exponential separation of two trajectories with very similar initial conditions. This effect is manifested in the quantum analog, by this aforementioned dissimilarity between the initial and final states (in state space) produced by shifting the Hamiltonian parameter. As the time average of the LE contains this distribution through the sum over all states, we see a significant drop in the value in the SR phase, clearly pointing to delocalization and hence serving as a signature of chaos.

In our entire argument, we are adhering to a completely quantum mechanical formulation. It should be mentioned that there have been other studies using the semiclassical LE on some models to identify chaos and draw a connection to the classical Lyapunov exponent ([28,34,35] (for a review, see [36])). However, we are interested here in a fully quantum mechanical case where establishing such a connection is not apparently feasible. A semiclassical treatment of the DH with a finite j through the LE and establishing a connection to the classical Lyapunov exponent would be an interesting topic of future research.

VI. CONCLUSION

In conclusion, we have studied the signature of "chaos" in the SR phase of the DH using the ground-state quantum fidelity, the LE, and its time-averaged value for a finite j.

We have observed that the ground-state fidelity shows aperiodic and random oscillations in the SR phase, and the results are drastically different from the behavior of fidelity in the $j \to \infty$ limit, which we also present for the sake of comparison. On the other hand, the LE shows a similar aperiodic oscillation as a function of time in the SR phase followed by a rapid decay.

We argue that in the SR phase, many states contribute to the LE, leading to a faster decay. We also show that the time-averaged LE contains the information gained from the ground-state fidelity and stays much less than unity in the SR phase. In short, we concentrate on the delocalization produced in the state space by the onset of chaos as compared to the distribution of eigenenergies as studied in other works. We conclude that the delocalization of the wave function manifests itself in the time average of the LE, and the fidelity giving clear signs of the presence of chaos in the SR phase of the DH.

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