Erosion by a one-dimensional random walk

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We consider a model introduced by Baker *et al.* [Phys. Rev. E **88**, 042113 (2013)] of a single lattice random walker moving on a domain of allowed sites, surrounded by blocked sites. The walker enlarges the allowed domain by eroding the boundary at its random encounters with blocked boundary sites: attempts to step onto blocked sites succeed with a given probability and convert these sites to allowed sites. The model interpolates continuously between the Pólya random walker on the one-dimensional lattice and a "blind" walker who attempts freely, but always aborts, moves to blocked sites. We obtain some exact results about the walker's location and the rate of erosion.

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I. INTRODUCTION

The problem of a single random walker moving on an unbounded periodic lattice, or on a fragment of such a lattice with relatively simple fixed boundaries, has been extensively studied [1–7] and used in a variety of ways to model physical, biological, and social phenomena [1,3,4]. There has been much less study of walkers that interact with their boundaries in ways that may change the boundary location, with the principal exceptions being two models that consider the sequential release of walkers: diffusion-limited aggregation (DLA) [8,9] and internal DLA [10,11], which are models of cluster growth. In both DLA and internal DLA, each walker interacts with the boundary once only (causing the boundary to move) and then dies before the next walker is released into the system.

We consider here a single random walker that is permitted to interact with the boundary repeatedly, eroding the boundary to enlarge the domain within which the random walk takes place. This model was introduced by Baker *et al.* [12] as a simple model of a motile biological cell that remodels extracellular matrix as it moves through tissue. Baker *et al.* gave for one-dimensional cases some exact results for times associated with boundary growth events and some mean-field approximations and also reported on simulations in higher dimensions. We take up the model of Baker *et al.* [12] and address for its one-dimensional version the important questions of the current location of the boundary of the allowed interval, the position of the walker within the allowed interval, and the probability that at any given instant the walker is currently at the boundary of the allowed interval. Our results, based on generating functions, recover as special cases a number of elegant known asymptotic properties of random walks.

Following Baker *et al.*, we consider a one-dimensional random walk of Pólya type (unbiased, nearest-neighbor stepping) on an interval of "allowed" sites, bounded by an endless succession of "blocked" sites. If the walker attempts to step from an allowed site to an adjacent blocked site, then with probability ϖ (the snipping or erosion probability), the step is permitted to take place and the blocked site changes its status to become allowed, while with probability $1-\varpi$ the stepping attempt is aborted, the walker does not move on this time step and no sites change status. (The symbol ϖ, a variant of π, serves as a mnemonic to remind us that this quantity is a probability and avoids recycling symbols that are already overused in the random-walk literature.) If $\varpi = 1$, the model reduces to the familiar one-dimensional Pólya random walk without boundaries [13], whereas if $\varpi = 0$, we have a "blind" random walker in a bounded interval [14], who attempts to step onto blocked sites but always aborts such attempts.

The lattice sites are assigned integer coordinates in the standard way. We are interested in the integer-valued variables X_n, L_n, and R_n, defined as follows: X_n is the location of the walker after n steps, L_n is the leftmost allowed site, and R_n is the rightmost allowed site, so that $L_n \leqslant X_n \leqslant R_n$ for all $n \geqslant 0$. Averages of these and other random variables will be indicated by angle brackets. Random variables are capitalized.

In Fig. 1 we illustrate the model and our notational conventions for an attempted erosion event when a walker currently at the rightmost allowed site attempts to make a step to the right.

In each realization of the walk, the random sequence $\{(X_n, L_n, R_n)\}$ is, of course, implicitly dependent on the value of ϖ, though we only exhibit this explicitly in notation by writing such things as $R_n(\varpi)$ when it is useful to do so. The number of sites in the allowed interval is $1 - L_n(\varpi) + R_n(\varpi)$. It may be noted that if we commence with the initial condition

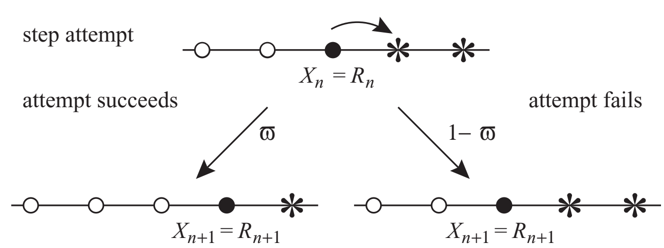

FIG. 1. Illustrating erosion or snipping for a walker currently at the rightmost allowed site ($X_n = R_n$) who attempts to step to the right for the $(n+1)$th step. The walker's location is shown as a black disk. White disks denote allowed sites and asterisks denote blocked sites. With probability ϖ the attempt succeeds (the blocked site becomes allowed and the walker steps onto it) and so $X_{n+1} = R_{n+1} = X_n + 1 = R_n + 1$. With probability $1-\varpi$ the attempt fails and the walker does not move and so $X_{n+1} = R_{n+1} = X_n = R_n$.

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$L_0 = X_0 = R_0 = 0$, then the random variable

$$S_n = 1 - L_n(1) + R_n(1) \tag{1}$$

is the number of distinct sites visited (sometimes called the "range" of the walk) for a walk of Pólya type on the set $\mathbb{Z}$ of integers. Some known properties of S_n [4] are useful for checking and interpreting results that we produce. For later reference we note the classical results [4] that

$$\langle X_n(1)\rangle = 0, \quad \langle X_n(1)^2\rangle = n, \tag{2}$$

$$\langle S_n\rangle = 2\langle R_n(1)\rangle + 1 = \sqrt{\frac{8n}{\pi}} + o(1) \quad \text{as} \quad n \to \infty. \tag{3}$$

The last result is valid with any initial prescription of $L_0 \leqslant 0$, provided that $X_0 = R_0 = 0$. There has been a little work on the set of sites visited by a number of simultaneous, independent random walkers (starting with the "moles' labyrinth" of Herrmann [15]), but in the present paper we never consider more than one walker being present.

In Sec. II we consider a one-sided erosion problem: we take as our initial condition $L_0 = -\infty$ and $X_0 = R_0 = 0$. In this case we are considering erosion of a half-line of blocked sites, with the walk commencing at the rightmost allowed site. There is no loss of generality in assuming that $X_0 = R_0 = 0$. If we were to start with $X_0 < R_0$, then since a Pólya random walk on $\mathbb{Z}$ is certain to visit all sites eventually [4,13], the problem reduces to the composition of two consecutive random walks: a walk from the starting site to R_0, then a walk (with a shifted time origin) starting at the right boundary. We are able to determine exactly the generating function over the step number n of the joint probability distribution of X_n and R_n, from which several attractive results follow.

In Sec. III we start from a single allowed site in a boundless sea of blocked sites: $L_0 = X_0 = R_0 = 0$. Here the generating function analysis is much harder than in Sec. II, perhaps unexpectedly so, but we are able to use generating function techniques to compute the asymptotic form of $\langle R_n(\varpi)\rangle$ as $n \to \infty$.

To extract the asymptotic behavior of some quantities of interest for walks of long duration, we have recourse to a Tauberian Theorem [4,16] and to a Theorem of Darboux [4,17]. Let $\Gamma(\rho)$ denote the usual gamma function. The Tauberian Theorem states that if $\sum_{n=0}^{\infty} a_n\xi^n$ converges for $0 \leqslant \xi < 1$, $a_n > 0$, a_n is monotonic and $0 < \rho < \infty$, then the following are equivalent as $\xi \to 1^-$ or $n \to \infty$, respectively:

$$\sum_{n=0}^{\infty} a_n\xi^n \sim (1-\xi)^{-\rho} \quad \text{and} \quad a_n \sim \frac{n^{\rho-1}}{\Gamma(\rho)} \quad \text{as} \quad n \to \infty.$$

Darboux's Theorem is a stronger result that bypasses the need to establish positivity and monotonicity but requires more analytic information. The special case of Darboux's Theorem that we use is the following. Let the functions $a(\xi) = \sum_{n=0}^{\infty} a_n\xi^n$ and $b(\xi) = \sum_{n=0}^{\infty} b_n\xi^n$ both be holomorphic in the unit circle except for isolated singularities at $\xi = -1$ and $\xi = 1$ and bounded in the neighborhood of $\xi = -1$, but not in the neighborhood of $\xi = 1$, with $a(\xi) - b(\xi) = O[(1-\xi)^{\sigma-1}]$ for some $\sigma > 0$. Then

$$a_n - b_n = o(1) \quad \text{as} \quad n \to \infty.$$

In several places we evaluate infinite series using the geometric series identity $\sum_{n=0}^{\infty} Z^n = (1-Z)^{-1}$ ($|Z| < 1$) and Z derivatives of this identity. The details are straightforward and are not discussed. A number of analytical results are checked against simulations performed using MATLAB. We have departed from some of the notational conventions used by Baker *et al.* [12]: they used R_n for the walker position, L_n for the length of the allowed interval, and p_s for the erosion probability.

II. ONE-SIDED EROSION OF A SEMI-INFINITE LINEAR CHAIN

Here we commence with the initial condition $L_0 = -\infty$ (sites to the left of the walker are all allowed), so it is unnecessary for us to retain L_n in our analysis. We introduce the joint distribution of the current walker location X_n and the rightmost allowed site R_n, taking the initial condition $X_0 = R_0 = 0$ (that is, starting on the rightmost allowed site, which we take as the origin of coordinates). Where x is an integer and r is a nonnegative integer, we write

$$P_n(x,r) = \Pr\{X_n = x, R_n = r\} \quad \text{for} \quad x \leqslant r, \tag{4}$$

so that the initial condition is simply expressed in terms of the usual Kronecker δ symbol:

$$P_0(x,r) = \delta_{x,0}\delta_{r,0}. \tag{5}$$

If the walker is not currently at the rightmost allowed site, then it has necessarily arrived at its current location by a step from an adjacent allowed site (that is, it has just executed a Pólya random walk step). Hence,

$$P_{n+1}(x,r) = \tfrac{1}{2}P_n(x-1,r) + \tfrac{1}{2}P_n(x+1,r) \quad (x < r). \tag{6}$$

If the walker is currently at the rightmost allowed site it could have arrived there in one of only three ways: (i) it stepped there from an allowed site on its left; (ii) it was already at the rightmost allowed site and made an unsuccessful attempt to move right; (iii) it was already at the rightmost allowed site and made a successful attempt to step onto a blocked site, bringing it to its current location and shifting the boundary. The third type of move cannot occur if the rightmost occupied site still retains its initial value 0. Hence,

$$P_{n+1}(r,r) = \tfrac{1}{2}P_n(r-1,r) + \tfrac{1}{2}(1-\varpi)P_n(r,r) + \tfrac{1}{2}\varpi P_n(r-1,r-1)(1-\delta_{r,0}). \tag{7}$$

A. Lattice Green function for one-sided erosion

Where $|\xi| < 1$, we introduce the generating function for the site occupation probability (the lattice Green function for one-sided erosion),

$$\mathcal{P}(x,r;\xi) = \sum_{n=0}^{\infty} P_n(x,r)\xi^n. \tag{8}$$

Generating functions for the expectation of powers of X_n or R_n will also be useful: where κ and λ are nonnegative integers,

$$\sum_{n=0}^{\infty}\langle X_n^{\kappa}R_n^{\lambda}\rangle\xi^n = \sum_{r=0}^{\infty}\sum_{x=-\infty}^{r}\sum_{n=0}^{\infty}x^{\kappa}r^{\lambda}P_n(x,r)\xi^n = \sum_{r=0}^{\infty}\sum_{x=-\infty}^{r}x^{\kappa}r^{\lambda}\mathcal{P}(x,r;\xi). \tag{9}$$

We multiply Eqs. (6) and (7) by ξ^n and sum over all possible values of n, apply Eq. (8) and the initial condition Eq. (5), obtaining the following difference equations for the lattice Green function:

$$\mathcal{P}(x,r;\xi) = \frac{\xi}{2}\mathcal{P}(x-1,r;\xi) + \frac{\xi}{2}\mathcal{P}(x+1,r;\xi), \tag{10}$$

for $x < r$, and

$$\mathcal{P}(r,r;\xi) = \delta_{r,0} + \frac{\xi}{2}\mathcal{P}(r-1,r;\xi) + \frac{\xi}{2}(1-\varpi)\mathcal{P}(r,r;\xi) + \frac{\xi}{2}\varpi\mathcal{P}(r-1,r-1;\xi)(1-\delta_{r,0}). \tag{11}$$

First we consider Eq. (10): this is a second-order difference equation in one variable x. Two independent solutions can be found in the standard way using a trial solution $\mathcal{P}(x,r;\xi) = z^x$, giving the quadratic equation

$$z^2 - 2\xi^{-1}z + 1 = 0, \tag{12}$$

for which the two solutions are

$$\frac{1 \pm \sqrt{1-\xi^2}}{\xi} = \left(\frac{1-\sqrt{1-\xi^2}}{\xi}\right)^{\mp 1}.$$

Therefore, we write

$$\mathcal{P}(x,r;\xi) = a(r;\xi)z(\xi)^{x-r} + b(r;\xi)z(\xi)^{r-x} \quad \text{for} \quad x < r,$$

where

$$z(\xi) = \frac{1}{\xi}(1-\sqrt{1-\xi^2}), \tag{13}$$

and $0 \leqslant z(\xi) \leqslant 1$ for $0 \leqslant \xi \leqslant 1$. For frequent later use we note the identity

$$[1-z(\xi)]^2 = \frac{2}{\xi}(1-\xi)z(\xi). \tag{14}$$

To keep our solution finite for all $x \leqslant r$, exponential growth of the solution as $x \to -\infty$ must be ruled out, so $a(r;\xi)$ must be zero. Hence,

$$\mathcal{P}(x,r;\xi) = b(r;\xi)z(\xi)^{r-x}, \quad x < r. \tag{15}$$

Notice that we can apply Eq. (10) as far as $x = r-1$. We do this and use Eq. (15) to obtain $\mathcal{P}(r,r;\xi) = b(r;\xi)$, so that

$$\mathcal{P}(x,r;\xi) = \mathcal{P}(r,r;\xi)z(\xi)^{r-x}, \quad x \leqslant r. \tag{16}$$

Next we substitute Eq. (16) into Eq. (11) to obtain a first-order difference equation for the unknown quantity $\mathcal{P}(r,r;\xi)$. This difference equation can be tidied up using the quadratic equation (12) satisfied by $z(\xi)$, and we find that

$$\mathcal{P}(r,r;\xi) = \frac{2z(\xi)}{\xi[1-z(\xi)(1-\varpi)]} \times \left[\delta_{r,0} + \frac{\xi\varpi}{2}(1-\delta_{r,0})\mathcal{P}(r-1,r-1;\xi)\right],$$

and it follows at once that

$$\mathcal{P}(r,r;\xi) = \frac{2}{\xi}\left[\frac{z(\xi)}{1-z(\xi)(1-\varpi)}\right]^{r+1}\varpi^r, \quad r \geqslant 0. \tag{17}$$

Consequently, we find that for $x \leqslant r$,

$$\mathcal{P}(x,r;\xi) = \frac{2}{\xi\varpi}\left[\frac{z(\xi)\varpi}{1-z(\xi)(1-\varpi)}\right]^{r+1} z(\xi)^{r-x}. \tag{18}$$

It is straightforward to verify by substitution that this is indeed a solution to Eqs. (10) and (11) and satisfies the normalization requirement

$$\sum_{r=0}^{\infty}\sum_{x=-\infty}^{r}\mathcal{P}(x,r;\xi) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{x=-\infty}^{r}P_n(x,r)\xi^n = \sum_{n=0}^{\infty}\xi^n = \frac{1}{1-\xi}. \tag{19}$$

We have not been able to extract a simple formula for the general term in the series expansion of $\mathcal{P}(x,r;\xi)$ in powers of ξ, so that closed-form expressions for $P_n(x,r)$ for general values of x, r, and n are not available. However, the expansion can be computed as far as one's patience extends using MATHEMATICA or other symbolic algebra packages. For example,

$$\mathcal{P}(x,r;\xi) = \varpi^r\left(\frac{\xi}{2}\right)^{2r-x}\left[1 + (1+r)(1-\varpi)\frac{\xi}{2} + O(\xi^2)\right]. \tag{20}$$

However, as we now show, a number of less ambitious but interesting questions can be answered using Eq. (18).

B. Is the walker at the wall?

We seek the probability E_n that after n steps the walker's current location is the rightmost allowed site. For the ordinary Pólya walk on $\mathbb{Z}$, covered by taking $\varpi = 1$, this is the probability that the walker is now as far right as it has ever been. Since

$$E_n = \sum_{r=0}^{\infty}P_n(r,r), \tag{21}$$

we see that

$$\sum_{n=0}^{\infty}E_n\xi^n = \sum_{r=0}^{\infty}\mathcal{P}(r,r;\xi). \tag{22}$$

Using Eq. (17) and summing the resulting geometric series we find that

$$\sum_{n=0}^{\infty}E_n\xi^n = \frac{2z(\xi)}{\xi[1-z(\xi)]}. \tag{23}$$

The absence of the erosion probability ϖ from the right-hand side may come as a surprise, until we note that if the walker is currently at the rightmost allowed site, then the result of an attempt to step right is to place the walker at the rightmost allowed site, whether this is a new site (this is the case with probability ϖ), or the same site (this is the case with probability $1-\varpi$).

Using Eq. (14) we find that

$$\sum_{n=0}^{\infty} E_n \xi^n = \frac{2z(\xi)[1-z(\xi)]}{\xi[1-z(\xi)]^2} = \frac{1}{\xi}\left[\frac{1+\xi}{\sqrt{1-\xi^2}} - 1\right]. \tag{24}$$

We do not need the Tauberian Theorem or Darboux's Theorem here to extract the asymptotic behavior of E_n, as we can extract two explicit formulas for E_n for even or odd n. If we recall the binomial expansion

$$(1-z)^{-\alpha} = \sum_{k=0}^{\infty} \frac{(\alpha)_k z^k}{k!}, \tag{25}$$

where

$$(\alpha)_k = \frac{\Gamma(k+\alpha)}{\Gamma(\alpha)} = \begin{cases} 1, & k=0, \\ \alpha(\alpha+1)\cdots(\alpha+k-1), & k \in \mathbb{N}, \end{cases} \tag{26}$$

we find that

$$E_{2m} = \frac{(1/2)_m}{m!}, \quad E_{2m+1} = \frac{(1/2)_{m+1}}{(m+1)!}. \tag{27}$$

In view of the standard results that

$$k! = \Gamma(k+1) \quad \text{and} \quad \Gamma(k+\alpha)/\Gamma(k+\beta) \sim k^{\alpha-\beta} \text{ as } k \to \infty,$$

we find that for both odd and even n,

$$E_n \sim \frac{2^{1/2}}{(\pi n)^{1/2}} \quad \text{as} \quad n \to \infty. \tag{28}$$

C. Where is the wall?

The mean location of the rightmost visited site, $\langle R_n \rangle$, can be computed by noting from Eq. (9) that

$$\sum_{n=0}^{\infty} \langle R_n \rangle \xi^n = \sum_{r=0}^{\infty} \sum_{x=-\infty}^{r} r \mathcal{P}(x,r;\xi). \tag{29}$$

When we insert the solution (18) the sums over x and r can be evaluated using geometric series identities. Some routine algebra, including making use of Eq. (14), leads us to

$$\sum_{n=0}^{\infty} \langle R_n \rangle \xi^n = \frac{2\varpi z(\xi)^2}{\xi[1-z(\xi)]^3} = \frac{\varpi}{2}\left[\frac{\sqrt{1-\xi^2}}{(1-\xi)^2} - \frac{1}{1-\xi}\right]. \tag{30}$$

If we exhibit the value of ϖ explicitly by writing the rightmost visited location as $R_n(\varpi)$, we observe that

$$\sum_{n=0}^{\infty} \langle R_n(\varpi) \rangle \xi^n = \varpi \sum_{n=0}^{\infty} \langle R_n(1) \rangle \xi^n, \tag{31}$$

and so

$$\langle R_n(\varpi) \rangle \equiv \varpi \langle R_n(1) \rangle. \tag{32}$$

As a check on the analysis, we note that from the exact relation (3) between $\langle R_n(1) \rangle$ and $\langle S_n \rangle$ (the mean number of distinct sites visited in a Pólya walk on $\mathbb{Z}$), it follows that

$$\sum_{n=0}^{\infty} \langle S_n \rangle \xi^n = 2\sum_{n=0}^{\infty} \langle R_n(1) \rangle \xi^n + \frac{1}{1-\xi},$$

and we can use this together with the known result [4] that

$$\sum_{n=0}^{\infty} \langle S_n \rangle \xi^n = \frac{\sqrt{1-\xi^2}}{(1-\xi)^2} \tag{33}$$

to verify that the generating function (30) is correct for $\varpi = 1$. Expanding the generating function in powers of $1-\xi$,

$$\sum_{n=0}^{\infty} \langle R_n(\varpi) \rangle \xi^n = \varpi \left[\frac{1}{\sqrt{2}(1-\xi)^{3/2}} - \frac{1}{2(1-\xi)} + O\left(\frac{1}{(1-\xi^2)^{1/2}}\right)\right],$$

so using Darboux's Theorem[1] and Eq. (25), we find that

$$\langle R_n(\varpi) \rangle = \varpi \left[\sqrt{\frac{2n}{\pi}} - \frac{1}{2} + o(1)\right] \quad \text{as} \quad n \to \infty. \tag{34}$$

For $\varpi = 1$, this agrees with a general result of Comtet and Majumdar [18] on the average value of the maximum of a (not necessarily lattice-based) one-dimensional random walk. Also, since $\sqrt{1-\xi^2}/(1-\xi)^2 = (1+\xi)^2(1-\xi^2)^{-3/2}$, we can use the binomial expansion (25) to recover the closed form expressions of Henze [19],

$$\langle S_{2n} \rangle = \frac{(4n+1)}{2^{2n}}\binom{2n}{n}, \quad \langle S_{2n+1} \rangle = \frac{(4n+2)}{2^{2n}}\binom{2n}{n}, \tag{35}$$

and corresponding results of Katzenbeisser and Panny [20] for the rightmost site visited in the ordinary Pólya walk, yielding for one-sided erosion

$$\langle R_{2n}(\varpi) \rangle = \frac{\varpi}{2}\left[\frac{(4n+1)}{2^{2n}}\binom{2n}{n} - 1\right], \tag{36}$$

$$\langle R_{2n+1}(\varpi) \rangle = \frac{\varpi}{2}\left[\frac{(4n+2)}{2^{2n}}\binom{2n}{n} - 1\right]. \tag{37}$$

These elegant exact representations suppress the simplicity of the asymptotic behavior for long walks.

To quantify the fluctuations in the location of the rightmost visited site, we compute its variance,

$$\mathrm{Var}\{R_n(\varpi)\} = \langle R_n(\varpi)^2 \rangle - \langle R_n(\varpi) \rangle^2. \tag{38}$$

[1] Within the unit circle in the complex ξ plane the generating function is bounded except as $\xi \to 1$, so that if we develop its asymptotic expansion about this point, we can determine the asymptotic behavior of $\langle R_n(\varpi) \rangle$ obtaining all terms that do not decay as $n \to \infty$, using the special version of Darboux's Theorem stated in Sec. I.

Similar algebra to that used for analyzing $\langle R_n \rangle$ establishes that

$$\sum_{n=0}^{\infty} \langle R_n(\varpi)^2 \rangle \xi^n = \sum_{r=0}^{\infty} \sum_{x=-\infty}^{r} r^2 \mathcal{P}(x,r;\xi)$$
$$= \frac{2\varpi z(\xi)^2}{\xi[1-z(\xi)]^4}[1+(2\varpi-1)z(\xi)]$$
$$= \frac{\varpi\xi}{2(1-\xi)^2}\left[1+\frac{(2\varpi-1)}{\xi}(1-\sqrt{1-\xi^2})\right],$$

and using Eq. (30), we have

$$\sum_{n=0}^{\infty} \langle R_n(\varpi)^2 \rangle \xi^n = \frac{\varpi^2\xi}{(1-\xi)^2} + (1-2\varpi)\sum_{n=0}^{\infty}\langle R_n(\varpi)\rangle \xi^n.$$

Hence, we have the exact result that

$$\langle R_n(\varpi)^2 \rangle = \varpi^2 n + (1-2\varpi)\langle R_n(\varpi)\rangle. \quad (39)$$

Closed form expressions for $\langle R_n(\varpi)^2 \rangle$ for even and odd n can be deduced from Eqs. (36)–(39), but we refrain from writing them out here. Using Eqs (34), (38), and (39), we find that as $n \to \infty$,

$$\mathrm{Var}\{R_n(\varpi)\} = \varpi^2\left(1-\frac{2}{\pi}\right)n + \varpi(1-\varpi)\sqrt{\frac{2n}{\pi}} + O(1). \quad (40)$$

The asymptotic expansion (40) can easily be compared numerically against the exact solution for $\mathrm{Var}\{R_n(\varpi)\}$ or simulations. We show in Fig. 2 only the comparison between the asymptotic form and simulations. At the resolution of the figure, the exact and asymptotic solutions cannot be distinguished. The fluctuations in $R_n(\varpi)$ are characterized by the square root of the variance, and we find that they are of the same order as the mean.

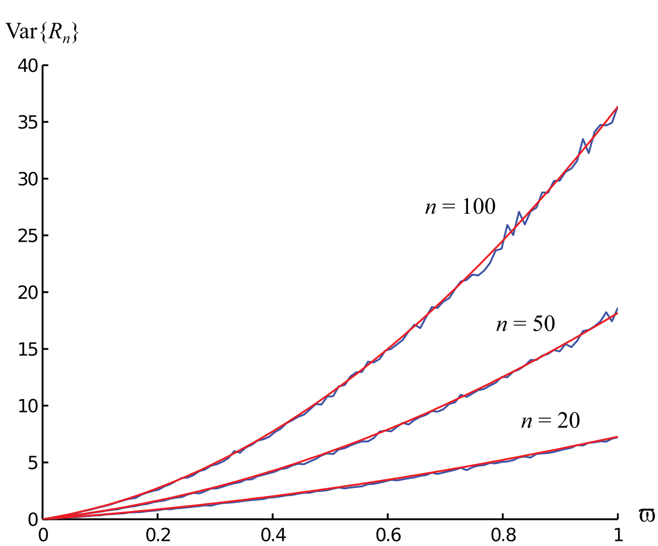

FIG. 2. (Color online) The asymptotic expansion (40) for $\mathrm{Var}\{R_n(\varpi)\}$ [red (smooth) curves] agrees very well with simulations (blue curves: 10 000 walk realizations for each value of n and ϖ; ϖ increments 0.01). Fluctuations in the simulation-based estimates increase with n and ϖ.

If we sum over x in Eq. (18), we obtain the generating function for the probability distribution of the rightmost allowed site:

$$\sum_{n=0}^{\infty} \Pr\{R_n(\varpi) = r\}\xi^n$$
$$= \frac{2}{\xi\varpi[1-z(\xi)]}\left[\frac{z(\xi)\varpi}{1-z(\xi)(1-\varpi)}\right]^{r+1}. \quad (41)$$

We can recover $\Pr\{R_n(\varpi) = r\}$, if desired, for modest values of n by expanding the right-hand side in powers of ξ [cf. the discussion below Eq. (19)]. For the special case $\varpi = 1$, a theorem of Erdös and Kac [21] on limit theorems in probability implies that for fixed $\alpha > 0$,

$$\lim_{n\to\infty} \Pr\{R_n(1) < \alpha\sqrt{n}\} = \mathrm{erf}\left(\frac{\alpha}{\sqrt{2}}\right), \quad (42)$$

where "erf" is the usual error function [22].

D. Where is the walker?

The mean displacement $\langle X_n(\varpi)\rangle$ of the walker has the generating function

$$\sum_{n=0}^{\infty} \langle X_n(\varpi)\rangle \xi^n = \sum_{r=0}^{\infty}\sum_{x=-\infty}^{r} x\mathcal{P}(x,r;\xi). \quad (43)$$

Using Eq. (18) it can be shown that

$$\sum_{n=0}^{\infty} \langle X_n(\varpi)\rangle \xi^n = -\frac{2z(\xi)^2(1-\varpi)}{\xi[1-z(\xi)]^3} \quad (44)$$
$$= -\frac{(1-\varpi)}{2}\left[\frac{\sqrt{1-\xi^2}}{(1-\xi)^2} - \frac{1}{1-\xi}\right]. \quad (45)$$

We observe at once that $\langle X_n(\varpi)\rangle = (1-\varpi)\langle X_n(0)\rangle$ and setting $\varpi = 1$ we recover the elementary result that for a Pólya random walk on $\mathbb{Z}$ the mean displacement from the starting site is always zero. Moreover, if we compare Eqs. (30) and (44), we conclude that

$$\frac{\langle X_n(\varpi)\rangle}{1-\varpi} \equiv -\frac{\langle R_n(\varpi)\rangle}{\varpi}. \quad (46)$$

Hence, from Eq. (34), we have

$$\langle X_n(\varpi)\rangle = -(1-\varpi)\left[\sqrt{\frac{2n}{\pi}} - \frac{1}{2} + o(1)\right] \text{ as } n \to \infty. \quad (47)$$

It can be seen from Fig. 3 that the asymptotic expansion (47) is in excellent agreement with simulations, even when n is as small as 20. Drift to the left is enhanced when the erosion probability ϖ is small.

Proceeding in a similar manner, we find the generating function for the mean-square displacement $\langle X_n^2(\varpi)\rangle$ to be

$$\sum_{n=0}^{\infty} \langle X_n(\varpi)^2\rangle \xi^n = \sum_{r=0}^{\infty}\sum_{x=-\infty}^{r} x^2\mathcal{P}(x,r;\xi)$$
$$= \frac{2z(\xi)^2[1+\varpi+z(\xi)(1-3\varpi+2\varpi^2)]}{\xi[1-z(\xi)]^4}$$

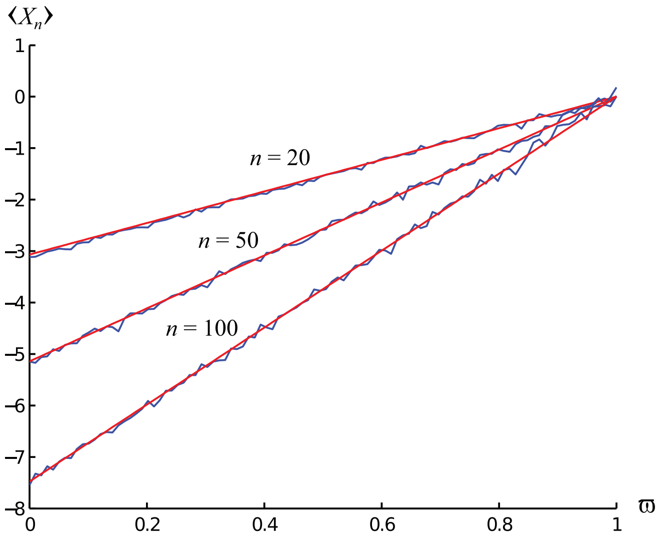

FIG. 3. (Color online) The mean displacement $\langle X_n(\varpi)\rangle$. The blue curves give the average of 10 000 realizations of the random walk (ϖ increments 0.01) and the red (smooth) curves show our asymptotic expansion (47).

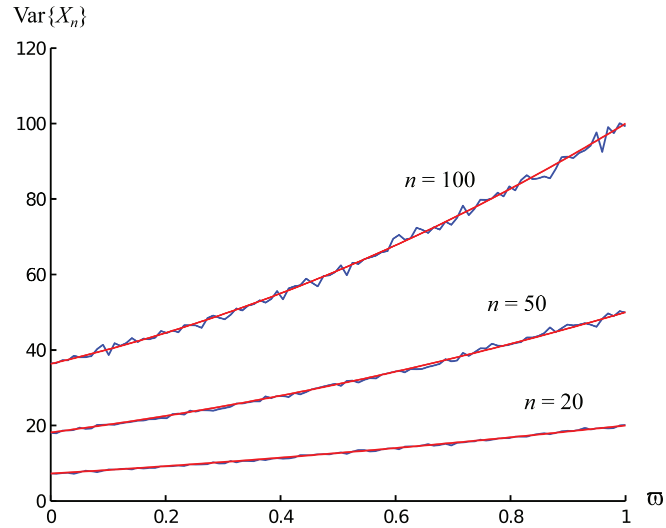

FIG. 4. (Color online) The asymptotic expansion (51) for $\mathrm{Var}\{X_n(\varpi)\}$ [red (smooth) curves] agrees very well with simulations (blue curves: 1000 walk realizations for each value of n and ϖ; ϖ increments 0.01). Fluctuations in the simulation-based estimates increase with n and ϖ.

$$= \frac{\xi}{2(1-\xi)^2}\left[1+\varpi+\frac{1}{\xi}(1-\sqrt{1-\xi^2})\right.$$
$$\left.\times\ (1-3\varpi+2\varpi^2)\right]. \quad (48)$$

If we rewrite the generating function in the equivalent form

$$\sum_{n=0}^{\infty}\langle X_n(\varpi)^2\rangle\xi^n$$
$$= \frac{(1-\varpi+\varpi^2)\xi}{(1-\xi)^2}$$
$$-\ \frac{(1-2\varpi)(1-\varpi)}{2}\left[\frac{\sqrt{(1-\xi)^2}}{(1-\xi)^2}-\frac{1}{1-\xi}\right], \quad (49)$$

the first term on the right is a multiple of the generating function for the sequence $\{n\}$, while the second is a multiple of the generating function Eq. (45) for $\langle X_n(\varpi)\rangle$. We can therefore deduce the exact relation

$$\langle X_n(\varpi)^2\rangle = (1-\varpi+\varpi^2)n + (1-2\varpi)\langle X_n(\varpi)\rangle. \quad (50)$$

We can now deduce the asymptotic behavior of the variance of the walker's position:

$$\mathrm{Var}\{X_n(\varpi)\} = \langle X_n(\varpi)^2\rangle - \langle X_n(\varpi)\rangle^2$$
$$= (1-\varpi+\varpi^2)n$$
$$-\ (1-2\varpi)(1-\varpi)\left[\frac{(2n)^{1/2}}{\pi^{1/2}}-\frac{1}{2}+o(1)\right]$$
$$-\left[\frac{(2n)^{1/2}}{\pi^{1/2}}-\frac{1}{2}+o(1)\right]^2$$
$$= \left[1-\varpi+\varpi^2-\frac{2}{\pi}(1-\varpi)^2\right]n$$
$$+\ \varpi(1-\varpi)\frac{(2n)^{1/2}}{\pi^{1/2}}+O(1). \quad (51)$$

The asymptotic formula for the variance in the walker position is compared against simulation in Fig. 4. We note that $\lim_{n\to\infty} n^{-1}\mathrm{Var}\{X_n(\varpi)\}$ is an increasing function of ϖ on the interval $0 \leqslant \varpi \leqslant 1$, and that

$$\lim_{n\to\infty}\frac{\mathrm{Var}\{X_n(0)\}}{n} = 1-\frac{2}{\pi}, \qquad \lim_{n\to\infty}\frac{\mathrm{Var}\{X_n(1)\}}{n} = 1.$$

The latter result is, of course, the standard result for a Pólya random walk on $\mathbb{Z}$. Unsuccessful walk erosion attempts reduce the fluctuations in position, as one would expect.

Using Eqs. (36), (37), (46), and (50) it is possible to determine $\mathrm{Var}\{X_n(\varpi)\}$ exactly in terms of binomial coefficients, but we refrain from writing out the resulting formulas here.

III. TWO-SIDED EROSION FROM A SINGLE SITE

Now we consider the two-sided erosion model, where the walker is initially placed on the only allowed site of the lattice (at $x = 0$) and on both sides is surrounded by blocked sites, so that the initial conditions on the leftmost allowed site, initial walker position, and rightmost allowed site are $L_0 = X_0 = R_0$ and thereafter $L_n \leqslant X_n \leqslant R_n$.

Baker *et al.* [12] studied the time (that is, the number of steps, including failed attempts to erode) that elapses until the allowed interval length takes a prescribed value s. If we denote this random time by $T_s(\varpi)$, then the exact results obtained by Baker *et al.* are

$$\langle T_s(\varpi)\rangle = \frac{s(s-1)}{2\varpi}, \quad (52)$$

$$\mathrm{Var}\{T_s(\varpi)\} = \frac{s(s-1)}{12\varpi^2}[\varpi s^2 + (4-5\varpi)s - 2]. \quad (53)$$

The results of Baker *et al.* concerning averages of X_n and S_n at given n values were limited to mean-field arguments and empirical formulas based on simulations. We shall determine exactly the leading-order large-n behavior of $\langle S_n(\varpi)\rangle$, the mean interval length at time n, by determining first the behavior of $\langle R_n(\varpi)\rangle$, the mean rightmost allowed site.

We write

$$P_n(x,l,r) = \Pr\{X_n(\varpi) = x, L_n(\varpi) = l, R_n(\varpi) = r\}, \quad (54)$$

so the initial condition is

$$P_0(x,l,r) = \delta_{x,0}\delta_{l,0}\delta_{r,0}. \quad (55)$$

It is easy to determine the probability that the walker always fails to enlarge the allowed interval and remains at the initial site: $P_{n+1}(0,0,0) = (1-\varpi)P_n(0,0,0)$, so

$$P_n(0,0,0) = (1-\varpi)^n. \quad (56)$$

If we have $L_n < R_n$, then unless the walker is at either end of the interval of allowed sites, it moves like a normal Pólya walker. Hence, for $l < x < r$, the site occupation probability evolution equation is

$$P_{n+1}(x,l,r) = \tfrac{1}{2}P_n(x-1,l,r) + \tfrac{1}{2}P_n(x+1,l,r). \quad (57)$$

At the boundaries (provided that $l < r$), the site occupation probability evolution equations are

$$\begin{aligned} P_{n+1}(x,l,r) &= \tfrac{1}{2}P_n(x+1,l,r) + \tfrac{1}{2}(1-\varpi)P_n(x,l,r) \\ &\quad + \tfrac{1}{2}\varpi P_n(x+1,l+1,r)(1-\delta_{l,0}), \\ x &= l, \end{aligned} \quad (58)$$

$$\begin{aligned} P_{n+1}(x,l,r) &= \tfrac{1}{2}P_n(x-1,l,r) + \tfrac{1}{2}(1-\varpi)P_n(x,l,r) \\ &\quad + \tfrac{1}{2}\varpi P_n(x-1,l,r-1)(1-\delta_{r,0}), \\ x &= r. \end{aligned} \quad (59)$$

Unlike the one-sided erosion model considered above, this problem contains an element of symmetry. In particular, we expect that

$$P_n(x,l,r) = P_n(-x,-r,-l). \quad (60)$$

In fact, if we let $r = -l$ and $l = -r$ in Eq. (58), then apply Eq. (60), we obtain Eq. (59), and vice versa.

A. Lattice Green function for two-sided erosion

We commence as in Sec. II, by introducing the generating function of the site occupation probability

$$P(x,l,r;\xi) = \sum_{n=0}^{\infty} P_n(x,l,r)\xi^n. \quad (61)$$

If we follow closely the approach of Sec. II, we arrive at a solution of Eq. (57) of the form

$$P(x,l,r;\xi) = a(l,r;\xi)z(\xi)^{x-l} + b(l,r;\xi)z(\xi)^{r-x}, \quad (62)$$

for $l \leqslant x \leqslant r$, where $z(\xi)$ is given by Eq. (13). Unlike the one-sided erosion problem, both of the coefficient functions $a(l,r;\xi)$ and $b(l,r;\xi)$ are nonzero, and the relations between them that follow from Eqs. (58) and (59) are difficult to analyze.

We adopt a different approach, escaping the awkward constraint that $l \leqslant x \leqslant r$ by extending the problem to define $P_n(x,l,r;\xi)$ for $-\infty < x < \infty$. This extension is only acceptable if we can ensure that $P_n(x,l,r)$ evolves in such a way that the requirement that $P_n(x,l,r) = 0$ for $x < l$ and for $x > r$ is preserved. The required evolution equation is

$$\begin{aligned} P_{n+1}(x,l,r) &= \frac{1}{2}P_n(x-1,l,r) + \frac{1}{2}P_n(x+1,l,r) - \frac{1}{2}\delta_{x,l-1}P_n(l,l,r) - \frac{1}{2}\delta_{x,r+1}P_n(r,l,r) \\ &\quad + \frac{\delta_{x,l}}{2}\Big[(1-\varpi)P_n(l,l,r) + \varpi P_n(l+1,l+1,r)(1-\delta_{l,0}) - P_n(l-1,l,r)\Big] \\ &\quad + \frac{\delta_{x,r}}{2}\Big[(1-\varpi)P_n(r,l,r) + \varpi P_n(r-1,l,r-1)(1-\delta_{r,0}) - P_n(r+1,l,r)\Big]. \end{aligned} \quad (63)$$

The six Kronecker δ symbols are inserted to ensure that the evolution equation is valid for $l \leqslant 0$, $r \geqslant 0$ and $-\infty < x < \infty$. The terms in braces with prefactors $\delta_{x,l}$ and $\delta_{x,r}$, respectively, deal with the case when the walker is at the leftmost allowed site or the rightmost allowed site. The two terms containing $\delta_{x,l-1}$ and $\delta_{x,r+1}$ ensure that there is no "leakage" of probability into $x < l$ or $x > r$ (and so also ensure that the last term of each of the expressions enclosed in braces is zero). The approach we have taken is a novel extension of the simpler defect technique used in lattice dynamics [23] and simple random walk problems [1,4], where a small number of sites in an otherwise translationally invariant system have different parameters.

If we define the discrete Fourier transform of the generating function $P(x,l,r;\xi)$ with respect to x by

$$\tilde{P}(\theta,l,r;\xi) = \sum_{x=-\infty}^{\infty} P(x,l,r;\xi)e^{i\theta x} \quad (64)$$

$$= \sum_{n=0}^{\infty}\sum_{x=-\infty}^{\infty} P_n(x,l,r)e^{i\theta x}\xi^n, \quad (65)$$

then we find from Eq. (63) that

$$\tilde{P}(\theta,l,r;\xi) = \frac{\delta_{l,0}\delta_{r,0}}{1-\xi\cos\theta} + \frac{\xi}{2}\Bigg[-\frac{e^{i(l-1)\theta}}{1-\xi\cos\theta}P(l,l,r;\xi) - \frac{e^{i(r+1)\theta}}{1-\xi\cos\theta}P(r,l,r;\xi) + \frac{(1-\varpi)e^{il\theta}}{1-\xi\cos\theta}P(l,l,r;\xi) + \frac{\varpi e^{il\theta}}{1-\xi\cos\theta}P(l+1,l+1,r;\xi)(1-\delta_{l,0}) + \frac{(1-\varpi)e^{ir\theta}}{1-\xi\cos\theta}P(r,l,r;\xi) + \frac{\varpi e^{ir\theta}}{1-\xi\cos\theta}P(r-1,l,r-1;\xi)(1-\delta_{r,0})\Bigg]. \quad (66)$$

We now recall [4] that for $x \in \mathbb{Z}$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-ix\theta}d\theta}{1-\xi\cos\theta} = \frac{z(\xi)^{|x|}}{\sqrt{1-\xi^2}}, \quad (67)$$

where $z(\xi)$ is given by Eq. (13). This is simply the lattice Green function for the Pólya walk on $\mathbb{Z}$, the evaluation of the integral being a simple exercise in contour integration.[2] Hence, if we apply

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ix\theta}\{\cdots\}d\theta$$

to invert the transform with the specific choices $x = l$ and $x = r$, respectively, we obtain

$$P(l,l,r;\xi) = \frac{\delta_{l,0}\delta_{r,0}}{\sqrt{1-\xi^2}} + \frac{\xi}{2}\Bigg[-\frac{z(\xi)}{\sqrt{1-\xi^2}}P(l,l,r;\xi) - \frac{z(\xi)^{r-l+1}}{\sqrt{1-\xi^2}}P(r,l,r;\xi) + \frac{(1-\varpi)}{\sqrt{1-\xi^2}}P(l,l,r;\xi) + \frac{\varpi}{\sqrt{1-\xi^2}}P(l+1,l+1,r;\xi)(1-\delta_{l,0}) + \frac{(1-\varpi)z(\xi)^{r-l}}{\sqrt{1-\xi^2}}P(r,l,r;\xi) + \frac{\varpi z(\xi)^{r-l}}{\sqrt{1-\xi^2}}P(r-1,l,r-1;\xi)(1-\delta_{r,0})\Bigg], \quad (68)$$

$$P(r,l,r;\xi) = \frac{\delta_{l,0}\delta_{r,0}}{\sqrt{1-\xi^2}} + \frac{\xi}{2}\Bigg[-\frac{z(\xi)^{r-l+1}}{\sqrt{1-\xi^2}}P(l,l,r;\xi) - \frac{z(\xi)}{\sqrt{1-\xi^2}}P(r,l,r;\xi) + \frac{(1-\varpi)z^{r-l}}{\sqrt{1-\xi^2}}P(l,l,r;\xi) + \frac{\varpi z(\xi)^{r-l}}{\sqrt{1-\xi^2}}P(l+1,l+1,r;\xi)(1-\delta_{l,0}) + \frac{(1-\varpi)}{\sqrt{1-\xi^2}}P(r,l,r;\xi) + \frac{\varpi}{\sqrt{1-\xi^2}}P(r-1,l,r-1;\xi)(1-\delta_{r,0})\Bigg]. \quad (69)$$

We now write $\ell = -l$ so that $\ell \geqslant 0$, and

$$A(\ell,r;\xi) = P(-\ell,-\ell,r;\xi), \quad B(\ell,r;\xi) = P(r,-\ell,r;\xi), \quad (70)$$

and note the reflection symmetry property

$$A(\ell,r;\xi) = P(-\ell,-\ell,r;\xi) = P(\ell,-r,\ell;\xi) = B(r,\ell;\xi). \quad (71)$$

If we introduce the convenient functions

$$\alpha(\xi) = z(\xi)^{-1} - (1-\varpi), \quad \beta(\xi) = z(\xi) - (1-\varpi),$$
$$\gamma(\xi) = \beta(\xi)/\alpha(\xi),$$

two of which are used immediately, while $\gamma(\xi)$ is used later, we find after a modest amount of algebra that

$$\alpha(\xi)A(\ell,r;\xi) - \varpi A(\ell-1,r;\xi)(1-\delta_{\ell,0}) = \frac{2}{\xi}\delta_{\ell,0}\delta_{r,0} - z(\xi)^{r+\ell}[\beta(\xi)B(\ell,r;\xi) - \varpi B(\ell,r-1;\xi) \times (1-\delta_{r,0})]. \quad (72)$$

Similarly,

$$\alpha(\xi)B(\ell,r;\xi) - \varpi B(\ell,r-1;\xi)(1-\delta_{r,0}) = \frac{2}{\xi}\delta_{\ell,0}\delta_{r,0} - z(\xi)^{r+\ell}[\beta(\xi)A(\ell,r;\xi) - \varpi A(\ell-1,r;\xi) \times (1-\delta_{\ell,0})]. \quad (73)$$

It can now be seen from the symmetry property Eq. (71) that Eqs. (72) and (73) are equivalent. In order to determine $\tilde{P}(\theta,l,r;\xi)$ we need to construct the solution for one of $A(\ell,r;\xi)$ or $B(\ell,r;\xi)$ from these equations and this appears to be a very difficult problem. If we only seek the generating function for the mean location of the rightmost allowed site, we obtain something more tractable.

[2] See Ref. [4], pp. 140–141. The integrand is an even function of x, so that we can replace $-x$ by $|x|$ and make the standard substitution $Z = e^{i\theta}$ to produce a contour integral around the unit circle, enclosing one simple pole of the integrand.

B. A functional equation and its solution

The reflection symmetry has an important consequence, since it enables us to evaluate exactly a generating function needed to infer the asymptotic growth of $\langle R_n(\varpi)\rangle$ as $n \to \infty$. If we define

$$\varphi(\kappa;\xi) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \kappa^{p+q} A(p,q;\xi) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \kappa^{p+q} B(p,q;\xi), \tag{74}$$

we see that on multiplying Eq. (73) by $\kappa^{\ell+r}$ and summing over ℓ and r, we have

$$[\alpha(\xi) - \varpi\kappa]\varphi(\kappa;\xi) = \frac{2}{\xi} - [\beta(\xi) - \varpi\kappa z(\xi)]\varphi(\kappa z(\xi);\xi). \tag{75}$$

If we write $[\alpha(\xi) - \varpi\kappa]\varphi(\kappa;\xi) = \frac{2}{\xi}\Phi(\kappa;\xi)$ then

$$\Phi(\kappa;\xi) = 1 - \frac{\beta(\xi) - \varpi\kappa z(\xi)}{\alpha(\xi) - \varpi\kappa z(\xi)}\Phi(\kappa z(\xi);\xi). \tag{76}$$

Since $\Phi(\kappa;\xi)$ is holomorphic with respect to κ for $|\kappa| < 1$, we know that $\Phi(\kappa;\xi) \to \Phi(0;\xi) = \alpha(\xi)/[\alpha(\xi) - \beta(\xi)]$ as $\kappa \to 0$. Iterating Eq. (76) produces a convergent infinite series for $|\xi| < 1$ and we find that

$$\varphi(\kappa;\xi) = \frac{2}{\xi[\alpha(\xi) - \varpi\kappa]} \times \left\{1 + \sum_{i=1}^{\infty}(-1)^i \prod_{j=1}^{i}\left(\frac{\beta(\xi) - \varpi\kappa z(\xi)^j}{\alpha(\xi) - \varpi\kappa z(\xi)^j}\right)\right\}. \tag{77}$$

Where $|q| < 1$, the q-Pochhammer symbol is defined by

$$(a;q)_n = \begin{cases} \prod_{m=0}^{n-1}(1 - aq^m), & n \in \mathbb{N}, \\ 1, & n = 0, \end{cases} \tag{78}$$

and Fine's basic hypergeometric series is defined by [24]

$$F(a,b;t:q) = \sum_{n=0}^{\infty} \frac{(aq;q)_n t^n}{(bq;q)_n}, \tag{79}$$

we deduce that

$$\varphi(\kappa;\xi) = \frac{2}{\xi[\alpha(\xi) - \varpi\kappa]} F\left[\frac{\varpi\kappa}{\beta(\xi)}, \frac{\varpi\kappa}{\alpha(\xi)}; -\gamma(\xi) : z(\xi)\right]. \tag{80}$$

It is also straightforward to solve the functional equation (75) by writing $\varphi(\kappa;\xi) = \sum_{m=0}^{\infty}\varphi_m(\xi)\kappa^m$, extracting a recurrence relation for the coefficients $\varphi_m(\xi)$ and solving it, leading to

$$\varphi(\kappa;\xi) = \frac{1}{1 - \xi(1-\varpi)} F\left[-1, -\gamma(\xi); \frac{\varpi\kappa}{\alpha(\xi)} : z(\xi)\right].$$

The equivalence of this solution and Eq. (80) is a manifestation of the identity (6.3) of Fine [24].

C. Where is the right wall?

We shall now determine the generating function for $\langle R_n(\varpi)\rangle$, the mean position of the rightmost allowed site, in terms of Fine's basic hypergeometric series. If we set $\theta = 0$ in Eq. (66), we have

$$\begin{aligned}\tilde{P}(0,l,r;\xi) &= \sum_{n=0}^{\infty} \Pr\{L_n = l, R_n = r\}\xi^n \\ &= \frac{\delta_{l,0}\delta_{r,0}}{1-\xi} + \frac{\xi}{2}\left\{-\frac{P(l,l,r;\xi)}{1-\xi} - \frac{P(r,l,r;\xi)}{1-\xi} + \frac{(1-\varpi)P(l,l,r;\xi)}{1-\xi} + \frac{\varpi P(l+1,l+1,r;\xi)}{1-\xi}(1-\delta_{l,0})\right. \\ &\quad \left. + \frac{(1-\varpi)P(r,l,r;\xi)}{1-\xi} + \frac{\varpi P(r-1,l,r-1;\xi)}{1-\xi}(1-\delta_{r,0})\right\} \\ &= \frac{\delta_{l,0}\delta_{r,0}}{1-\xi} + \frac{\varpi\xi}{2(1-\xi)}\{P(l+1,l+1,r;\xi)(1-\delta_{l,0}) + P(r-1,l,r-1;\xi)(1-\delta_{r,0}) - P(l,l,r;\xi) - P(r,l,r;\xi)\}.\end{aligned} \tag{81}$$

If we multiply Eq. (81) by r and sum over l and r, then the first and third terms in braces cancel, and using Eqs. (74) and (80) we find that

$$\sum_{n=0}^{\infty}\langle R_n(\varpi)\rangle\xi^n = \frac{\varpi\xi}{2(1-\xi)}\varphi(1;\xi) \tag{82}$$

$$= \frac{\varpi}{(1-\xi)[\alpha(\xi) - \varpi]} \times F\left[\frac{\varpi}{\beta(\xi)}, \frac{\varpi}{\alpha(\xi)}; -\gamma(\xi) : z(\xi)\right]. \tag{83}$$

It is easy to verify from Eq. (30) that

$$\frac{1}{(1-\xi)[\alpha(\xi) - \varpi]} = \sum_{n=0}^{\infty}\langle R_n(1)\rangle, \tag{84}$$

where $\langle R_n(1)\rangle$ is the mean location of the rightmost allowed site for one-sided erosion with $\varpi = 1$. However, when $\varpi = 1$, the solutions for the one- and two-sided problems coincide (there are never aborted steps), and we now have the elegant result for the two-sided problem that

$$\sum_{n=0}^{\infty}\langle R_n(\varpi)\rangle\xi^n = \mathcal{F}(\varpi;\xi)\sum_{n=0}^{\infty}\langle R_n(1)\rangle\xi^n, \tag{85}$$

where

$$\mathcal{F}(\varpi;\xi) = \varpi F\left[\frac{\varpi}{\beta(\xi)}, \frac{\varpi}{\alpha(\xi)}; -\gamma(\xi) : z(\xi)\right]. \tag{86}$$

Since $\langle R_n(\varpi)\rangle$ is positive and grows monotonically, if we can show that

$$\mathcal{F}(\varpi) = \lim_{\xi\to 1} \mathcal{F}(\varpi;\xi) \quad (87)$$

exists, then from the Tauberian Theorem we will have a rigorous proof of the exact asymptotic formula

$$\langle R_n(\varpi)\rangle \sim \mathcal{F}(\varpi)\langle R_n(1)\rangle \sim \mathcal{F}(\varpi)\left(\frac{2n}{\pi}\right)^{1/2} \quad \text{as } n\to\infty. \quad (88)$$

Moreover, as $\langle S_n(\varpi)\rangle = 2\langle R_n(\varpi)\rangle + 1$, it will also follow that

$$C(\varpi) = \lim_{n\to\infty} \frac{\langle S_n(\varpi)\rangle^2}{n} \quad (89)$$

exists, and that

$$C(\varpi) = \frac{8}{\pi}\mathcal{F}(\varpi)^2. \quad (90)$$

Baker *et al.* [12] observed from simulation that to a rough approximation (though only truly correct at the endpoints $\omega = 0$ and $\omega = 1$), $C(\varpi) \approx 8\varpi/\pi$, while both simulation and a mean-field argument led them to conjecture that $C(\varpi) \sim 2\varpi$ as $\varpi \to 0$.

We shall compute $\mathcal{F}(\varpi)$ explicitly in terms of the gamma function and prove the conjecture of Baker *et al.* [12]. It is helpful to rewrite our formula for $\mathcal{F}(\varpi)$ in terms of the basic hypergeometric function ${}_2\phi_1$, rather than Fine's function F. Since [25]

$${}_2\phi_1(a,b;c;q,t) = \sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n} t^n \quad (91)$$

we find that

$$\mathcal{F}(\varpi) = \lim_{z\to 1^-} \varpi\, {}_2\phi_1\left[\frac{\varpi z}{z-(1-\varpi)},z;\frac{\varpi z}{z^{-1}-(1-\varpi)};z,\; -\frac{z-(1-\varpi)}{z^{-1}-(1-\varpi)}\right]. \quad (92)$$

We cannot evaluate the limit by inserting $z = 1$, since the first four arguments of the basic hypergeometric function are all 1 in this case, while the last is -1, and the resulting series does not converge. However ([25], p. 3),

$$\lim_{q\to 1} {}_2\phi_1(q^a,q^b;q^c;q,t) = {}_2F_1(a,b;c;t), \quad (93)$$

where ${}_2F_1$ is the ordinary (Gauss) hypergeometric function. If we observe that

$$z^a = \frac{\varpi z}{z-(1-\varpi)}, \quad z^c = \frac{\varpi z}{z^{-1}-(1-\varpi)}$$

correspond, respectively, to

$$a = 1 - \frac{\log[1-(1-z)/\varpi]}{\log[1-(1-z)]} \sim 1 - \frac{1}{\varpi} \quad \text{as } z\to 1^-,$$

$$c = 1 - \frac{\log[1+(1-z)/(\varpi z)]}{\log[1-(1-z)]} \sim 1 + \frac{1}{\varpi} \quad \text{as } z\to 1^-,$$

we find that

$$\mathcal{F}(\varpi) = \varpi\, {}_2F_1(1-1/\varpi,1;1+1/\varpi;-1). \quad (94)$$

An identity for hypergeometric functions ([22], p. 557, (15.1.21)) enables us to evaluate the right-hand side in terms of the gamma function and we conclude at last that

$$\mathcal{F}(\varpi) = \frac{\varpi\sqrt{\pi}\,\Gamma(1+1/\varpi)}{2\Gamma(1/2+1/\varpi)}. \quad (95)$$

We note that $\mathcal{F}(1) = 1$, while from the asymptotic relation

$$\frac{\Gamma(n+a)}{\Gamma(n+b)} \sim n^{a-b} \quad \text{as} \quad n\to\infty$$

(a simple consequence of Stirling's approximation) we find that

$$\mathcal{F}(\varpi) \sim \frac{\sqrt{\pi}}{2}\varpi^{1/2} \quad \text{as} \quad \varpi\to 0. \quad (96)$$

Hence,

$$C(\varpi) = \frac{2\varpi^2\Gamma(1+1/\varpi)^2}{\Gamma(1/2+1/\varpi)^2} \sim \begin{cases} 2\varpi & \text{as} \quad \varpi\to 0, \\ 8/\pi & \text{as} \quad \varpi\to 1, \end{cases} \quad (97)$$

establishing the truth of the conjecture of Baker *et al.* [12].

IV. DISCUSSION

A characteristic feature of the simplest unbiased one-dimensional random walk is the slow decay with time of the probability that the walker is found in a finite interval containing the origin. This is closely connected to the well-known certainty of such walkers to revisit the starting site infinitely often, while making excursions to either side which may be of very long duration. For one-sided erosion, the right boundary continues to be eroded away, even though the expected walker location drifts off to $-\infty$. In contrast, for

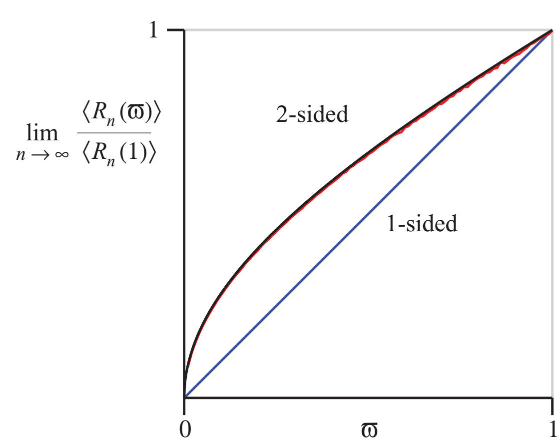

FIG. 5. (Color online) We compare the ϖ-dependence of the asymptotic form of the mean rightmost allowed site for one-sided and two-sided erosion. The blue (straight) line corresponds to one-sided erosion with the walker starting at 0 with all sites to the right initially blocked. The black (concave down) curve corresponds to the exact limit that we have obtained for two-sided erosion, with the walker starting at 0, with all other sites initially blocked. Simulation data for two-sided erosion based on 10 000 random walks of 10 000 steps of Baker *et al.* [12] for the mean allowed interval length, converted to the mean rightmost point using $\langle S_n(\varpi)\rangle = 2\langle R_n(\varpi)\rangle + 1$, has been plotted in red and is almost indistinguishable from the black (smooth) theoretical curve.

two-sided erosion the walker bounces off both walls and the frequency of collision with walls is enhanced, especially for small ϖ, leading to faster erosion. We show the long-time limit of the ratio $\langle R_n(\varpi)\rangle/\langle R_n(1)\rangle$ in Fig. 5. The correctness of the exact computation of the limiting ratio is confirmed by simulation data from Baker *et al.* [12]. We have been unable to obtain exact results for other quantities of interest related to boundary location such as $\mathrm{Var}\{R_n(\varpi)\}$ or $\mathrm{Var}\{S_n(\varpi)\}$, where $S_n(\varpi) = R_n(\varpi) - L_n(\varpi) + 1$. Weiss and Rubin [26] have discussed the large-n asymptotic distribution of $S_n(1) = R_n(1) - L_n(1) + 1$, but there does not appear to be any simple way to adapt their analysis to cover $\varpi < 1$.

To date, we have also been unable to obtain companion results for the mean-square displacement $\langle X_n(\varpi)^2\rangle$ in two-sided erosion. The generating function for $\langle X_n(\varpi)^2\rangle$ requires the evaluation at $\lambda = \rho = 1$ the first derivative with respect to λ of

$$\mathcal{A}(\lambda,\rho;\xi) = \sum_{\ell=0}^{\infty}\sum_{r=0}^{\infty} A(\ell,r;\xi)\lambda^{\ell}\rho^{r}.$$

A series solution for $\mathcal{A}(\lambda,\rho;\xi)$ reminiscent of the solution for $\varphi(\kappa;\xi)$ can be obtained, but it appears that new results in the theory of basic hypergeometric functions are required to finish the calculation.

It may be noted that the stochastic model discussed in this paper is a discrete analog of a deterministic moving boundary value problem for the diffusion or heat conduction equation [27,28]. Such discrete stochastic models may give insight into statistical fluctuations in moving boundary problems.

The biological question that motivated Baker *et al.* [12] to introduce the model we have discussed arose from cell motion within tissue. There are other contexts in which models of this type may be of interest: for example, modeling an individual animal which moves freely within a finite home range, but occasionally ventures beyond its previous borders, exploring new territory and so enlarging its range.

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[1] M. N. Barber and B. W. Ninham, *Random and Restricted Walks: Theory and Applications* (Gordon and Breach, New York, 1970).
[2] F. Spitzer, *Principles of Random Walk*, 2nd ed. (Springer-Verlag, New York, 1976).
[3] G. H. Weiss, *Aspects and Applications of the Random Walk* (North-Holland, Amsterdam, 1994).
[4] B. D. Hughes, *Random Walks and Random Environments*, Vol. 1 (Oxford University Press, Oxford, UK, 1995).
[5] W. Woess, *Random Walks on Infinite Graphs and Groups* (Cambridge University Press, Cambridge, UK, 2000).
[6] G. F. Lawler and V. Limic, *Random Walk: a Modern Introduction* (Cambridge University Press, Cambridge, UK, 2010).
[7] J. Klafter and I. M. Sokolov, *First Steps in Random Walks* (Oxford University Press, Oxford, UK, 2011).
[8] T. A. Witten and L. M. Sander, Phys. Rev. Lett. **47**, 1400 (1981).
[9] R. Jullien and R. Botet, *Aggregation and Fractal Aggregates* (World Scientific, Singapore, 1987).
[10] P. Diaconis and W. Fulton, Rend. Sem. Mat. Univ. Politec. Torino **49**, 95 (1991).
[11] G. F. Lawler, M. Bramson, and D. Griffeath, Ann. Probab. **20**, 2117 (1992).
[12] C. M. Baker, B. D. Hughes, and K. A. Landman, Phys. Rev. E **88**, 042113 (2013).
[13] G. Pólya, Math. Annalen **84**, 149 (1921).
[14] B. D. Hughes, *Random Walks and Random Environments*, Vol. 2 (Oxford University Press, Oxford, UK, 1996).
[15] H. Herrmann, J. Phys. A **16**, L611 (1983).
[16] W. Feller, *An Introduction to Probability Theory and its Applications*, 2nd ed. (Wiley, New York, 1971), Vol. 2.
[17] F. W. J. Olver, *Asymptotics and Special Functions* (Academic Press, New York, 1974).
[18] A. Comtet and S. N. Majumdar, J. Stat. Mech. (2005) P06013.
[19] N. Henze, *Irrfahrten und verwandte Zufälle: ein elementarer Einstieg in die stochastischen Prozesse* (Springer Spektrum, Weisbaden, Germany, 2013).
[20] W. Katzenbeisser and W. Panny, J. Stat. Plan. Infer. **101**, 149 (2002).
[21] P. Erdös and M. Kac, Bull. Amer. Math. Soc. **52**, 292 (1946).
[22] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, New York, NY, 1965).
[23] E. Montroll and R. Potts, Phys. Rev. **100**, 525 (1955).
[24] N. Fine, *Basic Hypergeometric Functions and Applications* (American Mathematical Society, Providence, RI, 1988).
[25] G. Gaspar and M. Rahman, *Basic Hypergeometric Series*, 2nd ed. (Cambridge University Press, Cambridge, UK, 2004).
[26] G. Weiss and R. Rubin, Adv. Chem. Phys. **52**, 363 (1983).
[27] J. Crank, *The Mathematics of Diffusion*, 2nd ed. (Oxford University Press, Oxford, UK, 1975).
[28] H. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, 2nd ed. (Oxford University Press, Oxford, UK, 1959).