Bifurcation study of phase oscillator systems with attractive and repulsive interaction

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We study a model of globally coupled phase oscillators that contains two groups of oscillators with positive (synchronizing) and negative (desynchronizing) incoming connections for the first and second groups, respectively. This model was previously studied by Hong and Strogatz (the Hong-Strogatz model) in the case of a large number of oscillators. We consider a generalized Hong-Strogatz model with a constant phase shift in coupling. Our approach is based on the study of invariant manifolds and bifurcation analysis of the system. In the case of zero phase shift, various invariant manifolds are analytically described and a new dynamical mode is found. In the case of a nonzero phase shift we obtained a set of bifurcation diagrams for various systems with three or four oscillators. It is shown that in these cases system dynamics can be complex enough and include multistability and chaotic oscillations.

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I. INTRODUCTION

Systems of coupled phase oscillators of the Kuramoto type have been widely used as models of synchronization processes in physics, chemistry, biology, and social science [1–3]. Their complex behaviors (synchronous states with special phase relations between oscillators, multistability, various regular and chaotic oscillations) have been a subject of many studies (see reviews [4,5]). In some of these studies an extended version of the Kuramoto system has been considered that is called the Kuramoto-Sakaguchi system [6–10]. This system is characterized by a more general form of the interaction function than the simple sinusoidal function initially used by Kuramoto. The interaction function of the Kuramoto-Sakaguchi oscillator contains an additive phase shift component. The introduction of this phase shift results in complex behavior of the system and significantly increases the number of dynamical modes that it can exhibit.

In earlier works, systems of phase oscillators with a uniform coupling parameter were studied. Later on the systems with heterogeneous coupling strengths attracted the attention of researchers (in particular, the systems that consist of several groups of oscillators coupled with different strengths) [11–14].

Stiller and Radons [15] studied systems with a symmetric matrix of coupling strengths that were chosen independently from the same probability distribution. The systems analyzed in [16] have strengths of outgoing connections that can take on one of two values, one positive and another negative, where each value is associated with a particular probability. Both of these models with randomly chosen connection strengths exhibit conventional mean-field behavior and display a second-order phase transition like that found in the standard Kuramoto model.

The situation is different for the Hong-Strogatz model [17,18], where the strengths of incoming connections are of different signs. In this model the set of oscillators is divided into two groups. The oscillators of the first group (the so-called conformists) receive positive (i.e., synchronizing) connections and tend to synchronize their activity with other oscillators. The oscillators of another group (the so-called contrarians) receive negative (i.e., desynchronizing) connections and are repelled by the mean field, preferring to run with a phase opposed to other oscillators. Using the generalized Watanabe-Strogatz theory [7,8] and Ott-Antonsen ansatz [19], the authors found that systems with large enough numbers of oscillators were able to generate all of the major types of dynamical behavior (stable equilibrium states, time dependent states, traveling wave states, etc.). A special case when the subset of contrarians contains only one unit and when the attracting oscillators interact locally is considered in [20].

In this paper we use bifurcation analysis to study invariant manifolds of the finite system with the aim to extend the results of Hong and Strogatz in two directions. First, for the Hong-Strogatz case we analytically describe the structure of the invariant manifolds of the system's dynamics, determine the types of bifurcations that occur, and compute bifurcation boundaries in parametric space. We also describe a new dynamical regime in which one contrarian is synchronized by the group of conformists. Second, we present the results of the detailed bifurcation analysis of small systems (two, three, or four oscillators) with a phase shift. We demonstrate that several types of dynamics (slow switching, chaotic behavior, etc.), which are impossible in the original Hong-Strogatz model, can appear even for small phase shifts.

In Sec. II we present the model description and its equivalent formulation in terms of phase differences, determine the symmetries of the system, and describe its invariant sets. In Sec. III we prove a theorem that gives a description of all equilibria for the Hong-Strogatz model. Section IV is devoted to the study of stability of the dynamical modes of the Hong-Strogatz model. In Sec. V we describe the results of bifurcation analysis of the system in the case when a phase shift is present. The results of our studies are summarized and discussed in Sec. VI.

II. MODEL DESCRIPTION

We consider a system of globally coupled identical phase oscillators whose dynamics are described by the ordinary differential equations

$$\dot{\theta}_i = \omega - \frac{K_s}{N}\sum_{j=1}^{N}\sin(\theta_i - \theta_j - \alpha), \quad i = 1,\ldots,N, \tag{1}$$

where $\theta_i \in \mathbb{T}^1 = \mathbb{R}/(2\pi\mathbb{Z})$ is a phase variable of the ith oscillator, ω is the natural frequency of the oscillators, K_s is the coupling strength, and $\alpha \in \mathbb{T}^1$ is a phase shift. The phase space of the whole system is the N-dimensional torus $\mathbb{T}^N$. We assume that K_s can take one of two values: $K_1 \neq 0$ or K_2. (The case $K_1 = K_2 = 0$ is trivial; it corresponds to the system without any interaction between oscillators.) The set of all oscillators $\mathcal{H}$ is split into two subsets, $\mathcal{H}_1$ and $\mathcal{H}_2$, which consist of the oscillators with the incoming coupling strengths K_1 and K_2, respectively.

Let N_1 and N_2 be the number of oscillators in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. It is assumed that both subsets are not empty; that is, $N_1 > 0$, $N_2 > 0$. Without loss of generality the oscillators in the subpopulation $\mathcal{H}_1$ can be indexed as $\{1,2,\ldots,N_1\} = J_1$ and in the subpopulation $\mathcal{H}_2$ as $\{N_1 + 1,\ldots,N\} = J_2$. The architecture of connections between two groups of globally coupled oscillators is shown in Fig. 1(a).

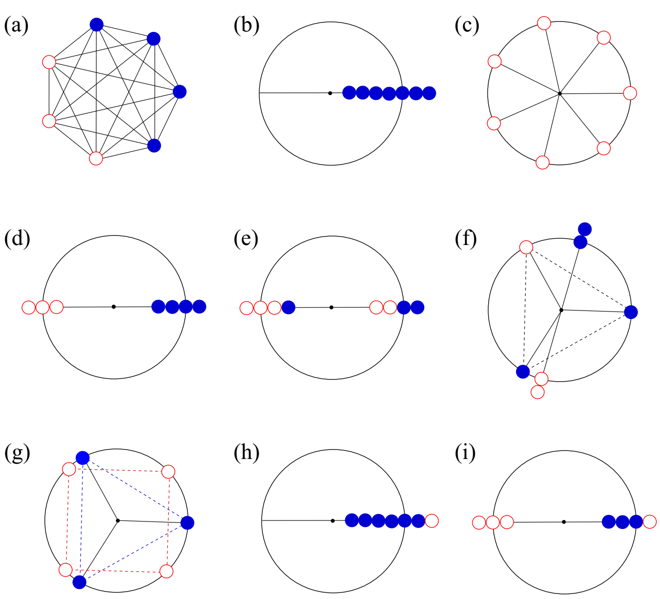

FIG. 1. (Color online) (a) Connections between two groups of oscillators (solid blue circles denote conformists; open red circles denote contrarians). (b)–(i) Distributions of conformists and contrarians on the circle for different stable regimes: (b) All the oscillators are conformists ($R = r_1 = 1$); (c) all the oscillators are contrarians ($R = r_2 = 0$); (d) the π state ($r_1 = r_2 = 1, \delta = \pi$); (e) the πS_M state, a π-like state with $R = 0$ when $N/2$ different oscillators are located at 0 and at π; (f) the blurred π state ($N_1 r_1 = N_2 r_2$, $\delta = \pi$); (g) the incoherent state, IS ($r_1 = r_2 = 0$); (h) the S_1 state, synchronization of a single contrarian with conformists ($R = r_1 = 1$); (i) the πS_1 state, a π-like state with one renegade contrarian [$r_1 = 1, r_2 = (N_2 - 2)/N_2$, $\delta = \pi$].

Substituting $\theta_i \to \theta_i + \omega t$, we obtain

$$\begin{aligned}\dot{\theta}_i &= -\frac{K_1}{N}\sum_{j=1}^{N}\sin(\theta_i - \theta_j - \alpha), \quad i \in J_1,\\ \dot{\theta}_i &= -\frac{K_2}{N}\sum_{j=1}^{N}\sin(\theta_i - \theta_j - \alpha), \quad i \in J_2.\end{aligned} \tag{2}$$

When $K_1 = K_2$, system (2) is known as the Kuramoto-Sakaguchi model [6] with identical natural frequencies. If, in addition, $\alpha = 0$, then the system is called the Kuramoto model [1]. If $K_1 > 0$, $K_2 < 0$, $\alpha = 0$, system (2) becomes the Hong-Strogatz model [17,18]. In this model the oscillators of the subset $\mathcal{H}_1$ are called *conformists* and the oscillators from the subset $\mathcal{H}_2$ are called *contrarians*. We keep these terms also for the oscillators of system (2). If both coupling parameters K_1, K_2 are positive, then the system comprises two groups of conformists. If both coupling parameters are negative, then the system comprises two groups of contrarians.

A. The order parameter

Conventionally, the order parameter R [1] is used as a measure of synchronization of a phase oscillator system. It is the amplitude of the complex mean field defined by the equality

$$Re^{\imath\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{\imath\theta_j}, \tag{3}$$

where $\imath = \sqrt{-1}$. If the Kuramoto model consists of conformists only, it will be fully synchronized. In this case $R = 1$ [Fig. 1(b)]. The opposite limit case is when this model consists of contrarians only. In this case the oscillators repel each other and their phases are homogeneously spread around the circle, which implies $R = 0$ [Fig. 1(c)].

Suppose that the system comprises both conformists and contrarians. Following [18] let us introduce the order parameters for each of the subpopulations $\mathcal{H}_1$ and $\mathcal{H}_2$ and some other useful parameters: the relative number of conformists and contrarians in the whole system, p, q, the average phase difference between the subpopulations δ, the relative intensity of coupling C,

$$r_s e^{\imath\psi_s} = \frac{1}{N_s}\sum_{j\in J_s} e^{\imath\theta_j}, \quad s = 1,2,$$

$$p = \frac{N_1}{N}, \quad q = 1 - p = \frac{N_2}{N}$$

$$\delta = \psi_2 - \psi_1, \quad C = \frac{K_1}{K_1 - K_2}.$$

In the Hong-Strogatz model the oscillators compete in an attempt to impose their own style of behavior on all oscillators in the system. The conformists not only try to synchronize their own subpopulation ($r_1 = 1$) but also try to engage some contrarians in this synchronization. On the other hand, the contrarians try to spread the phases of all the oscillators around the circle. The resulting dynamics depend on the competitive influence of the groups of conformists and contrarians on the oscillators and this influence is determined by the relations between the parameters N_1, N_2, K_1, K_2. If a phase shift is present ($\alpha \neq 0$), the dynamics of the system may be very complex.

B. Reduction to phase differences

To investigate synchronization properties of (2), it is reasonable to reduce the dimension of the system by one using new variables (note that we have already assumed that the system has at least one conformist):

$$\phi_i = \theta_1 - \theta_{i+1}, \quad i = 1,\ldots,N-1.$$

Without loss of generality we can also multiply all equations by N/K_1 (assuming that $K_1 > 0$). Then introducing a new parameter,

$$k = \frac{K_2}{K_1} = 1 - \frac{1}{C}, \tag{4}$$

and rescaling time (time is also reversed if $K_1 < 0$), we can rewrite the system in phase differences as

$$\begin{aligned}\dot{\phi}_i &= -\sin(\phi_i+\alpha) - \sum_{j=1}^{N-1}\sin(\phi_j-\alpha)\\ &\quad - \sum_{j=1,j\neq i}^{N-1}\sin(\phi_i-\phi_j+\alpha), \quad i=1,\ldots,N_1-1,\\ \dot{\phi}_i &= -k\sin(\phi_i+\alpha) - \sum_{j=1}^{N-1}\sin(\phi_j-\alpha)\\ &\quad -k\sum_{j=1,j\neq i}^{N-1}\sin(\phi_i-\phi_j+\alpha) - (k-1)\sin\alpha,\\ i &= N_1,\ldots,N-1.\end{aligned} \tag{5}$$

Note that if $K_1 < 0$, stable regimes of system (2) become unstable regimes of system (5) and vice versa. Since our bifurcation analysis of system (5) describes all of its dynamical behavior, the inversion of time does not decrease the generality of our results.

Below we consider system (5) with negative as well as positive values of the parameter k. The case of positive $k \neq 1$ corresponds to two groups of conformists or contrarians with two different strengths of interaction. We show that for nonzero values of the phase shift the system with such parameter values can have interesting nontrivial dynamics including chaos.

C. Symmetries

The theory of bifurcations of systems with symmetries (for example, [21,22]) provides (among other valuable results) a methodology for investigating such systems based on the analysis of invariant manifolds and determination of bifurcation curves. We apply this approach to systems (2) and (5), which allows us to describe their synchronization and clustering properties.

System (2) has the permutation symmetry S_{N_1} for the oscillators from the subpopulation $\mathcal{H}_1$ and it has the permutation symmetry S_{N_2} for the oscillators from the subpopulation $\mathcal{H}_2$. It also has an S^1 phase shift symmetry,

$$(\theta_1,\ldots,\theta_N) \longmapsto (\theta_1+\varepsilon,\ldots,\theta_N+\varepsilon),$$

for all $\varepsilon \in \mathbb{T}^1$.

The properties of the function $\sin(\phi-\alpha)$ imply the following symmetries for system (5)

$$\begin{aligned}\gamma_1 &: (\phi_1,\ldots,\phi_{N-1},\alpha,k,t)\\ &\longmapsto (-\phi_1,\ldots,-\phi_{N-1},-\alpha,k,t),\\ \gamma_2 &: (\phi_1,\ldots,\phi_{N-1},\alpha,k,t)\\ &\longmapsto (\phi_1,\ldots,\phi_{N-1},\alpha+\pi,k,-t).\end{aligned}$$

These symmetries allow us to consider only the values of $\alpha \in [0,\pi/2]$, instead of an arbitrary $\alpha \in \mathbb{T}^1$.

If $K_2 \neq 0$, an additional symmetry arises since Eqs. (2) can be written using the phase differences between the last oscillator (or any other contrarian oscillator due to S_{N_2} isotropy) and the other oscillators:

$$\begin{aligned}\overline{\phi}_i &= \theta_N - \theta_{N-i} = \phi_{N-i-1} - \phi_{N-1}, \quad i=1,\ldots,N-2,\\ \overline{\phi}_{N-1} &= \theta_N - \theta_1 = -\phi_{N-1}.\end{aligned} \tag{6}$$

The system in new phase differences $\overline{\phi}_i$ represents the same original equations (2) and its form is similar to (5), with the only difference being that the parameter $k = K_2/K_1$ is replaced by the parameter $k_1 = 1/k = K_1/K_2$. Therefore, the action

$$\begin{aligned}\gamma_3 &: (\phi_1,\ldots,\phi_{N-2},\phi_{N-1},\alpha,k,t)\\ &\longmapsto \left(\phi_{N-2}-\phi_{N-1},\ldots,\phi_1-\phi_{N-1},-\phi_{N-1},\alpha,\frac{1}{k},-t\right)\end{aligned} \tag{7}$$

transforms the vector field of the system with a given number of conformists to a similar system with the same number of contrarians. Using this property we can obtain information about the behavior of system (5) for $N_1 = 1,\ldots,N-1$ if we study this system for $N_1 = 1,\ldots,[N/2]$ only. In particular, the bifurcation (α,k) diagram for the system with n conformists will be the same as the bifurcation $(\alpha,1/k)$ diagram for the system with n contrarians. The bifurcation diagram for $N_1 = N_2 = N/2$ has a bifurcation point with coordinates $(\alpha,1/k)$ if it also has a bifurcation point with coordinates (α,k).

D. Invariant sets

1. Clusters

The motion of each variable θ_i is determined on the 2π periodic circle $\mathbb{T}^1$, which is the factorization of the real line $\mathbb{R}$ by the modulo $2\pi\mathbb{Z}$. It may also be reasonable to consider the motion of θ_i on the whole line $\mathbb{R}$ (without factorization). For example, this is necessary to distinguish between two essentially different types of behaviors of θ_i which are called *phase-locked* and *phase-unlocked*. To say that the dynamics are phase-locked means that the function $\theta_i(t)$ is bounded on $\mathbb{R}$. Conversely, phase-unlocked dynamics correspond to the condition where $\theta_i(t)$ is unbounded on $\mathbb{R}$. The same can be applied to the phase difference $\theta_i - \theta_j$. The phase locking of $\theta_i - \theta_j$ leads to frequency synchronization between two oscillators, $\Omega_{ij} = \Omega_i - \Omega_j = 0$, where the average frequency of the ith oscillator is defined as $\Omega_i = \lim_{t\to\infty} \theta_i(t)/t$, $\theta_i(t) \in \mathbb{R}$ [23].

The permutation symmetries S_{N_1} and S_{N_2} give rise to corresponding invariant manifolds.

Statement 1. System (2) has two types of invariant manifolds,

$$\widetilde{\mathcal{P}}^1_{i,j} = \{(\theta_1,\ldots,\theta_N) : \theta_i = \theta_j\}, \quad i,j \in J_1, \quad i \neq j,$$
$$\widetilde{\mathcal{P}}^2_{i,j} = \{(\theta_1,\ldots,\theta_N) : \theta_i = \theta_j\}, \quad i,j \in J_2, \quad i \neq j,$$

for all values of the parameters K_1, K_2, α.

Proof. Let both oscillators θ_i, θ_j belong to one of the sets $\widetilde{\mathcal{P}}^1_{i,j}$ or $\widetilde{\mathcal{P}}^2_{i,j}$. This means that $\theta_i - \theta_j = 0$. Then we have

$$\begin{aligned}\dot\theta_i - \dot\theta_j &= -\frac{K_s}{N}\sum_{l=1}^{N}\sin(\theta_i - \theta_l - \alpha) \\ &\quad + \frac{K_s}{N}\sum_{l=1}^{N}\sin(\theta_j - \theta_l - \alpha) \\ &= -2\frac{K_s}{N}\sin\left(\frac{\theta_i-\theta_j}{2}\right) \\ &\quad \times \sum_{l=1}^{N}\cos\left(\frac{\theta_i+\theta_j}{2} - \theta_l - \alpha\right) = 0, \\ s &= 1,2.\end{aligned}$$

This proves the invariance of $\widetilde{\mathcal{P}}^s_{i,j}$, $s = 1,2$.

It is easy to check that the last equality is not satisfied in the general case when $\theta_i \in \mathcal{H}_1$ and $\theta_j \in \mathcal{H}_2$.

The existence of the invariant manifold $\theta_i = \theta_j$ leads to the phase locking of the ith and jth oscillators since the points $2\pi k$ form impenetrable barriers on $\mathbb{R}$ for the phase flow $\phi_{ij} = \theta_i - \theta_j$. The manifolds $\mathcal{P}^s_{i,j}$ are $(N-1)$-dimensional sets in the phase space $\mathbb{T}^N$. These manifolds are also $(N-2)$-dimensional hyperplanes for the $(N-1)$-dimensional system (5) in phase differences,

$$\begin{aligned}\mathcal{P}^1_{i,j} &= \{(\phi_1,\ldots,\phi_{N-1}) : \phi_i = \phi_j\}, \\ &\quad i,j = 0,1,\ldots,N_1-1,\ i \neq j, \\ \mathcal{P}^2_{i,j} &= \{(\phi_1,\ldots,\phi_{N-1}) : \phi_i = \phi_j\}, \\ &\quad i,j = N_1,\ldots,N-1,\ i \neq j,\end{aligned}$$

where $\phi_0 \overset{df}{=} 0$. The manifolds split the whole phase space $\mathbb{T}^{N-1}$ of (5) into invariant regions. We note that in the limit cases of identical globally coupled phase oscillators, $N_1 = N$ (conformists only) and $N_1 = 0$ (contrarians only), all invariant regions are bounded in $\mathbb{R}^{N-1}$ [there are $(N-1)!$ invariant regions in the toroidal phase space $\mathbb{T}^{N-1}$]. In general, the invariant regions of system (5) are not bounded in $\mathbb{R}^{N-1}$ for $N_1 \in [1,\ldots,N-1]$; that is, the phase differences $\phi_i - \phi_j = \theta_{j+1} - \theta_{i+1}$ can be phase unlocked if $i = 1,\ldots,N_1-1$, $j = N_1,\ldots,N-1$. Thus, there can be a regime of two frequency clusters when the average frequencies of conformists and contrarians do not coincide.

We define the m-dimensional sets $\mathcal{P}^1_m$ or $\mathcal{P}^2_m$ in $\mathbb{T}^{N-1}$ as the intersection of $N-m-1$ invariant sets $\mathcal{P}^1_{i,j}$ or $\mathcal{P}^2_{i,j}$ (up to a permutation inside of the subpopulations $\mathcal{H}_1$, $\mathcal{H}_2$). These manifolds describe all possible $(N-m)$ clusters of conformist or contrarians. Hence, we can obtain some partial information about the system's dynamics by considering a reduced m-dimensional system on a proper manifold. Note that transversal stability on such a manifold may vary from point to point.

2. *An invariant manifold of dispersed phases*

A nontrivial invariant manifold is described by the following statement.

Statement 2. The manifold

$$\widetilde{\mathcal{M}} = \left\{(\theta_1,\ldots,\theta_N) : \sum_{j=1}^{N} e^{\imath\theta_j} = 0\right\}$$

is an invariant manifold of system (2) for all values of the parameters K_1, K_2, α.

Proof. If a point $(\theta_1,\ldots,\theta_N)$ belongs to the set $\widetilde{\mathcal{M}}$, then it is an equilibrium of system (2). The proof is given by the following sequence of equalities where the first term is the right-hand side of Eqs. (2) and Im denotes the imaginary part:

$$\begin{aligned}\frac{K_s}{N}\sum_{j=1}^{N}\sin(\theta_l - \theta_j - \alpha) &= \frac{K_s}{N}\,\mathrm{Im}\left(\sum_{j=1}^{N} e^{\imath(\theta_l-\theta_j-\alpha)}\right) \\ &= \mathrm{Im}\left(\frac{K_s}{N}e^{\imath(\theta_l-\alpha)}\sum_{j=1}^{N}e^{-\imath\theta_j}\right) = 0.\end{aligned}$$

These equalities are fulfilled for $l = 1,\ldots,N$ and all values of the parameters.

The manifold $\widetilde{\mathcal{M}}$ is $(N-2)$-dimensional [24]. It consists of fixed points of the system and corresponds to zero value of the order parameter R.

An example of $\widetilde{\mathcal{M}}$ is a uniform distribution of the phases of oscillators around the circle (the splay state), as in Fig. 1(c). Less trivial examples are shown in Figs. 1(e)–1(g). Here contrarians win the competition. All oscillators of the system are irregularly distributed around the circle so that the order parameter $R = 0$.

III. THE HONG-STROGATZ CASE: EQUILIBRIA

In this section we consider system (2) in the case of zero phase shift $\alpha = 0$. This version of the model was studied in detail by Hong and Strogatz [18]. We consider an essentially finite-dimensional system and use a different approach to find dynamical regimes.

Equilibria. To describe the steady states of system (5), let us consider the system of $N-1$ algebraic equations,

$$\begin{aligned}g_1(\theta_1,\ldots,\theta_N) - g_i(\theta_1,\ldots,\theta_N) &= 0, \quad i = 2,\ldots,N_1, \\ g_1(\theta_1,\ldots,\theta_N) - k g_i(\theta_1,\ldots,\theta_N) &= 0, \quad i = N_1+1,\ldots,N,\end{aligned} \tag{8}$$

where

$$g_i(\theta_1,\ldots,\theta_N) = \sum_{j=1}^{N}\sin(\theta_i - \theta_j).$$

The next lemma helps us characterize all solutions of (8).

Lemma 1. For $k \neq 0$, $k \neq -N_2/N_1$ a point $(\theta_1,\ldots,\theta_N)$ satisfies system (8) if and only if one of the following two conditions is fulfilled:

(1) $\sum_{j=1}^{N} e^{\imath\theta_j} = 0$,

(2) $\theta_i - \theta_j \in \{0,\pi\}$, $i,j = 1,\ldots,N$.

The proof of the lemma is given in the Appendix.

As a corollary we obtain the following theorem that localizes all equilibria for the system in phase differences.

Theorem. For $\alpha = 0$ and any values of the parameter k such that $k \neq 0$, $k \neq -N_2/N_1$ all equilibria of system (5) are described by the set

$$\mathrm{FP} = \mathrm{FP}_{0\pi} \cup \mathcal{M},$$

where

$$\mathrm{FP}_{0\pi} = \{(\phi_1,\ldots,\phi_{N-1}):\ \phi_j \in \{0,\pi\},\ j = 1,\ldots,N-1\},$$

$$\mathcal{M} = \left\{(\phi_1,\ldots,\phi_{N-1}):\ \sum_{j=1}^{N-1} e^{\imath\phi_j} = -1\right\}.$$

The theorem implies that the locations of fixed points of (5) do not depend on the parameter k, $k \neq 0$, $k \neq -N_2/N_1$. The system has only 2^N fixed points with coordinates 0 and π represented by the set $\mathrm{FP}_{0\pi}$ and the $(N-3)$-dimensional manifold $\mathcal{M} \in \mathbb{T}^{N-1}$. $\mathcal{M} \in \mathcal{T}^{N-1}$ represents the same set for system (5) as $\widetilde{\mathcal{M}} \in \mathbb{T}^N$ does for system (2). Note that in the even-dimensional case the sets $\mathrm{FP}_{0\pi}$ and $\mathcal{M}$ intersect in the points $(\phi_1,\ldots,\phi_{N-1})$ with $N/2-1$ coordinates equal to 0 and $N/2$ coordinates equal to π.

The theorem shows that the set of regimes that correspond to equilibrium points $\mathrm{FP}_{0\pi}$ of system (5) is wider than the set of such regimes described in [18]. Though some of the points of $\mathrm{FP}_{0\pi}$ are not stable for arbitrary values of the parameter k, we show that the system has more stable regimes than those presented in [18].

The theorem has two restrictions: $k \neq 0$ and $k \neq -N_2/N_1$. The condition $k = 0$ means that $K_2 = 0$; i.e., the conformists have no influence on contrarians. The system has a bifurcation when $k = 0$; here the system becomes degenerate and the N_2 equations are identical. In the second case, where $k = -N_2/N_1$, at least two of the equations are linearly dependent. We show that a bifurcation also occurs at this value of the parameter k.

The theorem puts restrictions on possible bifurcations of fixed points. The coordinates of all fixed points of $\mathrm{FP}_{0\pi}$ are constant under variation of parameter k; therefore, saddle-node, transcritical, and pitchfork bifurcations are not possible. However, a fixed point can change its stability as a result of a bifurcation. When an isolated fixed point changes stability, for a critical parameter value the invariant manifold of the point becomes neutral and consists of nonisolated fixed points. More precisely, the normal form on the invariant manifold contains zero terms only. This special type of degenerate bifurcation is similar to the degenerate Andronov-Hopf bifurcation (also known as the "vertical Hopf bifurcation"). To simplify notation, we denote this special degenerate bifurcation as DB.

The following example specifies the DB in the case $N_1 = 1$, $N_2 = 2$. In this case the two-dimensional system in phase differences (ϕ_1,ϕ_2) has two neutral one-dimensional invariant manifolds for the critical parameter value $k = -2$. The straight line $\phi_2 = \phi_1$ contains two fixed points (0,0), (π,π) and their invariant manifolds. The second curve $\phi_2 = \phi_1 + 2\arctan[(2+\cos\phi_1)/\sin\phi_1]$ contains four fixed points $(0,\pi)$, $(4\pi/3,2\pi/3)$, $(\pi,0)$, $(2\pi/3,4\pi/3)$ and their invariant manifolds. The straight line is unstable and the curve is stable in the transversal direction, but both of them are neutral for $k = -2$. The fixed points $(4\pi/3,2\pi/3)$, $(2\pi/3,4\pi/3)$ belong to the manifold $\mathcal{M}$ (but not to $\mathrm{FP}_{0\pi}$) and they are isolated in phase space. Under parameter variation, the stability of all six fixed points changes at the critical parameter value $(k = -2)$. The invariant manifolds are neutral (consist of nonisolated fixed points) at the critical parameter value. Another DB corresponds to the critical parameter value $k = 0$ with the invariant manifold $\phi_2 = -\phi_1$.

In the case of a system with a nonzero phase shift the degenerate bifurcation becomes a codimension 2 bifurcation. For example, under the variation of two parameters the degenerate bifurcation corresponds to the intersection of two saddle-node curves. We also show that under parameter variation the points of the invariant manifold $\mathcal{M}$ can change their stability in the two directions transversal to the manifold.

Remark. The symmetry γ_2 implies that the statement of the theorem is also correct for $\alpha = \pi$.

IV. STABLE REGIMES IN THE HONG-STROGATZ SYSTEM

In this section we consider all possible stable regimes for the system in phase differences when $\alpha = 0$. There are two sets of stable regimes: fixed points with a constant value of the order parameter ($R =$ const) and nonequilibrium regimes when the order parameter varies in time $[R = R(t)]$. Both of these regimes were found in [18]. We show that there are other stable equilibria for finite-dimensional systems besides those described in [18]. We also describe some nonequilibrium regimes and indicate the smallest number of oscillators that allow such regimes to occur.

A. Stable points of the system in phase differences

The theorem shows that the set $\mathrm{FP}_{0\pi}$ includes all possible fixed points. Some of these points are stable; others are not. The stability also depends on the value of the parameter k. Our aim is to find the possible stable regimes and their regions of stability.

1. Stable states with coordinates 0 and π

The coordinates of a point of $\mathrm{FP}_{0\pi}$ can only be 0 or π. In general, such a point can be represented as

$$\Phi(N_1,N_2,n_1,n_2) = (\underbrace{\overbrace{0,\ldots,0}^{n_1},\pi,\ldots,\pi}_{N_1-1},\underbrace{\overbrace{0,\ldots,0}^{n_2},\pi,\ldots,\pi}_{N_2}).$$

The Jacobian matrix at this point has a block form,

$$J(N_1,N_2,n_1,n_2) = \begin{pmatrix} A_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{-2} & A_2 & \mathbf{-2} & \mathbf{2} \\ \mathbf{-1+k} & \mathbf{1-k} & A_3 & \mathbf{1-k} \\ \mathbf{-1-k} & \mathbf{1+k} & \mathbf{-1-k} & A_4 \end{pmatrix},$$

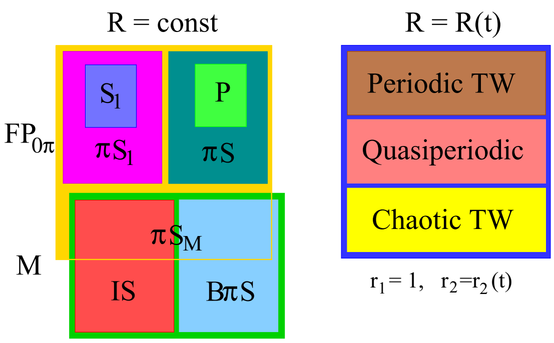

FIG. 2. (Color online) Schematic diagram of possible sets of stable solutions. Notation: $\mathrm{FP}_{0\pi}$, fixed points of (5) with coordinates 0 and π; $\mathcal{M}$, the invariant manifold with $R=0$; P, the parity state with $N_1=N_2$; S_1, synchronization of one contrarian with the conformists; $\pi\mathrm{S}$, the π state; $\pi\mathrm{S}_1$, the π-like state with one renegade contrarian; $\pi\mathrm{S}_{\mathcal{M}}=\mathrm{FP}_{0\pi}\cap\mathcal{M}$; IS, the incoherent state; BπS, the blurred π state; TW, the traveling wave state.

where A_1, A_2, A_3, A_4 are square matrices of dimension n_1, (N_1-n_1-1), n_2, and (N_2-n_2), respectively:

(1) A_1 has $N-2n-2$ on the diagonal and 0 elsewhere;

(2) A_2 has $-N+2n+4$ on the diagonal and 2 elsewhere;

(3) A_3 has $k(N-2n-1)-1$ on the diagonal and $k-1$ elsewhere;

(4) A_4 has $k(-N+2n+3)+1$ on the diagonal and $k+1$ elsewhere.

Here we use the notation $n=n_1+n_2$. By the bold font we denote the rectangular matrices filled by the corresponding elements: 0, -2, $-1+k$, etc.

The type of the Jacobian matrix allows us to transform it to a block-triangular or block-diagonal form and to calculate eigenvalues. These calculations show that some points of $\mathrm{FP}_{0\pi}$ are stable for some values of k and other points are unstable (unstable nodes, saddles, or degenerate saddles) for all values of k. Stable states can be subdivided into several groups [Fig. 2 (left)].

Synchronization with one contrarian (S_1). The origin $\mathcal{O}=(0,\dots,0)$ represents the full synchronization of all oscillators (the conformists win the competition; $R=1$ implies $r_1=r_2=1$, $\delta=0$). We use k_{S_1} to denote bifurcation values of the parameter k such that the origin is stable for $k>k_{S_1}$. The values of k_{S_1} are then given by the following formula:

$$k_{S_1}=\begin{cases}0, & N_1=1,\dots,N-2,\\ -\frac{1}{N-1}, & N_1=N-1,\\ -\infty, & N_1=N.\end{cases}$$

This means that if there is only one contrarian in the system ($N_1=N-1$) it can be attracted by the synchronous group of $N-1$ conformists [as in Fig. 1(h)]. This regime occurs when the coupling parameter $k\in[-1/(N-1),0]$. The regime is only possible for a finite-dimensional system with a single contrarian, because for positive and negative oscillator fractions we have $p\to 1, q\to 0$ when $N\to\infty$. Note that the above formula includes an evident case of full synchronization that corresponds to the system with two groups of conformists $K_1>0$, $K_2>0$.

The π state (πS). Following [18] we define the π state as that which occurs when the relationship between two synchronous groups is such that $r_1=r_2=1$ and $\delta=\pi$. In other words, the π state is characterized by strict antiphase polarization between conformists and contrarians [Fig. 1(d) represents an example of this state for $N_1=4$, $N_2=3$]. The points corresponding to the π state are described by the formula

$$\Phi(N_1,N_2)=(\underbrace{0,\dots,0}_{N_1-1},\underbrace{\pi,\dots,\pi}_{N_2})\in\mathbb{T}^{N-1},$$
$$N_1=1,\dots,N-1.$$

They are asymptotically stable when

$$k<k_{\pi S}=-\frac{N_2}{N_1}\quad\text{for}\quad N_1>N_2,\quad N\geqslant 3.$$

Thus, the π state is only possible when conformists dominate over contrarians. It is easy to see that the general order parameter is $R=(N_1-N_2)/N=2p-1$ when $N_1>N_2$. Our result is in agreement with [18], where this result was presented in terms of the parameters (p,C).

The π-like state (πS_1): One renegade contrarian. In addition to the state of full synchronization and the π states, system (5) can have some other asymptotically stable points in the set $\mathrm{FP}_{0\pi}$. These points are described by the expression

$$\Phi_1(N_1,N_2)=(\underbrace{0,\dots,0}_{N_1-1},\underbrace{0,\pi,\dots,\pi}_{N_2})\in\mathbb{T}^{N-1},$$
$$N_1=1,\dots,N-2.$$

They are stable if

$$k\in\left(-\frac{N_2}{N_1};-\frac{N_2-2}{N_2}\right)\quad\text{for}\ 2\leqslant N_2\leqslant N_1+\frac{1-(-1)^N}{2}.$$

This is the situation when all conformists synchronize and N_2-1 contrarians synchronize, too, but one "renegade" contrarian joins the group of conformists [Fig. 1(i)]. Note that this case is missing in the classification of stable states in [18]. The order parameter for conformists is $r_1=1$, while the order parameter for contrarians is $r_2=(N_2-2)/N_2$, and $\delta=\pi$. The general order parameter R is

$$R=\frac{|(N_1+1)-(N_2-1)|}{N}=\frac{N_1-N_2+2}{N}$$
$$=2p-1-\frac{2}{N}.$$

This parameter depends not only on the fraction sizes p, q, but also on the number of oscillators N. The bifurcation value $k_{\pi S_1}=-(N_2-2)/N_2$ cannot be expressed in terms of the fraction sizes. Thus, the $\pi\mathrm{S}_1$ states are only possible for *finite-dimensional* systems.

The π state with order parameter zero ($\pi S_{\mathcal{M}}$). All points of the invariant manifold $\mathcal{M}$ with coordinates 0 and π belong to the intersection $\pi\mathrm{S}_{\mathcal{M}}=\mathcal{M}\cap\mathrm{FP}_{0\pi}$. The set $\pi\mathrm{S}_{\mathcal{M}}$ is nonempty only when N is an even number [Fig. 1(e)], and the equilibrium points have $N/2-1$ zeros and $N/2$ coordinates π (with any permutation of 0, π). Such points are stable in one direction transversal to $\mathcal{M}$ for the parameter values

$$k < k_{\pi S_{\mathcal{M}}} = -\frac{N_1}{N_2}, \quad N_2 = 1, \ldots, N-1,$$

and they are neutral in the other $N-2$ directions. Calculations of the order parameters show that the equalities $r_1 = r_2 = 0$ or $N_1 r_1 = N_2 r_2$ are satisfied, except in the special case $N_1 = N_2 = N/2$ for which we have $r_1 = r_2 = 1, R = 0, \delta = \pi$. This means that stable points of $\pi S_{\mathcal{M}}$ belong to the set $\text{IS} \cup \text{B}\pi\text{S}$ (this set is described below) or to the set P of the points with an equal number of conformists and contrarians.

Remark. The set of stable points of system (5) is much smaller than the set FP. There are many fixed points of $\text{FP}_{0\pi}$ which are not stable for any value of the parameter k (sources, saddles, degenerate saddles). The set $\text{FP}_{0\pi}^{\text{stable}}(k) = \text{S}_1 \cup \pi\text{S} \cup \pi\text{S}_1 \cup \pi\text{S}_{\mathcal{M}}$ is a set of all points that can be stable for appropriate values of the parameter k.

The set of stable points $\text{FP}_{0\pi}$ would be enlarged if more general functions than the sine function were allowed as interaction functions. Suppose that an arbitrary odd and periodic function $f(x)$ such that $f(\pi) = 0$ is allowed as an interaction function. Computations suggest that all points in $\text{FP}_{0\pi}$ become stable under a proper selection of the parameters k, $a = \frac{df(x)}{dx}|_{x=0}$, $b = \frac{df(x)}{dx}|_{x=\pi}$. Unfortunately, analytical proof of this fact is absent.

2. *The invariant manifold $\mathcal{M}$: Contrarians win*

The second subset (besides $\text{FP}_{0\pi}$) that was described by the theorem is the manifold $\mathcal{M}$ of fixed points with the general order parameter $R = 0$. This manifold is the union of two sets $\mathcal{M} = \text{IS} \cup \text{B}\pi\text{S}$ [18], where IS is the incoherent state and $\text{B}\pi\text{S}$ is the so-called blurred π state.

We recall that $\dim \mathcal{M} = N - 3$ in $\mathbb{T}^{N-1}$. Thus, two coordinates of a point on the manifold can be written as functions of the other $N-3$ free coordinates [25]:

$$\phi_{N-2} = \arctan\left(\frac{f_1}{f_2}\right) - \frac{1}{2}\arccos\left(\frac{f_1^2+f_2^2}{2} - 1\right) + \frac{\pi}{2}[1 - \text{sgn}(f_2)],$$

$$\phi_{N-1} = \arctan\left(\frac{f_1}{f_2}\right) + \frac{1}{2}\arccos\left(\frac{f_1^2+f_2^2}{2} - 1\right) + \frac{\pi}{2}[1 - \text{sgn}(f_2)],$$

where

$$f_1(\phi_1, \ldots, \phi_{N-3}) = -\sum_{j=1}^{N-3} \sin\phi_j,$$

$$f_2(\phi_1, \ldots, \phi_{N-3}) = -1 - \sum_{j=1}^{N-3} \cos\phi_j.$$

The following lemma will be helpful in studying manifold stability.

Lemma 2. The Jacobian rank of system (5) is

$$\text{rank}(\mathcal{J}) = \begin{cases} 1 & \text{if} \quad (\phi_1, \ldots, \phi_{N-1}) \in \pi S_{\mathcal{M}}, \\ 2 & \text{if} \quad (\phi_1, \ldots, \phi_{N-1}) \in \mathcal{M}/\pi S_{\mathcal{M}}, \end{cases}$$

where

$$\mathcal{J} = \mathcal{J}(\phi_1, \ldots, \phi_{N-1}, k, \alpha) = \frac{\partial(G_1(\phi_1, \ldots, \phi_{N-1}), \ldots, G_{N-1}(\phi_1, \ldots, \phi_{N-1}))}{\partial(\phi_1, \ldots, \phi_{N-1})},$$

$G_i(\phi_1, \ldots, \phi_{N-1})$, $i = 1, \ldots, N-1$, are right hand sides of Eqs. (5).

The proof of the Lemma is analogous to the proof of Lemma 3 in [25].

Lemma 2 implies that $N-3$ eigenvalues for the points of the manifold are

$$\lambda_1 = \cdots = \lambda_{N-3} = 0.$$

Another two eigenvalues are functions of $N-3$ coordinates of the points on the manifold:

$$\lambda_{N-2} = h_1(\phi_1, \ldots, \phi_{N-3}),$$
$$\lambda_{N-1} = h_2(\phi_1, \ldots, \phi_{N-3}).$$

In the two-cluster states $\pi S_{\mathcal{M}}$, one more eigenvalue vanishes ($\lambda_{N-2} = 0$). Note that Lemma 2 is correct for $\alpha \neq 0$, which is a more general condition than the Hong-Strogatz case.

The manifold $\mathcal{M}$ is a continuous set of nonisolated fixed points. According to Lemma 2, the stability of each point inside the manifold is neutral. Stability of a point in two directions transversal to the manifold depends on the point location in the manifold and can vary for different points even for the same parameter value.

To describe bifurcations on the manifold, we need a function that depends on $N-3$ variables $k = k(\mathcal{M}) = k(\phi_1, \ldots, \phi_{N-3})$. Therefore, at least one point on the manifold is transversally stable if $k < k_{\max}(\mathcal{M}) = \max_{\phi_j \in \mathcal{M}} k(\phi_1, \ldots, \phi_{N-3})$ and all points of the manifold are transversally stable if $k < k_{\min}(\mathcal{M}) = \min_{\phi_j \in \mathcal{M}} k(\phi_1, \ldots, \phi_{N-3})$.

We also note that the *bifurcation without parameters* theory (see [26] and references there) can be applied to understand the qualitative structure of the manifold $\mathcal{M}$.

The incoherent state (IS). According to [18] a regime with $r_1 = r_2 = 0$ is called the incoherent state [Fig. 1(g)]. There is a manifold of fixed points of Eqs. (5) with the property $r_1 = r_2 = 0$ and this manifold is a subset of $\mathcal{M}$, because the general order parameter is $R = 0$ in this case. Denote by $\mathcal{M}_s \in \mathbb{T}^{N_s}$ the sets that correspond to $r_s = 0$, $s = 1,2$. Each of these sets is $(N_s - 2)$-dimensional in $\mathbb{T}^{N_s}$. Hence, the set

$$\mathcal{M}_{\text{IS}} = \{(\theta_1, \ldots, \theta_N) : (\theta_1, \ldots, \theta_{N_1}) \in \mathcal{M}_1, (\theta_{N_1+1}, \ldots, \theta_N) \in \mathcal{M}_2\}$$

is m-dimensional, where

$$m = m_1 + m_2, \quad m_s = \begin{cases} 0, & N_s = 0, \\ 1, & N_s = 2, \\ N_s - 2, & N_s \geqslant 3, \end{cases} \quad N_s \neq 1.$$

The submanifold $\mathcal{M}_{IS}$ is $(m-1)$-dimensional for $(N-1)$-dimensional system (5) in phase differences; $\dim(\mathcal{M}_{IS}) < \dim(\mathcal{M})$ when $N_s \neq N$.

Different fixed points of $\mathcal{M}_{IS}$ can have different stability in the last two directions transversal to the manifold depending on the location of the points. The bifurcation parameter $k = k_{IS}$ is a function $k_{IS} = k_{IS}(\Phi)$, where Φ consists of $(m-1)$ coordinates ϕ_i. There is an interval $[k_{\min(IS)},k_{\max(IS)}] = [\min_{\Phi\in\mathcal{M}_{IS}} k(\Phi), \max_{\Phi\in\mathcal{M}_{IS}} k(\Phi)]$ of the values of the parameter k where the manifold $\mathcal{M}_{IS}$ gradually loses the stability of its fixed points.

As an example, consider the case $N=7$, $N_1=4$, $N_2=3$. The manifold $\mathcal{M}_{IS}$ is two-dimensional in the phase space $\mathbb{T}^6$. It can be described by the two-parametric set $\mathcal{M}_{IS} = (\pi,\phi_1,\phi_1+\pi,\phi_2,\phi_2+2\pi/3,\phi_2+4\pi/3) \in \mathbb{T}^6$ with $r_1 = r_2 = 0$. Calculations show that the bifurcation parameter only depends on the first of these parameters and $\mathcal{M}_{IS}$ is stable when

$$k < k(\mathcal{M}_{IS}) = k_{IS}(\phi_1) = -4(1+|\cos\phi_1|)/3;$$

therefore, $[k_{\min(IS)},k_{\max(IS)}] = [-8/3,-4/3]$.

The blurred π state (BπS). Another state described in [18] is the regime that satisfies the equalities $pr_1 = qr_2$ (equivalently, $N_1 r_1 = N_2 r_2$) and $\delta = \pi$ [see Fig. 1(f) for an example]. The blurred π state is a subset of the invariant manifold $\mathcal{M}$. More exactly, $\mathcal{M}_{B\pi S} = \mathcal{M}\backslash\mathcal{M}_{IS}$. $\dim(\mathcal{M}_{B\pi S}) = N-3$ in the phase space $\mathbb{T}^{N-1}$. Therefore, BπS is a $(N-3)$-dimensional set of nonisolated fixed points for the system in phase differences. All these fixed points have $(N-3)$ zero eigenvalues in the directions inside the manifold and two other eigenvalues (that are functions of the point coordinates on this manifold) which characterize the stability of the fixed point in directions transversal to the manifold. As in the previous case, there are two bifurcation parameters $k_{\min(B\pi S)}$ and $k_{\max(B\pi S)}$.

As an example, consider a system with four conformists and three contrarians, $\dim(\mathcal{M}_{B\pi S}) = 4$ in the phase space of system (5). We check the stability of the one-parametric set $\mathcal{M}_{B\pi S}$, which is presented by the set $\mathcal{M}_{B\pi S}(\phi) = (0,\pi,\phi,2\pi/3,4\pi/3,\phi+\pi) \in \mathbb{T}^6$. The local order parameters are

$$r_1 = \tfrac{1}{4}\sqrt{2(1+\cos\phi)}, \quad r_2 = \tfrac{1}{3}\sqrt{2(1+\cos\phi)}.$$

Each point of $\mathcal{M}_{B\pi S}$ is neutral in four directions inside the manifold and stable in two directions transversal to the manifold when

$$k < k_{B\pi S}(\phi) = -\frac{2[2\sin^2\phi + 6 + \sqrt{36 + 13\sin^2\phi - 4\sin^2(2\phi)}]}{4\cos^2\phi + 5}.$$

The parametric region is rather wide: $k_{B\pi S}(\phi) \in [-6,-8/3]$. This example shows that transversal stability, the order parameters, and parametric regions of stability significantly depend on the point location on the blurred π-state manifold.

Summarizing the results obtained in this section, we remark that no new regimes with $R=0$ were found in the finite-dimensional case besides those presented in [18]. Still we obtained more details about the structure of stable solutions with $R=0$: (1) Each regime with $R=0$ is not isolated; (2) all solutions with $R=0$ compose a continuous $(N-2)$-dimensional set (invariant manifold) $\mathcal{M}$; (3) $\mathcal{M}$ is a union of incoherent and blurred π states; (4) each regime is stable in two directions transversal to the manifold and neutral in other directions; (5) transversal stability of the regimes can vary across the manifold for a fixed value of the parameter k; (6) the bifurcation value of k depends on the location of the fixed point on the manifold.

B. Nonequilibrium regimes

All the regimes described above correspond to equilibria of the system in phase differences, which implies a constant value of the order parameter $R = \text{const}$. Now let us concentrate on the regimes with a nonconstant order parameter $R = R(t)$.

The traveling wave (TW) state. Consider a regime when all conformists are fully synchronized ($r_1 = 1$) but the mutual location of contrarians in phase space varies with time [the order parameter is not constant, $r_2 = r_2(t)$]. Then the angle $\delta = \delta(t)$ will not be constant either, in contrast to the previous regimes when $\delta = 0$ or $\delta = \pi$. The general order parameter $R = R(t)$ will also not be constant. A regime with such properties has been called the traveling wave state in [18]. It should be noted that the term "traveling wave" is correct for the case of a very large (or infinite) number of oscillators, especially contrarians (as in [18]). In the case when the number of oscillators is relatively small it is difficult to speak about "waves" in the usual meaning of the word. However, we keep this term for arbitrary sized systems because the properties of the regimes with $r_1 = 1$ and $r_2 = r_2(t)$ are the same for any number of oscillators and for similar parametric regions.

The condition $r_1 = 1$ reduces system (5) to the system on the N_2-dimensional manifold $\mathcal{P}^1_{N_2} \in \mathbb{T}^{N-1}$ ($\phi_i = 0$, $i = 1,\ldots,N_1-1$):

$$\dot{\phi}_i = -kN_1\sin\phi_i - \sum_{j=1}^{N_2}\sin\phi_j - k\sum_{j=1,j\neq i}^{N_2}\sin(\phi_i-\phi_j), \quad i = 1,\ldots,N_2. \tag{9}$$

The stability analysis of the TW state can be divided into two steps:

(1) finding a phase-unlocked limit cycle for the N_2-dimensional system (9) inside the manifold;

(2) investigating the stability of solutions in the directions transversal to the manifold.

The first $N_1 - 1$ equations of system (5),

$$\dot{\phi}_i = -\sin\phi_i - \sum_{j=1}^{N-1}\sin\phi_j - \sum_{j=1,j\neq i}^{N-1}\sin(\phi_i-\phi_j), \quad i = 1,\ldots,N_1-1, \tag{10}$$

are responsible for the transversal stability of the manifold. This subsystem has the variables $(\phi_1,\ldots,\phi_{N_1-1})$ and parameters $(\phi_{N_1},\ldots,\phi_{N-1})$. The Jacobian matrix of the system linearized at the origin (10) has the eigenvalues

$$\lambda_i = -N_1 - \sum_{j=N_1}^{N-1}\cos\phi_j, \quad i = 1,\ldots,N_1-1.$$

This implies that the manifold $\mathcal{P}^1_{N_2}$ is transversally stable for $N_1 \geqslant N_2$. Thus, all stable trajectories of (10) are stable for

(5) when $N_1 \geqslant N_2$. The last condition is sufficient but not necessary for transversal stability.

Note that the regime when all contrarians are synchronized ($r_2 = 1$) but at least one conformist is not synchronous with other ones ($r_1 \neq 1$) is impossible. This can be explained by the transversal instability of the invariant manifold $\mathcal{P}^2_{N_1}$ ($\phi_{N_1} = \cdots = \phi_{N-1}$) for all values of k when $r_1 \neq 1$.

System (9) has permutation symmetry of its variables ϕ_i. The TW state corresponds to phase-unlocked limit cycles (nonhomologous to the zero periodic orbit on the torus) for the system in phase differences. These cycles appear and disappear as a result of the fold bifurcations of stable and saddle unlocked cycles or saddle-connection bifurcations. The existence of a stable limit cycle of system (9) is a necessary condition for the TW state.

System (5) cannot have the TW state in the simplest nontrivial case $N = 3$, $N_1 = 2$, because the only possible TW state must be on the line $\phi_1 = 0$. However, according to Lemma 1 two equilibria are located on this line at the points 0 and π independent of the value of parameter k. The system with two conformists and an arbitrary number of contrarians cannot have the TW state for the same reason. We show later that a stable TW state is possible for $N = 3$ for any small value of the parameter α and a certain value of k. The analysis of system (9) in the two-dimensional case shows that it does not have stable limit cycles for arbitrary values of the parameters N_1, k. This system has two square invariant regions on $\mathbb{T}^2$ bounded by the invariant lines $\phi_2 = \pm\phi_1$. For the same reason, the system has TW states for an arbitrary number of oscillators only if $N_1 \in [3, N-1]$.

If all conformists are synchronized, periodic, quasiperiodic, and chaotic traveling waves are possible in the subset of contrarians [Fig. 2 (right)]. The identification of these types of dynamics as well as the determination of the regions where they occur was achieved through numerical analysis.

The periodic traveling wave (TW_P) state. In this regime all phase differences for contrarians $\phi_{N_1+1}, \ldots, \phi_N$ are periodic (the system has a stable limit cycle on the invariant manifold $\mathcal{P}^1_{N_2}$). The order parameter $r_2(t)$ is periodic; therefore, $R(t)$ is also periodic. A periodic TW appears when $k \in (k_P, k_Q)$ for $k_Q < 0$ (k_Q is usually near 0). The smallest dimensions for such a regime to exist are $N = 5$ and $N_1 = 2$. Additionally, a four-dimensional system with three contrarians has stable periodic regimes for small negative values of the parameter k. This regime can be formally considered as the TW state with only one conformist (of course, $r_1 = 1$ in this case).

The quasiperiodic traveling wave (TW_Q) state. The system can have stable quasiperiodic regimes [Fig. 3(b)] with quasiperiodic functions $r_2(t)$ and $R(t)$. This regime occurs in a very narrow parametric region $k \in (k_Q, 0)$. When $k = 0$, system (5) is degenerate and has N_2 equivalent equations. Therefore, the invariant subspace $\mathcal{P}^1_{N_2}$ is filled by straight lines at the bifurcation point $k = 0$. A narrow quasiperiodic attractor appears from these lines when k becomes negative. The second type of quasiperiodic TW regimes occurs as a transition from periodic to chaotic attractors according to [27] for $k \in (k_C, k_P)$. A transition from periodic to quasiperiodic and then to chaotic attractors is shown in Figs. 3(a)–3(c). The smallest number of oscillators for the quasiperiodic regime is $N_1 = 2$, $N_2 = 3$.

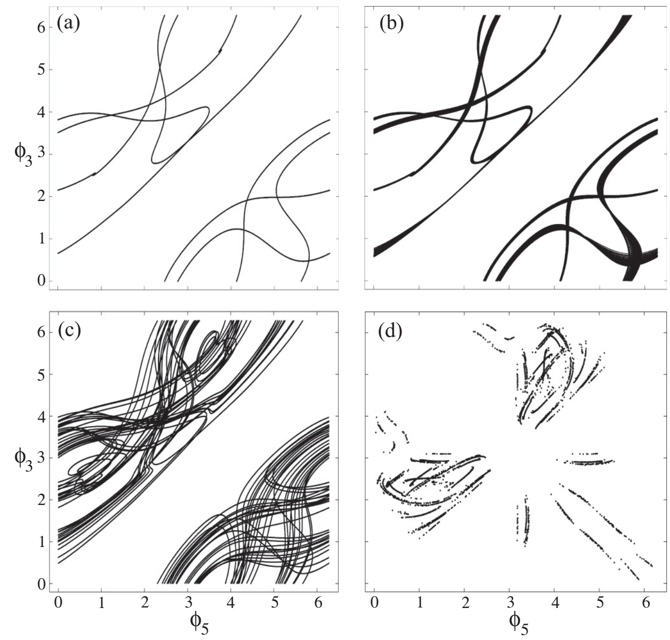

FIG. 3. Projections on the plane (ϕ_3,ϕ_5) of some trajectories of system (5) for $N_1 = N_2 = 3$: (a) periodic trajectory ($k = -0.21$), (b) quasiperiodic trajectory ($k = -0.216$), (c) chaotic trajectory ($k = -0.3$), (d) Poincaré section $\phi_4 = \pi$ ($k = -0.3$).

Traveling chaos (TW_C). Chaos is possible when the N_2-dimensional system (9) has phase-unlocked chaotic trajectories. A chaotic attractor arises from homoclinic tangency bifurcations or according to the Ruelle-Takens-Newhouse bifurcation scenario [27]. Chaos exists when $k \in (k_{C_1}, k_{C_2})$. The smallest number of oscillators required for traveling chaos to exist is $N_1 = N_2 = 3$ [Fig. 3(c)]. In this particular case, $k_{C_1} \approx -0.3554$, $k_{C_2} \approx -0.2174$.

V. THE SYSTEM WITH A PHASE SHIFT

In this section we study system (2) with a phase shift $\alpha \in \mathbb{T}^1$. A phase shift extends the set of interaction functions and makes the dynamics of the system much more complicated and rich. The Hong-Strogatz model is a special case of system (2) for $\alpha = 0$. If $\alpha = \pi$, the interaction between oscillators swaps conformists and contrarians.

A. The Kuramoto-Saskaguchi model ($k = 1$)

If system (2) contains only conformists ($N_1 = N$) or contrarians ($N_2 = N$), it is called the Kuramoto-Sakaguchi model. The dynamics of this model have been studied in [6] and [7]. In [25] it is shown that the corresponding system in phase differences,

$$\dot{\phi}_i = -\sin(\phi_i + \alpha) - \sum_{j=1}^{N-1} \sin(\phi_j - \alpha) - \sum_{j=1, j\neq i}^{N-1} \sin(\phi_i - \phi_j + \alpha), \quad i = 1, \ldots, N-1, \quad (11)$$

can have only three types of equilibria:

(1) the origin $\mathcal{O} = (0,\ldots,0)$;
(2) $2^{N-1} - 1$ saddle points that belong to the invariant one-dimensional lines

$$\phi_1 = \cdots = \phi_i \neq \phi_{i+1} = \cdots = \phi_{N-1} = 0,$$
$$i = 1,\ldots,N-1$$

(with any possible permutations);
(3) fixed points that form the invariant $(N-3)$-dimensional manifold $\mathcal{M}$.

The $(N-2)$-dimensional invariant manifolds $\mathcal{P}^1_{N-2}$ ($\phi_i = 0,\ \phi_i = \phi_j,\ i,j = 1,\ldots,N-1$) split up the whole phase space $\mathbb{T}^{N-1}$ into $(N-1)!$ invariant regions [22]. There are no equilibria inside these invariant manifolds. The dynamics inside the invariant manifolds of the Kuramoto-Sakaguchi system are well described by the Watanabe-Strogatz theory [7].

The origin $\mathcal{O}$ and the manifold $\mathcal{M}$ (in two transversal directions) have opposite types of stability for any value of the parameter α except $\alpha = \pm\pi/2$. Each saddle moves along an invariant line when the parameter α is varied. System (11) simultaneously has multiple transcritical bifurcations at the point $\mathcal{O}$ (all saddles meet at the origin) and a degenerate Andronov-Hopf bifurcation at all points on the manifold $\mathcal{M}$ when $\alpha = \pm\pi/2$. $(N-1)$-dimensional sets of nonisolated periodic orbits fill invariant regions at the bifurcation point and neutral sets of neutral heteroclinic cycles create borders of the invariant regions. The bifurcations change the stability of $\mathcal{O}$ and $\mathcal{M}$.

The situation is more complicated when the system has two parameters α, k because one has to consider the two parametric bifurcation plane. Our studies of system dynamics in this case are based on the known results about system dynamics in two partial cases when the parameters belong to the line $\alpha = 0$ (the Hong-Strogatz model) or $k = 1$ (the Kuramoto-Sakaguchi model). We start from the system with the lowest dimension.

B. Two oscillators

Consider the simplest system that contains one conformist and one contrarian ($N = 2,\ N_1 = N_2 = 1$). The bifurcation diagram for this case is shown in Fig. 4. The only phase difference $\phi_1 = \theta_1 - \theta_2$ obeys the equation

$$\dot{\phi}_1 = -\sin(\phi_1 - \alpha) - k\sin(\phi_1 + \alpha) - (k-1)\sin\alpha,$$

which can be rewritten as

$$\dot{\phi}_1 = -(k+1)\cos\alpha\sin\phi_1 - (k-1)\sin\alpha(1+\cos\phi_1).$$

The last equation has only two fixed points: $\phi_1 = \pi$ and

$$\phi_1^* = -2\arctan\left(\frac{k-1}{k+1}\tan\alpha\right).$$

Therefore, the system has three lines $k = -1,\ \alpha = \pm\pi/2$ of transcritical bifurcations which occur at the point $\phi_1 = \pi$.

The point $\phi_1 = \pi$ corresponds to the antiphase conformist-contrarian regime (with the order parameter $R = 0$). This regime is stable in the parametric regions $k < -1,\ \alpha \in (-\pi/2,\pi/2)$, and $k > -1,\ \alpha \in (\pi/2,3\pi/2)$. The point ϕ_1^*

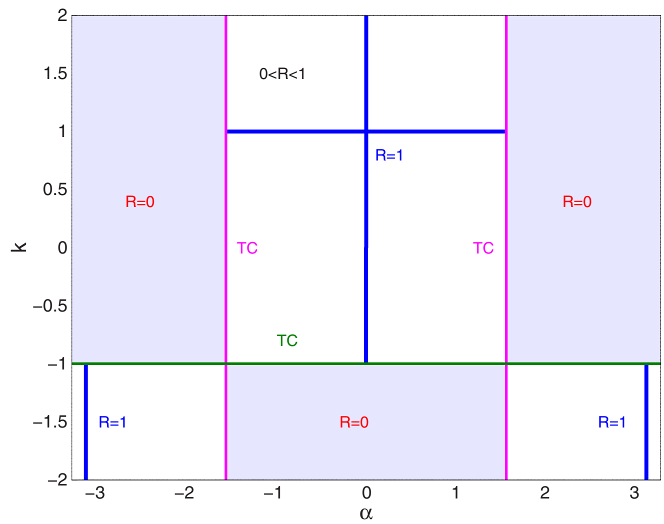

FIG. 4. (Color online) Bifurcation diagram on the (α,k) bifurcation plane for two oscillators $N_1 = N_2 = 1$: TC indicates transcritical bifurcations. The regions of stability of the antiphase state ($R = 0$) are marked by light gray (light blue online) and the lines of synchronization ($R = 1$) are shown in blue. The white color indicates the regions for $0 < R < 1$.

changes its location with parameter variation. The point ϕ_1^* is located at zero (synchronization of both oscillators, $R = 1$) if at least one of the following equalities holds $\alpha = 0$, $\alpha = \pi$, $k = 1$. This point is stable in the following one-dimensional intervals on the parametric plane (α,k):
(1) $\alpha = 0,\ k > -1$ (the Hong-Strogatz model);
(2) $\alpha = \pi,\ k < -1$ (the inverse Hong-Strogatz model);
(3) $k = 1,\ \alpha \in (-\pi/2,\pi/2)$ (the Kuramoto-Sakaguchi model).

C. Three oscillators

For $N = 3$ two opposing configurations are possible $N_1 = 1,\ N_2 = 2$ and $N_1 = 2,\ N_2 = 1$ (the cases $N_1 = 3$ and $N_1 = 0$ were considered in Sec. V A). Since the number of conformists in the first configuration is equal to the number of contrarians in the second configuration, the situation satisfies to the conditions mentioned in Sec. II C; therefore, the action γ_3 (7) transforms the trajectories of the case $N_1 = 2$ into the trajectories of the case $N_2 = 2$. This action is of the form

$$\gamma_3:\ (\phi_1,\phi_2,\alpha,k,t) \longmapsto (-\phi_1,\phi_2-\phi_1,\alpha,1/k,-t). \quad (12)$$

The parameter k of the first system is transformed to the parameter $1/k$ of the second system. Both systems have the zero-dimensional invariant manifold $\mathcal{M} \in \mathbb{T}^2$, which consists of two parts: $\mathcal{M}_1 = (2\pi/3,4\pi/3)$ and $\mathcal{M}_2 = (4\pi/3,2\pi/3)$. The action γ_3 transforms $\mathcal{M}_1 \to \mathcal{M}_2$ and vice versa.

System (5) has the invariant manifold $\mathcal{P}^1_1 = \{(\phi_1,\phi_2):\ \phi_1 = 0\}$ for $N_1 = 2$. It also has the invariant manifold $\mathcal{P}^2_1 = \{(\phi_1,\phi_2):\ \phi_1 = \phi_2\}$ for $N_2 = 2$. The action γ_3 transforms $\mathcal{P}^1_1$ for the first system into $\mathcal{P}^2_1$ for the second one. Therefore, the manifold $\mathcal{P}^i_1$ does not split the phase space $\mathbb{T}^2$ into invariant regions (or there exists only one invariant region). In contrast, the phase space of the standard Kuramoto-Sakaguchi model

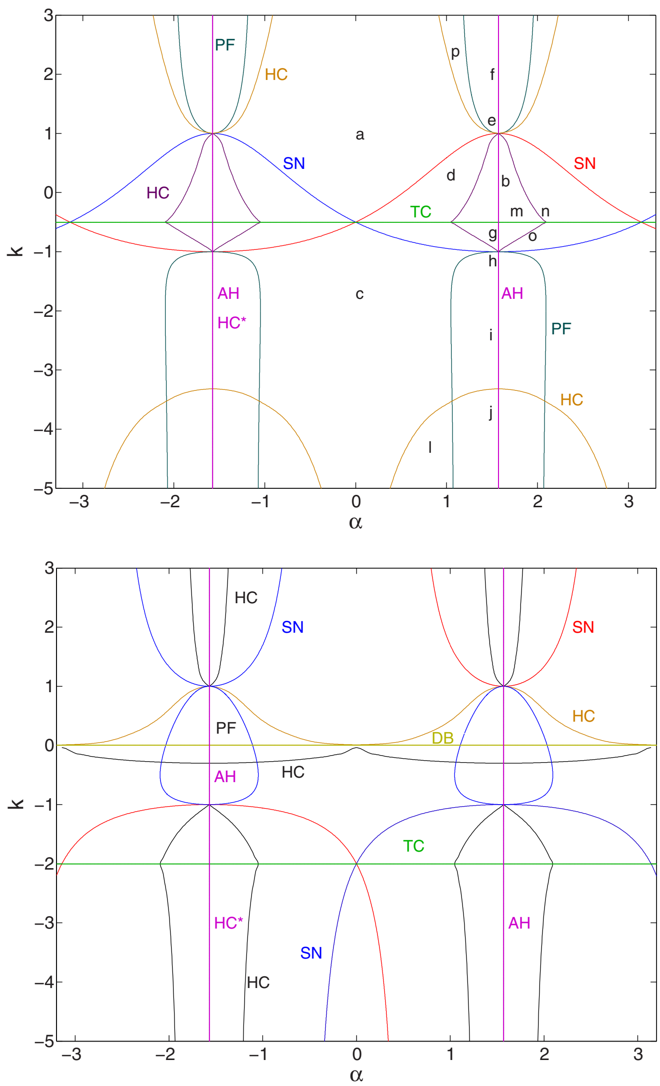

FIG. 5. (Color online) Bifurcation diagrams on the (α,k) bifurcation plane for three oscillators: (top) $N_1 = 2, N_2 = 1$; (bottom) $N_1 = 1, N_2 = 2$. Notation: AH, Andronov-Hopf; PF, pitchfork; SN, saddle-node; TC, transcritical;, HC, HC*, heteroclinic (saddle-connection); DB, degenerate bifurcations. The lines AH and HC* coincide in the diagrams but these two bifurcations occur in different places of the phase space at the same bifurcation moment.

is split into two invariant regions by the manifolds $\phi_1 = 0$, $\phi_2 = 0$, and $\phi_1 = \phi_2$.

Two conformists and one contrarian. Let $N_1 = 2$, $N_2 = 1$. The system has manifolds $\mathcal{P}_1^1$ (a straight line) and $\mathcal{M}$ (two points), which are invariant under variation of the parameters k and α. In contrast to the Hong-Strogatz model, the system can have many equilibria besides those in FP and these equilibria do not have to belong to the invariant manifolds $\mathcal{P}_1^1$, $\mathcal{M}$ (Fig. 6). Nevertheless, all bifurcations of equilibria occur at points that belong to these invariant manifolds. The system has Z_2 symmetry which is described by the action $\gamma : (\phi_1,\phi_2) \longmapsto (-\phi_1,\phi_2 - \phi_1)$.

The bifurcation lines in the case $N_1 = 2$, $N_2 = 1$ are shown in the diagram of Fig. 5 (top). These lines split the parameter (α,k) plane into the regions with different phase dynamics. Figure 6 shows 16 phase portraits.

The following notation is used in Fig. 5.

(1) AH at $\alpha = \pm\pi/2$ indicates a degenerate Andronov-Hopf bifurcation at the points $\mathcal{M}_1$, $\mathcal{M}_2$. The bifurcation changes the stability of $\mathcal{M}_i$ without the appearance of a limit cycle. At the moment of bifurcation the fixed point is surrounded by a nonisolated set of periodic orbits.

(2) TC at $k = -1/2$ indicates a transcritical bifurcation at the points $\mathcal{M}_i$. This bifurcation occurs for sources and saddles when $\alpha \in (-\pi/2,\pi/2)$ and it occurs for sinks and saddles when $\alpha \in (\pi/2,3\pi/2)$.

(3) SN denotes a saddle-node bifurcation on the invariant line $\mathcal{P}_1^1$ ($\phi_1 = 0$). This bifurcation generates a traveling wave, corresponding to the disappearance of fixed points on the line $\phi_1 = 0$ and its transformation into a stable [for $\alpha \in (-\pi/2,\pi/2)$] or unstable [for $\alpha \in (\pi/2,3\pi/2)$] limit cycle which is phase-unlocked along the variable ϕ_2. The degenerate bifurcation at the line $\alpha = \pm\pi/2$ changes the stability of this cycle. The bifurcation occurs simultaneously with the AH bifurcation.

(4) PF indicates a pitchfork bifurcation on the $\mathcal{P}_1^1$ invariant line and in the direction transversal to this line.

(5) HC are saddle-connection bifurcations that lead to the appearance of homoclinic and heteroclinic cycles at the bifurcation point. Homoclinic and heteroclinic cycles (which consist of two saddle or saddle-node points and their one-dimensional invariant manifolds) can be of two types: phase-locked [Figs. 6(b), 6(e)–6(h)] and phase-unlocked [Figs. 6(b), 6(f)–6(h), 6(n), 6(o)]. The same heteroclinic curve (connecting two saddle or saddle-node points) can simultaneously lead to phase locked and phase unlocked heteroclinic cycles, as shown in Figs. 6(c) and 6(f)–6(h). One can see in Fig. 6(h) that heteroclinic lines may fill the whole phase space $\mathbb{T}^2$ or form two-dimensional sets of phase-locked homoclinic and heteroclinic cycles. This happens at the points $(\alpha,k) = (\pm\pi/2, -1)$ of the intersection of several bifurcation lines.

Two bifurcations occur in different places of phase space when $\alpha = \pi/2$ (and symmetrically $\alpha = -\pi/2$): a degenerate Andronov-Hopf bifurcation AH at the points $\mathcal{M}_1$, $\mathcal{M}_2$, and a saddle-connection (heteroclinic, HC*) bifurcation of one-dimensional invariant manifolds of saddles or saddle nodes (depending on the value of the parameter k). These two simultaneous bifurcations are not a generic situation; they are only possible due to system symmetries. The points $\mathcal{M}_1$, $\mathcal{M}_2$ are neutral (centers) at the point of bifurcation and they are surrounded by sets of nonisolated periodic orbits. These points lose their neutrality if $\alpha \neq \pm\pi/2$. No limit cycle appears or disappears after the bifurcation. Homo- or heteroclinic cycles bound the sets of nonisolated periodic orbits and split phase-locked and phase-unlocked solutions.

These situations are shown in Figs. 6(b) and 6(f), respectively. In these figures, the regions of phase-locked periodic orbits are shown by the white regions and phase-unlocked solutions are drawn in blue, yellow, and pink. A saddle-connection bifurcation destroys homoclinic boundaries, so that neutral for $\alpha = \pi/2$ trajectories can leave their previous neutral region.

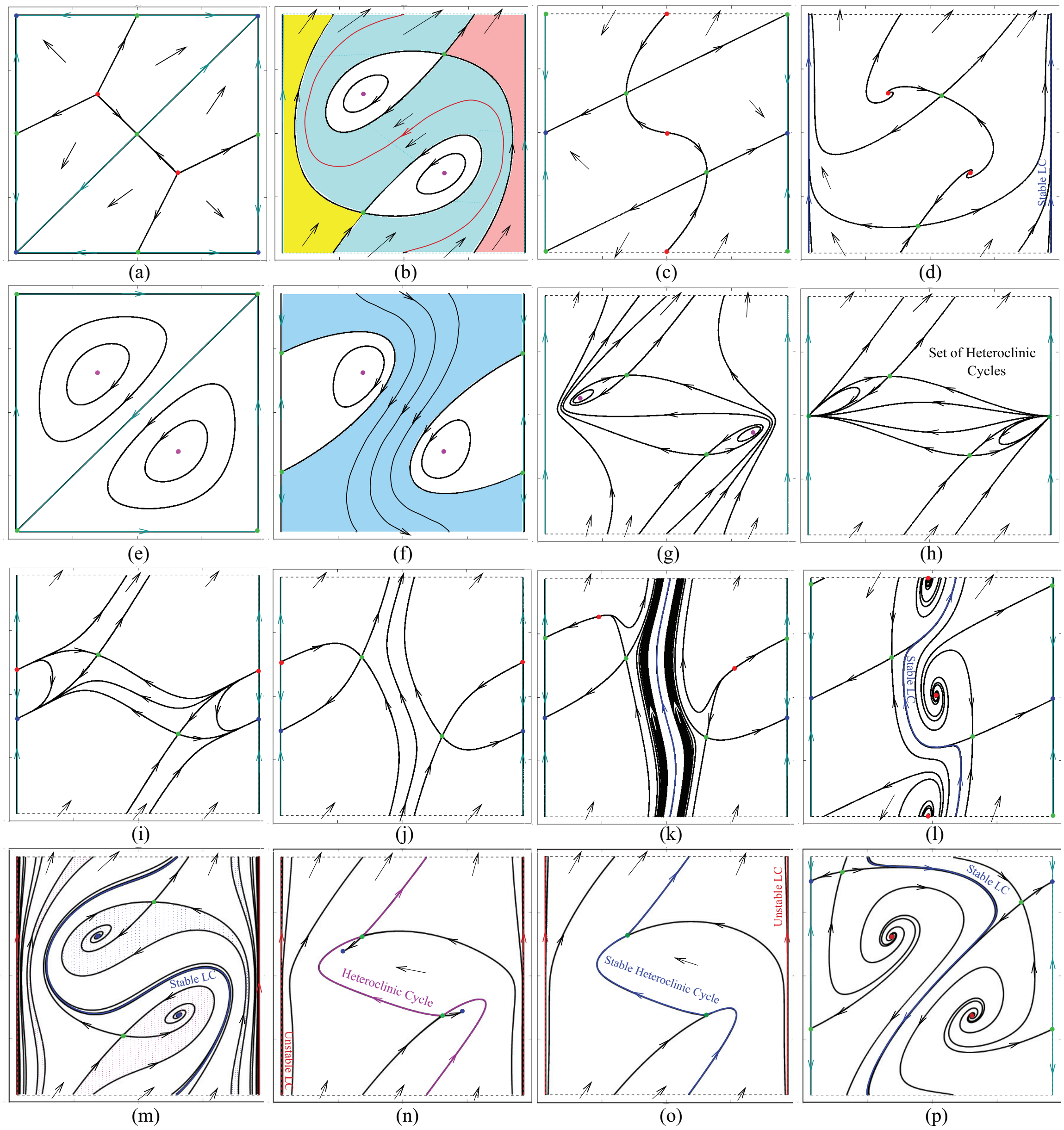

FIG. 6. (Color) Phase portraits for $\phi_1,\phi_2 \in [0,2\pi)$ for two conformists and one contrarian. The cases a, b,..., p of the bifurcation diagram of Fig. 5 (top) are represented for different values of the parameters α and k. Point types: red, source; blue, sink; green, saddle; magenta, center; dark green, saddle node. Stable limit cycles are shown by blue; unstable limit cycles by red. Two-dimensional color regions indicate the sets of nonisolated periodic orbits.

Different types of bifurcations occur along the bifurcation lines $\alpha = \pm\pi/2$ under the variation of the parameter k [Figs. 6(b) and 6(e)–6(j)]. For example, one can see a transition from homoclinic cycles of Fig. 6(b) to heteroclinic cycles of Fig. 6(f) at the bifurcation point $(\alpha,k) = (\pi/2,1)$ [Fig. 6(e)]. Saddle points move to the invariant region, meet at the origin $(\phi_1,\phi_2) = (0,0)$, creating a degenerate saddle (with three saddle regions), and then two new saddles move along the invariant line $\mathcal{P}_1^1$, creating phase-locked and phase-unlocked heteroclinic cycles.

Besides neutral heteroclinic cycles for $\alpha = \pm\pi/2$, the system has stable or unstable heteroclinic cycles [see Fig. 6(o) for an example] on other HC bifurcation lines [Fig. 5 (top)]. The transversal stability of each part of a heteroclinic cycle does not guarantee the stability of this cycle. Figure 6(n) shows a heteroclinic cycle that consists of two one-dimensional invariant manifolds of saddles. Each of these invariant manifolds is stable in the transversal direction. The heteroclinic cycle is unstable only at the saddle points and only in the direction of the single unstable one-dimensional invariant manifold W^u, which leads to the closest sink. This is enough for the whole heteroclinic cycle to be unstable.

Phase-unlocked limit cycles [Figs. 6(b), 6(d), 6(f), 6(g), 6(j)–6(m), and 6(p)] are typical for the system with a phase shift. This regime can correspond to the traveling wave state

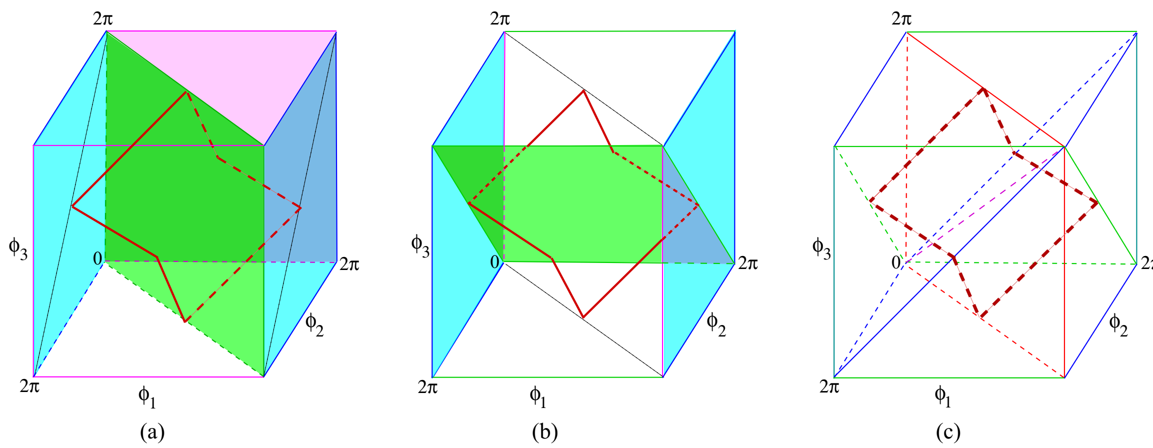

FIG. 7. (Color online) Schematic representations of the invariant sets $\mathcal{P}_2^i$ and $\mathcal{M}$ in the torus $\mathbb{T}^3$: (a) $N_1 = 3$, (b) $N_1 = 2$, (c) $N_1 = 1$. Different invariant planes are shown in different colors. Only the edges of three invariant planes are shown in panel (c). The invariant manifold $\mathcal{M}$ (connected straight segments) is shown in dark red in all panels.

(synchronization of conformists, $r_1 = 1$, $r_2 = r_2(t) \in [0,1]$ [Fig. 6(d)]). It can be more general when two order parameters for the groups of conformists and contrarians are functions of time $r_1(t)$, $r_2(t)$ and $\delta = \delta(t)$. This situation corresponds to the existence of a stable phase-unlocked limit cycle with a nonzero phase difference ϕ_1 [Figs. 6(j)–6(m) and 6(p)].

There can be many codimension 2 bifurcation points SN ∩ TC, AH ∩ TC, PF ∩ HC, etc. Many bifurcation lines can meet at one point, as happens at the points $(0, -1/2)$, $(\pi, -1/2)$, $(\pm\pi/2, -1/2)$, $(\pm\pi/2, 1)$, and $(\pm\pi/3, -1/2)$ of the bifurcation plane.

The phase shift $\alpha = \pi$ changes the system's dynamics by swapping contrarians and conformists. The phase shifts $\alpha = \pm\pi/2$ make all interaction functions even. As a result, nontrivial neutral structures can appear (sets of nonisolated periodic orbits, two-dimensional sets of homo- or heteroclinic cycles, etc.). Multistability of different types of attractors can occur. A stable unlocked limit cycle can coexist with fixed points on the invariant manifold $\mathcal{P}_1^1$ or with the stable antiphase points $\mathcal{M}_1, \mathcal{M}_2$ [Figs. 6(k), 6(l), 6(m), 6(p)]. This is illustrated by Fig. 6(m) where the basins of attraction of the points $\mathcal{M}_1$, $\mathcal{M}_2$ are filled by blue and magenta dots (color online), and the basin of attraction of the unlocked limit cycle is colored in white. Multistability regions can be found using the bifurcation diagram of Fig. 5 (top).

One conformist and two contrarians. The properties of the system in the case $N_1 = 1$, $N_2 = 2$ can be obtained from the previous results for $N_1 = 2$, $N_2 = 1$ using action (12). In particular, the symmetry action maps the invariant manifold $\phi_1 = 0$ of system (5) for $N_1 = 2$ into the invariant manifold $\phi_1 = \phi_2$ of the system for $N_2 = 2$. The manifold $\mathcal{M}$ is invariant under the same mapping. Any bifurcation curve on the bifurcation diagram of Fig. 5 (bottom) can be obtained from the bifurcation curve of the same type in Fig. 5 (top) using the map $(\alpha,k) \mapsto (\alpha,1/k)$. The bifurcations SN and PF now relate to the invariant manifold $\phi_1 = \phi_2$. The bifurcations AH and TC occur at the same points $\mathcal{M}_1, \mathcal{M}_2$. The bifurcation line $k = 0$ corresponds to the bifurcation line at infinity which we cannot see on the bifurcation diagram for $N_1 = 2$. Crossing this line means the transformation of contrarians into conformists and vice versa. If $k = 0$ the whole phase space is filled by straight line trajectories.

D. Four oscillators

This section is devoted to the system with four oscillators. There are three nontrivial cases when the number of conformists is $N_1 = 1,2,3$ (two degenerate cases correspond to the systems with conformists or contrarians only). Therefore, we consider three different three-dimensional systems in phase differences with two parameters k and α.

Invariant manifolds. Each of the systems has invariant manifolds $\mathcal{P}_2^1$ and/or $\mathcal{P}_2^2$ that correspond to clusters of two conformists or two contrarians. These manifolds are represented by the corresponding planes in $\mathbb{T}^3$ (Fig. 7). If $N_1 = 3$ or $N_1 = 1$, three invariant planes split the phase space into two triangular invariant regions [Figs. 7(a) and 7(c)]. If $N_1 = 2$, the system has only two invariant planes [Fig. 7(b)].

For $N = 4$ all systems have the same invariant manifold $\mathcal{M}$ that consists of one-parametric lines $\mathcal{M}(\phi) = (\phi,\pi,\phi + \pi)$ (up to variables permutation) in $\mathbb{T}^3$. As mentioned above, each $\mathcal{M}$ consists of fixed points that are neutral to each other (one eigenvalue for each point is zero) and can have different types of stability in two directions transversal to the manifold. The transversal eigenvalues for the lines of $\mathcal{M}$ are different in each case and depend on the variable ϕ: $\lambda_{1,2}(\phi) = (\tau \pm \sqrt{\tau^2 - 4\xi \sin^2 \phi})/2$, where $\tau = (N_1 + kN_2)\cos\alpha$, $\xi = (\xi_1 + \xi_2)(\xi_3 + \xi_4)$, and ξ_j $(j = 1,\ldots,4)$ is 1 or k such that $\sum_{j=1}^4 \xi_j = N_1 + kN_2$. Therefore, we can describe the transversal stability of $\mathcal{M}$ and find bifurcations of its points.

Three conformists and one contrarian. Let $N_1 = 3$, $N_2 = 1$. In this case three invariant manifolds $\mathcal{P}_2^1$ are described (in terms of phase differences) by the equalities $\phi_1 = 0$, $\phi_2 = 0$, $\phi_1 = \phi_2$. These manifolds split the phase space $\mathbb{T}^3$ into two invariant regions (Fig. 7). Phase differences between the conformists are bounded, while the phase difference between a contrarian and a conformist can increase to infinity in $\mathbb{R}^3$. The system has the Z_3 symmetry group of rotations around the central lines $\phi_1 = 2\pi/3$, $\phi_2 = 4\pi/3$ and $\phi_1 = 4\pi/3$, $\phi_2 = 2\pi/3$ of each invariant region.

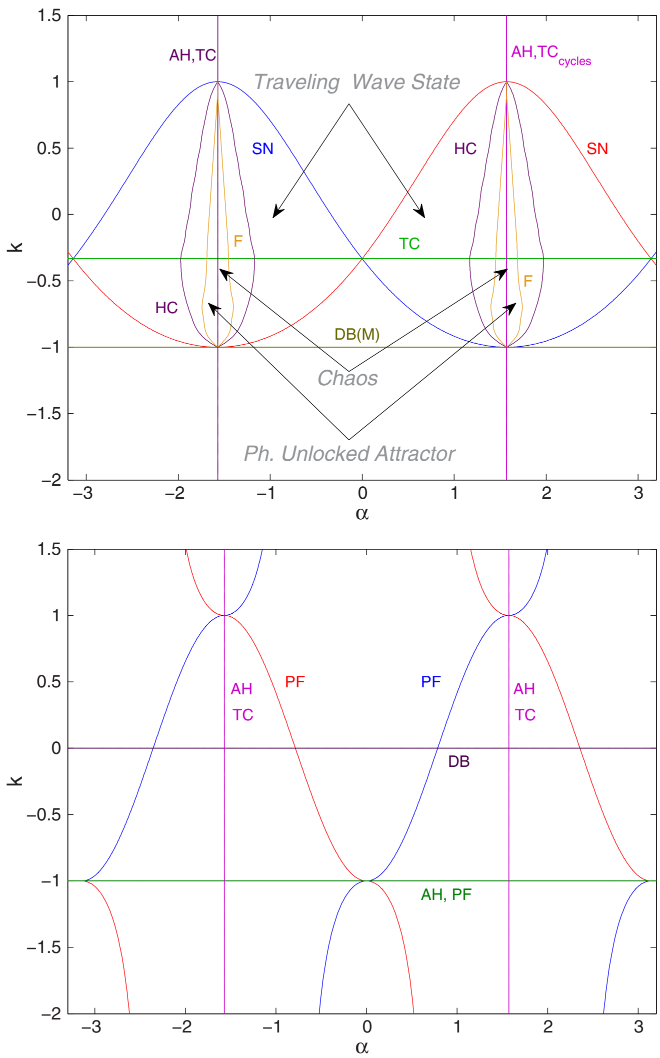

FIG. 8. (Color online) Bifurcation diagrams on the (α,k) bifurcation plane for four oscillators: (top) $N_1 = 3, N_2 = 1$; (bottom) $N_1 = N_2 = 2$. Notation: AH, Andronov-Hopf; SN, saddle node; PF, pitchfork; TC, transcritical; F, fold of cycles; HC, homoclinic; DB, degenerate bifurcations.

The bifurcation (α,k) diagram for the system is presented in Fig. 8 (top). From the theorem we know that in addition to the points of the manifold $\mathcal{M}$ the system has eight fixed points with coordinates 0 and π. Disappearance of the sink $(0,0,0)$ and saddle $(0,0,\pi)$ after a saddle-node (SN) bifurcation leads to the passability of the invariant line $\mathcal{P}_1^1$ $\phi_1 = \phi_2 = 0$. This line is a stable phase-unlocked limit cycle when system parameters are inside the quasitriangles that are formed by the SN bifurcation lines and transcritical bifurcation lines TC_{cycles}. The HC line corresponds to a homoclinic (saddle-connection) bifurcation of the saddle on each of three invariant planes $\mathcal{P}_2^i$. This bifurcation leads to the appearance of three symmetric saddle limit cycles inside each invariant plane. These saddles are stable inside the invariant manifolds and unstable in the directions transversal to the manifolds.

A symmetric transcritical bifurcation (TC_{cycles}) of the stable limit cycle $\mathcal{P}_1^1$ and three saddle limit cycles happens when $\alpha = \pm\pi/2$. This bifurcation is similar to the transcritical bifurcation

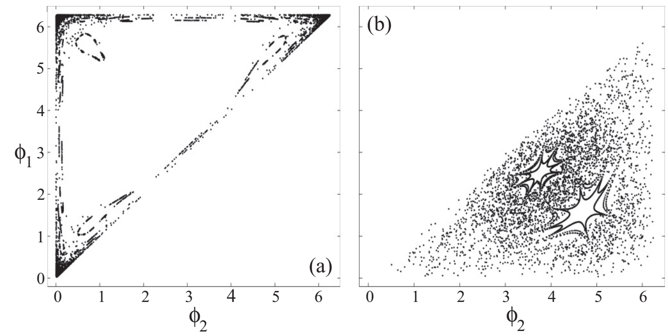

FIG. 9. Poincaré sections $\phi_3 = \pi$ of system (5) for $N_1 = 3$, $N_2 = 1$ when (a) $k = -0.1$, $\alpha = -1.58$ and (b) $k = 2$, $\alpha = \pi/2$.

of saddle points and sinks (see [25], Fig. 2). At the point of bifurcation the line $\mathcal{P}_1^1$ is a degenerate saddle limit cycle with six saddle regions. Globally this cycle creates a homoclinic cycle (heteroclinic if the phase space is $\mathbb{R}^3$ instead of $\mathbb{T}^3$); i.e., stable and unstable two-dimensional invariant manifolds are the invariant planes: $W_2^u(\mathcal{P}_1^1) = W_2^s(\mathcal{P}_1^1) = \mathcal{P}_2^1$.

A chaotic trajectory inside invariant regions appears at the point of bifurcation $\alpha = \pi/2$. This trajectory is phase-unlocked along the variable ϕ_3. It goes in the same direction as the degenerate saddle limit cycle $\mathcal{P}_1^1$, fidgeting near the invariant planes $\mathcal{P}_2^1$, trying to approach the saddle cycle and then running away from it. The chaotic attractor exists in a narrow region of the parameter $\alpha > \pi/2$ [a Poincaré section of a chaotic trajectory is shown in Fig. 9(a)], then it turns into a quasiperiodic attractor and later into a stable phase-unlocked limit cycle inside the invariant region (according to [27]). This stable limit cycle disappears together with a saddle cycle in a fold bifurcation (line F in Fig. 8).

All three phase-unlocked attractors (periodic, quasiperiodic, and chaotic) have a nonconstant order parameter for the conformists, $r_1 = r_1(t)$. This fact makes these regimes impossible in the Hong-Strogatz model, since for this model $r_1 = 1$. The system with a phase shift can show chaotic behavior even for two groups of conformists. A Poincaré section of such chaotic trajectory is shown in Fig. 9(b) ($k = 2$, $\alpha = \pi/2$). (Recall that if $\alpha = 0$ only full synchronization is possible for two groups of conformists, $K_1 > 0$, $K_2 > 0$.)

Any point on the manifold $\mathcal{M}$ represents the blurred π state with the order parameters $r_1 = 1/3$, $r_2 = 1$. These points have the same type of transversal stability with the exception of the points with coordinates 0 and π (which have one additional neutral stability for any values of the parameters). A degenerate Andronov-Hopf (AH) bifurcation occurs for each point of the manifold when $\alpha = \pm\pi/2$. However, a limit cycle does not appear after this bifurcation due to the odd symmetry of the system relative to the parameter $\alpha = \pm\pi/2$. Additionally, a degenerate bifurcation [DG($\mathcal{M}$)] happens for the points of the manifold when $k = -1$. This bifurcation changes the point's stability in one transversal direction. A transcritical (TC) bifurcation occurs in the invariant manifolds $\mathcal{P}_2^1$ at the points that correspond to π states of the system when $k = -1/3$.

Two conformists and two contrarians. In this case two invariant planes $\mathcal{P}_2^1$ and $\mathcal{P}_2^2$ are not symmetric to each other and they do not split the phase space into invariant regions [Fig. 7(b)]. Since the system has the same number of conformists and contrarians, there exist both bifurcation lines with coordinates (α,k) and $(\alpha,1/k)$ according to the symmetry γ_3 [Fig. 8 (bottom)]. Contrary to the previous case, not all lines of the invariant manifold $\mathcal{M}$ are mutually symmetric. One parametric set $\mathcal{M}$ represents two different regimes: incoherent states with coordinates $(\pi,\phi,\pi+\phi)$ and blurred π states with coordinates $(\phi,\pi,\pi+\phi)$. The order parameters in the blurred π state depend on the coordinate, $r_1 = r_2 = \sqrt{2(1+\cos\phi)}/2$. Transversal stability is the same for each point on the two invariant lines, except points with coordinates 0 and π, which represent the $\pi S_{\mathcal{M}}$ regime and have additional neutral stability (Lemma 2).

A degenerate AH bifurcation of blurred π states of the manifold $\mathcal{M}$ (without appearance of a limit cycle) occurs for $\alpha = \pm\pi/2$ (k is arbitrary). The whole phase space (with the exception of the manifold $\mathcal{M}$) is filled by a nonisolated set of limit cycles. Phase-locked and phase-unlocked limit cycles coexist on this bifurcation line for some values of k. Another AH bifurcation of BπS points occurs on the line $k = -1$. The BπS set is transversally stable for $k < -1$, $|\alpha| < \pi/2$ and for $k > -1$, $\pi/2 < \alpha < 3\pi/2$. The points of the incoherent state line are degenerate saddles (of positive, negative, and neutral stability) for all parameter values. A pitchfork (PF) bifurcation occurs for a point that belongs to the invariant plane $\mathcal{P}_2^2$ in the direction transversal to this plane. A stable fixed point gives birth to two new stable points. Fixed points disappear in a TC bifurcation on the lines $\alpha = \pm\pi/2$ (which coincide with AH lines) and change their stability in a degenerate bifurcation on the line $k = 0$. For $\alpha \neq \pm\pi/2$ a stable fixed point of the system exists in the bifurcation regions where the invariant manifold $\mathcal{M}$ is unstable in transversal directions. Homoclinic (HC) bifurcations also occur on the invariant planes $\mathcal{P}_2^1$ and $\mathcal{P}_2^2$ and they lead to the appearance of saddle limit cycles.

Thus, the bifurcations of the system's dynamics are absolutely different in the two presented cases, as confirmed by the bifurcation diagrams shown in Fig. 8.

One conformist and three contrarians. In this case three invariant planes split the phase space $\mathbb{T}^3$ into two triangular invariant regions [Fig. 7(c)]. System (5) has the Z_3 symmetry group of rotations around one of the regional central lines $L_1 = \{\phi_1 - \phi_2 = 2\pi/3, \phi_1 - \phi_3 = 4\pi/3\}$, $L_2 = \{\phi_1 - \phi_2 = 4\pi/3, \phi_1 - \phi_3 = 2\pi/3\}$ by the angle $2\pi/3$ with the simultaneous shift along this line by $2\pi/3$ (rotation around the torus). As we mentioned above, the bifurcation diagram (α,k) for the case $N_1 = 1$ can be obtained from the bifurcation diagram for $N_1 = 3$ by replacing k by $1/k$. In addition, the bifurcation diagram has the degenerate bifurcation line $k = 0$.

E. Regimes that are impossible in the Hong-Strogatz model

Summarizing the results presented above, we can say that for $\alpha \neq 0$ new regimes appear which are impossible in the original Hong-Strogatz model. The first difference is the possibility of mutual *rotation of two synchronous clusters*, conformists ($r_1 = 1$) and contrarians ($r_2 = 1$), with a free angle between the clusters $\delta = \delta(t)$. This rotation corresponds to a trajectory that moves along the one-dimensional line which is an intersection of two hyperplanes

$$\mathcal{P}_{N_2}^1 \cap \mathcal{P}_{N_1}^2 = \{(\phi_1,\ldots,\phi_{N-1}) : \phi_1 = \cdots = \phi_{N_1-1} = 0, \phi_{N_1} = \cdots = \phi_{N-1}\}.$$

According to the theorem, the origin and the fixed point $(0,\ldots,0,\pi,\ldots,\pi)$ (with $N_1 - 1$ coordinates 0 and N_2 coordinates π) belong to the invariant line $\mathcal{P}_{N_2}^1 \cap \mathcal{P}_{N_1}^2$ when $\alpha = 0$. This makes the mutual rotation state impossible in the Hong-Strogatz model. These two points move along the line when the parameter α is varied and disappear in a saddle-node bifurcation. As a result, clusters of conformists and contrarians that move towards each other become possible for $\alpha \neq 0$ and $\alpha \neq \pi$. The simplest case of such a regime is seen for $N = 4$, $N_1 = N_2 = 2$.

Another difference is the regime of *phase-unlocked attractors*. This regime assumes oscillations of not only contrarians but also conformists. If $\alpha = 0$, attractive coupling between the conformists forces them to stay together and periodic trajectories are stable only when they belong to the manifolds $\mathcal{P}_m^1$. In the general case the system can have two clusters of conformists and contrarians with complex dynamics inside each cluster and without synchronization between the clusters. The oscillators are phase locked inside each cluster, but there is no phase locking between the clusters. In addition to full synchronization inside the group of conformists with $r_1 = 1$, periodic, quasiperiodic, and chaotic regimes can also occur [with $r_1 = r_1(t)$] when $\alpha \neq 0$ and $\alpha \neq \pi$.

Stable *homoclinic and heteroclinic cycles* are also impossible for $\alpha = 0$. One can check that the heteroclinic cycle that consists of two saddle points (0,0) and (π,π) and their one-dimensional invariant manifolds exists in the two-dimensional manifold $\mathcal{P}_2^1$ (two contrarians) for an arbitrary number of conformists N_1 and a small enough negative k, but this cycle is unstable. Homo- or heteroclinic cycles also can exist in manifolds of higher dimensions [system (9)] but they are already unstable inside the manifolds. Introduction of a phase shift leads to the transformation of unstable homo- or heteroclinic cycles into stable ones. It also leads to the appearance of new stable homoclinic and heteroclinic cycles inside invariant manifolds (see, for example, Fig. 6).

Multidimensional sets of neutral limit or heteroclinic cycles are possible only for $\alpha = \pm\pi/2$. Different types of such two-dimensional structures are shown in Fig. 6. In the general case (an arbitrary number of oscillators) phase-unlocked limit cycles fill the whole invariant region (the structure of dimension $N - 1$) and coexist with heteroclinic or homoclinic cycles.

VI. DISCUSSION

In this paper we considered a system of identical phase oscillators with positive and negative coupling parameters and a phase shift in the coupling functions. Our interest in this system was originally stimulated by the paper of Hong and Strogatz [18], where the authors obtained amazing results about system dynamics using the Watanabe-Strogatz theory [7] and the Ott-Antonsen ansatz [19]. We used a different approach based on bifurcation analysis, which is an efficient way to study the dynamics of finite-dimensional (including low-dimensional) systems.

For the Hong-Strogatz model we proved the theorem that gave a full description of equilibria localization. Based on this theorem we obtained conditions of stability of fixed points and invariant manifolds, confirmed that the regimes discovered in [16] exist in the finite-dimensional case, and found a new stable regime that is absent in the Hong and Strogatz classification. For a more general system with phase shifts in connections we found two types of invariant manifolds that exist for all values of the coupling parameters and arbitrary phase shifts. Complete bifurcation analysis of the simplest cases of two, three, and four oscillators allowed us to find new dynamical regimes that are impossible in the Hong-Strogatz model. Our analysis shows that a small perturbation of the phase shift parameter can radically change system dynamics.

A. General remarks on the Hong-Strogatz model

The results obtained here for the original Hong-Strogatz model, rewritten in terms of phase differences (5), can be summarized in the following statements.

(1) The system has two types of stable regimes: fixed points with constant order parameters and traveling waves which are phase-unlocked nonequilibrium regimes with constant order parameter $r_1 = 1$ and nonconstant order parameters $r_2(t)$ and $R(t)$.

(2) Equilibria of the system can be subdivided into two subsets: $\mathrm{FP}_{0\pi}$, which is a set of isolated points with coordinates $0, \pi$, and the $(N-3)$-dimensional manifold $\mathcal{M}$ of nonisolated fixed points.

(3) The points in the subsets (πS, S_1, $\pi\mathrm{S}_1$) of the set $\mathrm{FP}_{0\pi}$ are stable in all directions, but the points in the subsets (IS, BπS) of the manifold $\mathcal{M}$ are stable in only two directions transversal to the manifold.

(4) Two regimes (S_1 and $\pi\mathrm{S}_1$) when a single contrarian is synchronized or phase locked by the group of conformists are only possible for finite-dimensional systems.

(5) The stability of fixed points on the invariant manifold $\mathcal{M}$ varies for a fixed value of the parameter k. The value of k at which the stability of a point on the manifold changes is different from point to point.

(6) The system has three types of traveling wave states: periodic, quasiperiodic, and chaotic.

(7) Multistability of different regimes is possible.

B. General remarks on the model with a phase shift

Influence of the phase shift for $N = 3$. As we have shown, bifurcation transitions in the Hong-Strogatz model ($\alpha = 0$) are rather simple. The system has only one bifurcation point for $N_1 = 2$ and two bifurcation points for $N_1 = 1$. However, the system has nontrivial bifurcation structure if both k and α are varied (Fig. 5). One can see radical dynamical alterations around the bifurcation point $\alpha = 0$ in both cases $N_1 = 1$ and $N_1 = 2$ because this point is a codimension 2 bifurcation point on the (α,k) plane and it is the intersection of three codimension 1 bifurcation lines. Therefore a small phase shift can completely change the dynamics of the system, adding new regimes. In the case of two conformists, system (2) cannot have the traveling wave state with the order parameter $r_1 = 1$ (rotation around the invariant line $\phi_1 = 0$) because according to the theorem the fixed points $(\phi_1,\phi_2) = (0,0)$ and $(\phi_1,\phi_2) = (\pi,0)$ exist for any value of the parameter k. However, the traveling wave regime is possible in a wide parametric region (see Fig. 5) between two saddle-node bifurcation lines (a saddle-node bifurcation kills two fixed points mentioned above). The intersection of two homoclinic bifurcation lines at the point $(\alpha,k) = (0,0)$ of the bifurcation plane for two contrarians also implies the appearance of a phase-unlocked limit cycle with a nontrivial structure.

Influence of the phase shift for $N = 4$. The study of the system with four oscillators also shows the influence of the phase shift on system dynamics as in the case $N = 3$. The bifurcation diagrams in Fig. 8 show the majority of different bifurcation curves. Only some of these bifurcation curves intersect when $\alpha = 0$. The system is sensitive to the phase shift around these intersections (particularly when $k = K_2/K_1$, $\alpha \approx 0$). One can see that phase shift variation implies the appearance of new regimes in the system and deviation of the dynamics of the system far away from the line $\alpha = 0$. The most interesting dynamics occur when $\alpha = \pm\pi/2$, i.e., when coupling functions are even. A few different bifurcations simultaneously happen when $\alpha = \pm\pi/2$ in different places of $\mathbb{T}^3$, which leads to global restructuring of the whole phase space. The most nontrivial dynamics of the system (chaos, complex heteroclinic structures) occur in this case. Complex dynamics also happen when we consider two groups of interacting conformists for $\alpha \neq 0$, in contrast to the case $\alpha = 0$ when full synchronization happens for all values of $k > 0$.

General properties of the model with a phase shift. In studying some low-dimensional systems with two parameters k and α we found many common properties which are also fulfilled for *the systems with an arbitrary number of oscillators*. The symmetries of the system and coupling functions (Sec. II C) also imply the existence of different invariant structures for the general case. In addition, we have information about the dynamics of the system in some special cases such as $\alpha = 0$, $\alpha = \pi$, $k = 1$ and an arbitrary value of α (the Kuramoto-Sakaguchi system). Based on this knowledge, we can make some conclusions about stability and bifurcation properties in the general case.

(1) The system has AH bifurcation lines $\alpha = \pm\pi/2$ at each point of the invariant manifold $\mathcal{M}$ for all values of N_1 and N_2.

(2) The system has two symmetric saddle-node bifurcation lines that intersect at the point $(\alpha,k) = (0,-N_2/N_1)$ (a codimension 2 bifurcation). This intersection corresponds to the appearance of a stable π state in the Hong-Strogatz model for smaller values of k.

(3) The system has a transcritical (or pitchfork for $N_1 = N_2$) bifurcation line $k = -N_2/N_1$. This line also intersects the codimension 2 point mentioned above.

(4) The system has a complex structure in the limit cases $\alpha = \pm\pi/2$. Continuous sets of phase-locked and phase-unlocked limit cycles, homo- or heteroclinic structures of saddle points or saddle limit cycles as well as chaotic trajectories are possible.

(5) Introduction of a phase shift radically changes the dynamics of the system. New stable regimes (phase-unlocked trajectories, heteroclinic cycles, etc.) appear when $\alpha \neq 0$. The system is sensitive to small changes of the shift parameter.

(6) Nontrivial dynamics (quasiperiodicity, chaos) for two different groups of conformists or two groups of contrarians are only possible for a nonzero phase shift.

Summarizing the results for the model with two types of interactions and a phase shift, we can say that this model demonstrates complex dynamical regimes which are impossible in the Hong-Strogatz model without phase shifts, and that these regimes can be found in the systems with an arbitrary finite number of oscillators.

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APPENDIX

Lemma 1. For any parameter $k \neq 0, k \neq -N_2/N_1$ the point $(\theta_1,\ldots,\theta_N)$ satisfies system (8) if and only if one of the following two conditions is fulfilled:

(1) $\sum_{j=1}^{N} e^{\imath\theta_j} = 0$,

(2) $\theta_i - \theta_j \in \{0,\pi\}, i,j = 1,\ldots,N$.

Proof. It is obvious that Eqs. (8) are valid if any of conditions (1) or (2) is fulfilled. Let us show that if Eqs. (8) are satisfied this implies that either (1) or (2) is true. We can rewrite (8) in the following way:

$$(\sin\theta_1 - \sin\theta_i)\sum_{j=1}^{N}\cos\theta_j - (\cos\theta_1 - \cos\theta_i) \times \sum_{j=1}^{N}\sin\theta_j = 0, \quad i = 2,\ldots,N_1, \tag{A1}$$

$$(\sin\theta_1 - k\sin\theta_i)\sum_{j=1}^{N}\cos\theta_j - (\cos\theta_1 - k\cos\theta_i) \times \sum_{j=1}^{N}\sin\theta_j = 0, \quad i = N_1+1,\ldots,N. \tag{A2}$$

Denote $X := \sum_{j=1}^{N}\cos\theta_j$, $Y := \sum_{j=1}^{N}\sin\theta_j$. If a solution of Eqs. (A1) and (A2) satisfies conditions $X = 0$, $Y = 0$, then condition (1) of the lemma is satisfied. Suppose that there is a solution of (A1) and (A2) that does not satisfy conditions $X = 0$, $Y = 0$. We prove that in this case condition (2) must be fulfilled.

Due to S_{N_1} and S_{N_2} symmetries it is enough to prove (2) for $\phi_1 = \theta_1 - \theta_2$ and for $\phi_{N_1-1} = \theta_1 - \theta_{N_1}$. Let us prove (2) for ϕ_1. The proof for ϕ_{N_1-1} can be done in a similar way.

If Eqs. (A1) and (A2) have a nonzero solution relative to (X,Y), the following expression must be fulfilled:

$$\det\begin{pmatrix}\sin\theta_1 - \sin\theta_2 & \cos\theta_1 - \cos\theta_2\\ \sin\theta_1 - \sin\theta_j & \cos\theta_1 - \cos\theta_j\end{pmatrix} = (\sin\theta_1 - \sin\theta_2)(\cos\theta_1 - \cos\theta_j) - (\sin\theta_1 - \sin\theta_j)(\cos\theta_1 - \cos\theta_2) = 0,$$
$$j = 1,\ldots,N_1,$$
$$\det\begin{pmatrix}\sin\theta_1 - \sin\theta_2 & \cos\theta_1 - \cos\theta_2\\ \sin\theta_1 - k\sin\theta_j & \cos\theta_1 - k\cos\theta_j\end{pmatrix} = (\sin\theta_1 - \sin\theta_2)(\cos\theta_1 - k\cos\theta_j) - (\sin\theta_1 - k\sin\theta_j)(\cos\theta_1 - \cos\theta_2) = 0,$$
$$j = N_1+1,\ldots,N.$$

These expressions can be transformed to

$$\sin(\theta_1 - \theta_2) + \sin(\theta_j - \theta_1) + \sin(\theta_2 - \theta_j) = 0, \quad j = 1,\ldots,N_1,$$
$$\sin(\theta_1 - \theta_2) + k\sin(\theta_j - \theta_1) + k\sin(\theta_2 - \theta_j) = 0, \quad j = N_1+1,\ldots,N.$$

From here we derive

$$\begin{aligned}\sin(\theta_1 - \theta_2) &= \sin(\theta_1 - \theta_j) - \sin(\theta_2 - \theta_j) = 0, \quad j = 1,\ldots,N_1,\\ \frac{1}{k}\sin(\theta_1 - \theta_2) &= \sin(\theta_1 - \theta_j) - \sin(\theta_2 - \theta_j) = 0, \quad j = N_1+1,\ldots,N.\end{aligned} \tag{A3}$$

From (8) for $i = 2$ we have

$$\sum_{j=1}^{N}\sin(\theta_1 - \theta_j) - \sum_{j=1}^{N}\sin(\theta_2 - \theta_j) = 0. \tag{A4}$$

Substituting (A3) into (A4) we obtain

$$N_1\sin(\theta_1 - \theta_2) + \frac{N_2}{k}\sin(\theta_1 - \theta_2) = \left(N_1 + \frac{N_2}{k}\right)\sin(\theta_1 - \theta_2) = 0.$$

Thus, $\theta_1 - \theta_2 \in \{0,\pi\}$ since $k \neq -N_2/N_1$ according to the lemma suggestions.

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