Double-maximum enhancement of signal-to-noise ratio gain via stochastic resonance and vibrational resonance

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This paper studies the signal-to-noise ratio (SNR) gain of a parallel array of nonlinear elements that transmits a common input composed of a periodic signal and external noise. Aiming to further enhance the SNR gain, each element is injected with internal noise components or high-frequency sinusoidal vibrations. We report that the SNR gain exhibits two maxima at different values of the internal noise level or of the sinusoidal vibration amplitude. For the addition of internal noise to an array of threshold-based elements, the condition for occurrence of stochastic resonance is analytically investigated in the limit of weak signals. Interestingly, when the internal noise components are replaced by high-frequency sinusoidal vibrations, the SNR gain displays the vibrational multiresonance phenomenon. In both considered cases, there are certain regions of the internal noise intensity or the sinusoidal vibration amplitude wherein the achieved maximal SNR gain can be considerably beyond unity for a weak signal buried in non-Gaussian external noise. Due to the easy implementation of sinusoidal vibration modulation, this approach is potentially useful for improving the output SNR in an array of nonlinear devices.

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I. INTRODUCTION

The studies on stochastic resonance have progressively changed the status of noise in physics and information sciences. It is now known that under certain circumstances a nonzero optimal noise level can enhance, rather than degrade, the nonlinear system performance measured by signal-to-noise ratio (SNR) [1–6], correlation coefficient [7,8], mutual information [9], detection probability [10], response power spectrum [11], etc. Moreover, this counterintuitive view attracts considerable attention in the area of improvement of the response of systems that might be operated in noisy environments, e.g., electronic devices [3,12,13], neuronal systems [7,8,11,14–17], and signal processors [9,10,18–23].

The primary feature of stochastic resonance is that the system response can reach a maximum at a nonzero optimal amount of noise [1–3,5]. In addition, Vilar and Rubí reported that in some cases the system output SNR can be enhanced at multiple values of the noise level, and exhibits a series of maxima; this is stochastic multiresonance [24]. This effect was then observed in threshold-crossing systems [5,25], coupled oscillators [26,27], and hierarchical networks [28], which extends the scope and perspectives of the noise-enhanced phenomenon [24–28]. Recently, in the generic bistable system, Landa and McClintock [29] reported the phenomenon of vibrational resonance that displays many analogies to the stochastic resonance effect, but with high-frequency vibrational modulation filling the constructive role usually played by noise. Experimental and analytical evidence of vibrational resonance has been demonstrated in analog electric circuits [30,31], excitable neurons [32], optical devices [33,34], and dynamical oscillators [35–40]. Interestingly, the occurrence of vibrational multiresonance was also found [29,39,40].

Motivated by the fact of large numbers of neurons in the nervous systems of animals and humans with variations in structure, function, and size, the potential exploitation of stochastic (vibrational) resonance in neuroscience becomes an interesting open question [6–9,14–17,22,23,32,37,41], especially in a summing parallel threshold-based sensory neuron model [8,9,22,23,41]. In spite of efforts devoted to understanding the constructive roles of external or internal noise components, the resonant mode of noise is not sufficiently considered, for instance, the optimal values of the noise level and the optimizing modes of the noise type that benefit signal transmission through nonlinear elements.

In this paper, we focus on enhancing the SNR gain, i.e., the ratio between the output SNR and the input SNR, for a periodic signal mixed with external input noise passing through a parallel array of nonlinear elements. It is usually argued that the level of external noise is not tunable, thus we inject internal noise into each element for further improvement of the SNR gain. We report that in a parallel array of threshold-based elements, the output-input SNR gain manifests double resonant peaks as the internal noise level increases. Within the limit of weak signals, we analytically address the condition under which stochastic resonance occurs for various external noise types and array sizes. The optimal internal noise

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levels and the external noise types that elicit the double resonance mode are theoretically addressed. The possibility of the maximal SNR gain exceeding unity is demonstrated for weak signals corrupted by the external non-Gaussian noise. The approach by which internal noise is added has received considerable attention for enhancing the array performance within the framework of stochastic resonance [9,16,22,23,42]. However, in many practical operating devices, the internal noise type or level may be not controllable. Inspired by the mechanism of vibrational resonance [29–40], in each element of an array, the random noise components are replaced by the high-frequency sinusoidal vibrations. Upon increasing the amplitude of sinusoidal vibrations, we find that the SNR gain also exhibits two maxima. Besides, our analysis shows that the locally maximum SNR gain obtained by vibrational resonance is higher than the one via the phenomenon of stochastic resonance. These analytical results not only show a new feature of array stochastic resonance, but also provide another practical realization of the SNR improvement in an array via tuning sinusoidal vibrations. We believe that these theoretical results presented in this paper will be valuable to a variety of systems ranging from threshold sensors to sensory neural networks.

II. MODEL AND MEASURE

The input $x(t) = s(t) + \xi(t)$ comprises the deterministic sinusoidal signal $s(t) = A\sin(2\pi t/T)$ with period T and amplitude A, and the stationary white noise $\xi(t)$ with probability density function (PDF) f_ξ and variance σ_ξ^2. The input SNR for $x(t)$ can be defined as the power contained in the spectral line at frequency $1/T$ divided by the power contained in the noise background in a small frequency bin ΔB around $1/T$ [5], this is

$$R_{\rm in} = \frac{|\langle s(t)\exp[-i2\pi t/T]\rangle|^2}{\sigma_\xi^2 \Delta B \Delta t} = \frac{A^2/4}{\sigma_\xi^2 \Delta B \Delta t}, \tag{1}$$

with Δt indicating the time resolution or the sampling time in a discrete-time implementation and the temporal average defined as $\langle \cdots \rangle = \frac{1}{T}\int_0^T \cdots dt$ [5].

Then, $x(t)$ is applied to a parallel array of N identical elements, which have the same memoryless characteristic g. The internal white noise terms $\eta_n(t)$, independent of $x(t)$, are injected into each element to yield the outputs

$$y_n(t) = g[x(t) + \eta_n(t)], \tag{2}$$

where the noise components $\eta_n(t)$ are mutually independent and identically distributed with the same PDF f_η and variance σ_η^2. Then, the response $y(t)$ of an array is the collective outputs as

$$y(t) = \frac{1}{N}\sum_{n=1}^{N} y_n(t). \tag{3}$$

Since $s(t)$ is periodic, $y(t)$ is in general a cyclostationary random signal with period T [5]. Similarly, the output SNR for $y(t)$ is given by

$$R_{\rm out} = \frac{|\langle \mathrm{E}[y(t)]\exp[-i2\pi t/T]\rangle|^2}{\langle \mathrm{var}[y(t)]\rangle \Delta B \Delta t} \tag{4}$$

with nonstationary expectation $\mathrm{E}[y(t)]$ and nonstationary variance $\mathrm{var}[y(t)] = \mathrm{E}[y^2(t)] - \mathrm{E}^2[y(t)]$ [5].

At time t, for a fixed value x of the input $x(t)$, the conditional expectations can be computed as

$$\mathrm{E}[y(t)|x] = \mathrm{E}[y_n(t)|x] = \int_{-\infty}^{\infty} g(x+u) f_\eta(u)du, \tag{5}$$

$$\mathrm{E}[y^2(t)|x] = \frac{1}{N}\mathrm{E}[y_n^2(t)|x] + \frac{N-1}{N}\mathrm{E}^2[y_n(t)|x], \tag{6}$$

with

$$\mathrm{E}[y_n^2(t)|x] = \int_{-\infty}^{\infty} g^2(x+u) f_\eta(u)du. \tag{7}$$

Here, for the input $x(t) = s(t) + \xi(t)$, we have the expectations

$$\mathrm{E}[y(t)] = \int_{-\infty}^{\infty} \mathrm{E}[y(t)|x] f_\xi(x - s(t))dx, \tag{8}$$

$$\mathrm{E}[y^2(t)] = \int_{-\infty}^{\infty} \mathrm{E}[y^2(t)|x] f_\xi(x - s(t))dx. \tag{9}$$

Based on the input SNR of Eq. (1) and the output SNR of Eq. (4), we have the output-input SNR gain of an array as

$$G_N = \frac{R_{\rm out}}{R_{\rm in}} = \frac{|\langle \mathrm{E}[y(t)]\exp[-i2\pi t/T]\rangle|^2}{\langle \mathrm{var}[y(t)]\rangle} \frac{\sigma_\xi^2}{A^2/4}. \tag{10}$$

Since $\xi(t)$ and $\eta_n(t)$ are independent, Eq. (2) can be rewritten as $y_n(t) = g[s(t) + z(t)]$, where the composite noise components $z_n(t) = \xi(t) + \eta_n(t)$ are with the same convolved PDF $f_z(z) = \int f_\eta(u) f_\xi(z-u)du$. We further consider a small periodic signal $s(t)$ with a maximal amplitude $A \to 0$ $[|s(t)| \leqslant A]$. Then, at a fixed time t, we can make a Taylor expansion of $f_\xi(x - s(t)) \approx f_\xi(x) - s(t) f_\xi'(x)$ up to the first order in Eqs. (8) and (9) with $f_\xi'(x) = df_\xi(x)/dx$. Then, the small-signal limit of Eq. (8) can be expressed as

$$\begin{aligned}
\mathrm{E}[y(t)] &\approx \int_{-\infty}^{\infty} \mathrm{E}[y(t)|x][f_\xi(x) - s(t) f_\xi'(x)]dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x+u) f_\eta(u) f_\xi(x)dudx - s(t) \\
&\quad \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x+u) f_\eta(u) f_\xi'(x)dudx \\
&= s(t)\int_{-\infty}^{\infty} g'(z)\int_{-\infty}^{\infty} f_\eta(u) f_\xi(z-u)dudz \\
&= s(t)\mathrm{E}_z[g'(z)],
\end{aligned} \tag{11}$$

where we assume that the derivative $g'(z) = dg(z)/dz$ and g has zero mean under f_z, i.e., $\mathrm{E}_z[g(z)] = 0$, which is not restrictive since any arbitrary g can always include a constant bias to cancel this average. Similarly, we obtain

$$\begin{aligned}
\mathrm{E}[y^2(t)] &\approx \int_{-\infty}^{\infty} \mathrm{E}[y^2(t)|x][f_\xi(x) - s(t) f_\xi'(x)]dx \\
&\approx \mathrm{E}_\xi\{\mathrm{E}_\eta[y^2(t)]\} \\
&= \mathrm{E}_\xi\left\{\frac{1}{N^2}\sum_{n=1}^{N}\sum_{m=1}^{N}\mathrm{E}_\eta[y_n y_m]\right\}
\end{aligned}$$

$$= \frac{1}{N^2}\mathrm{E}_\xi\{N\mathrm{E}_\eta[y_n^2] + N(N-1)\mathrm{E}_\eta[y_n y_m]\}\ (\forall n \neq m)$$
$$= \frac{1}{N}\mathrm{E}_z[g^2(z)] + \frac{N-1}{N}\mathrm{E}_\xi\{\mathrm{E}_\eta^2[g(\xi+\eta)]\}, \qquad (12)$$

where $\mathrm{E}_z[\cdot] = \int_{-\infty}^{\infty} \cdot f_z(z)dz = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdot f_\xi(\xi)d\xi\ f_\eta(\eta)d\eta = \mathrm{E}_\xi\{\mathrm{E}[\cdot]\}$ and $\mathrm{E}_\eta[y_n] = \mathrm{E}_\eta[y_m]$ for $n,m = 1,2,\ldots,N$. Therefore, based on Eqs. (11) and (12), we have

$$\begin{aligned}\mathrm{var}[y(t)] &= \mathrm{E}[y^2(t)] - \mathrm{E}^2[y(t)]\\ &\approx \mathrm{E}[y^2(t)] - s^2(t)\mathrm{E}_z^2[g'(z)]\\ &\approx \mathrm{E}[y^2(t)] \end{aligned} \qquad (13)$$

up to first order in the small signal $s(t)$. Substituting Eqs. (12) and (13) into Eq. (10), we obtain the expression of the SNR gain of a parallel array of memoryless nonlinearities, up to first order in the small signal $s(t)$, as

$$G_N \approx \frac{\sigma_\xi^2 \mathrm{E}_z^2[g'(z)]}{\frac{1}{N}\mathrm{E}_z[g^2(z)] + \frac{N-1}{N}\mathrm{E}_\xi\{\mathrm{E}_\eta^2[g(\xi+\eta)]\}}. \qquad (14)$$

III. DOUBLE-MAXIMUM RESONANCE EFFECTS

A. Array stochastic resonance by random noise

When the internal noise root-mean-square amplitude (RMS) σ_η increases, the positive derivative

$$\frac{\partial G_N}{\partial \sigma_\eta} > 0 \qquad (15)$$

indicates the occurrence of the stochastic resonance phenomenon. Furthermore, when the equality

$$\left.\frac{\partial G_N}{\partial \sigma_\eta}\right|_{\sigma_\eta = \sigma_\eta^{(k)}} = 0 \qquad (16)$$

holds at solutions of $\sigma_\eta = \sigma_\eta^{(k)}$, and in the neighborhood $[\sigma_\eta^{(k)} - \delta, \sigma_\eta^{(k)} + \delta]$ $(\delta > 0)$ of $\sigma_\eta^{(k)}$, the derivative $\partial G_N/\partial\sigma_\eta$ is positive to the left of $\sigma_\eta^{(k)}$ and negative to the right of this point, then G_N has a local maximum at $\sigma_\eta^{(k)}$. Thus, if there is more than one local maximum, the stochastic multiresonance phenomenon of SNR gain will appear.

We here consider the three-threshold characteristic

$$g(x) = \tfrac{1}{2}[\mathrm{sgn}(x-\lambda) + \mathrm{sgn}(x+\lambda)], \qquad (17)$$

with the threshold parameter λ and the signum (sgn) function $\mathrm{sgn}(\cdot)$. The internal noise components $\eta_n(t)$ are assumed to be uniformly distributed over $[-\sqrt{3}\sigma_\eta, \sqrt{3}\sigma_\eta]$ with PDF $f_\eta(u) = 1/(2\sqrt{3}\sigma_\eta)$ and cumulative distribution

$$F_\eta(u) = \begin{cases} 0, & \text{for } u \leqslant \sqrt{3}\sigma_\eta \\ \frac{u+\sqrt{3}\sigma_\eta}{2\sqrt{3}\sigma_\eta}, & \text{for } -\sqrt{3}\sigma_\eta < u < \sqrt{3}\sigma_\eta, \\ 1, & \text{for } u \geqslant \sqrt{3}\sigma_\eta. \end{cases} \qquad (18)$$

Then, based on Eq. (17), we have the explicit expression of expectation

$$\mathrm{E}_\eta[g(\xi+\eta)] = 1 - F_\eta(-\lambda-\xi) - F_\eta(\lambda-\xi). \qquad (19)$$

Thus, the asymptotic expression of the SNR gain of Eq. (14) can be expressed as

$$G_N \approx \frac{4\sigma_\xi^2 f_z^2(\lambda)}{\frac{2}{N}[1-F_z(\lambda)] + \frac{N-1}{N}\mathrm{E}_\xi\{[1-F_\eta(-\lambda-\xi)-F_\eta(\lambda-\xi)]^2\}} \qquad (20)$$

and the equality of Eq. (16) becomes

$$\begin{aligned}\frac{\partial G_N}{\partial \sigma_\eta} &= 2[f_\xi(\lambda-\sqrt{3}\sigma_\eta) + f_\xi(\lambda+\sqrt{3}\sigma_\eta) - 2f_z(\lambda)][1-F_z(\lambda)] + f_z(\lambda)[F_\xi(\lambda-\sqrt{3}\sigma_\eta) + F_\xi(\lambda+\sqrt{3}\sigma_\eta) - 2F_z(\lambda)]\\ &\quad + (N-1)[f_\xi(\lambda-\sqrt{3}\sigma_\eta) + f_\xi(\lambda+\sqrt{3}\sigma_\eta) - 2f_z(\lambda)]\mathrm{E}_\xi[(1-F_\eta(-\lambda-\xi)-F_\eta(\lambda-\xi))^2]\\ &\quad - 2(N-1)f_z(\lambda)\,\mathrm{E}_\xi\{[1-F_\eta(-\lambda-\xi)-F_\eta(\lambda-\xi)][(\lambda-x)f_\eta(\lambda-x) - (\lambda+x)f_\eta(\lambda+x)]\} = 0. \end{aligned} \qquad (21)$$

We further consider the generalized Gaussian noise $\xi(t)$ with PDF

$$f_\xi(x) = \frac{c_1}{\sigma_\xi}\exp\left(-c_2\left|\frac{x}{\sigma_\xi}\right|^\alpha\right), \qquad (22)$$

where $c_1 = \frac{\alpha}{2}\Gamma^{\frac{1}{2}}(\frac{3}{\alpha})/\Gamma^{\frac{3}{2}}(\frac{1}{\alpha})$, $c_2 = [\Gamma(\frac{3}{\alpha})/\Gamma(\frac{1}{\alpha})]^{\frac{\alpha}{2}}$ for a rate of decay exponent $\alpha > 0$, and the noise RMS is σ_ξ. Here, as the exponent α varies, we can conveniently consider a spectrum of densities ranging from the Gaussian ($\alpha = 2$) to those with relatively much faster ($\alpha > 2$) or slower ($\alpha < 2$) rates of exponential decay of their tails. In this case, the PDF f_z of the composite random variables $z_n(t) = \xi(t) + \eta_n(t)$ becomes

$$\begin{aligned} f_z(x) &= \int_{-\sqrt{3}\sigma_\eta}^{\sqrt{3}\sigma_\eta} f_\xi(x-u)\frac{1}{2\sqrt{3}\sigma_\eta}du\\ &= \frac{F_\xi(x+\sqrt{3}\sigma_\eta) - F_\xi(x-\sqrt{3}\sigma_\eta)}{2\sqrt{3}\sigma_\eta}, \end{aligned} \qquad (23)$$

with the cumulative distribution $F_\xi(x) = \int_{-\infty}^{x} f_\xi(u)du$.

Given the noise RMS amplitude $\sigma_\xi = 1/\sqrt{3}$ and the exponent $\alpha = 8$, we plot the SNR gain G_N as a function of the internal uniform noise level σ_η in Fig. 1(a) for different array sizes. It is seen that G_1 of an isolated characteristic monotonically decreases as σ_η increases, and the stochastic

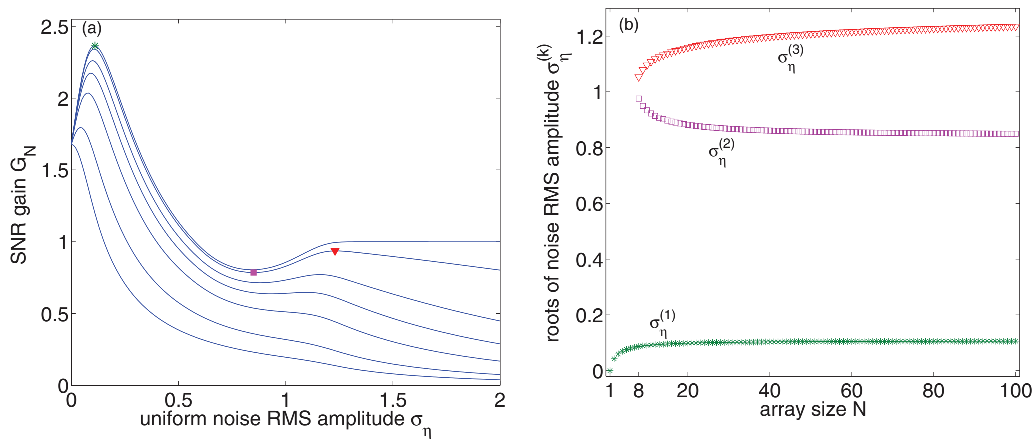

FIG. 1. (Color online) (a) Output-input SNR gain G_N of Eq. (20) versus the internal uniform noise level σ_η for different array sizes $N = 1,2,5,10,20,100$, and ∞ (from the bottom up); (b) Roots of uniform noise RMS amplitudes $\sigma_\eta^{(k)}$ in Eq. (21) versus the array size N. The optimal noise levels $\sigma_\eta^{(1)}$ and $\sigma_\eta^{(3)}$ that locally maximize G_N are marked by green asterisks ($*$) and red downward triangles ($\triangledown$), respectively. While, the magenta squares ($\square$) represent $\sigma_\eta^{(2)}$ that correspond to the locally minimum of G_N. Here, the threshold $\lambda = 1$ in Eq. (17), the external generalized Gaussian noise $\xi(t)$ is with RMS amplitude $\sigma_\xi = 1/\sqrt{3}$ and exponent $\alpha = 8$.

resonance effect does not occur. For the array size $N \geqslant 2$, the noise-enhanced effect appears and the SNR gain G_N exhibits the resonancelike behavior as σ_η increases. Moreover, as the array size $N \geqslant 8$, it is found in Fig. 1(b) that there are three roots of the noise RMS amplitudes $\sigma_\eta^{(k)}$ in Eq. (21). For solutions $\sigma_\eta^{(1)}$ and $\sigma_\eta^{(3)}$, we calculate the derivative $\partial G_N/\partial\sigma_\eta$ of Eq. (21) in the neighborhood $[\sigma_\eta^{(k)} - \delta, \sigma_\eta^{(k)} + \delta]$ of $\sigma_\eta^{(k)}$ ($\delta > 0$). It is found that the derivative $\partial G_N/\partial\sigma_\eta$ is positive to the left of $\sigma_\eta^{(k)}$ and negative to the right of this point, then G_N has local maxima at $\sigma_\eta^{(1)}$ and $\sigma_\eta^{(3)}$. This is the stochastic multiresonance effect that manifests in an array possessing nonlinear characteristics, as illustrated in Fig. 1(a). The mechanism giving rise to stochastic multiresonance depends on the interaction of internal noise components $\eta_n(t)$ in the nonlinear systems. It is shown in Fig. 2 that the first variance term $\mathrm{E}_z[g^2(z)]$ (solid line) of the denominator in Eq. (21) increases monotonically for the increase of uniform noise RMS amplitude σ_η, which represents the statistical fluctuation

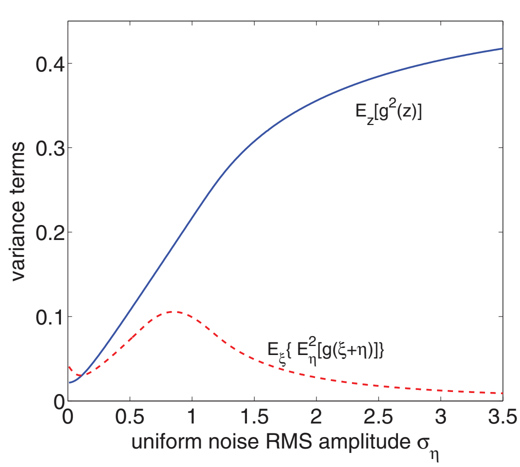

FIG. 2. (Color online) Variance terms $\mathrm{E}_z[g^2(z)]$ (solid line) and $\mathrm{E}_\xi\{\mathrm{E}_\eta^2[g(\xi+\eta)]\}$ (dashed line) in the denominator of Eq. (14) versus the internal uniform noise RMS amplitude σ_η. Other parameters are the same as in Fig. 1(a).

around the nonstationary mean $\mathrm{E}[y(t)]$ of each element. However, as the array size N increases, its contribution to the whole variance weakens with the proportional factor of $1/N$. While, as shown in Fig. 2, the term $\mathrm{E}_\xi\{\mathrm{E}_\eta^2[g(\xi+\eta)]\}$ (dashed line) first decreases, and then goes through a bell-type curve. By this mechanism, the addition of internal noise $\eta_n(t)$ first helps the input signal to overcome the nonlinear threshold, and the array output $y(t)$ carries more signal-ingredient of $\mathrm{E}[y(t)]$, leading to the decrease of this variance part. Further increase of noise results in the increase of $\mathrm{E}_\xi\{\mathrm{E}_\eta^2[g(\xi+\eta)]\}$. However, due to the independent characteristics of η_n, the quantity of $\mathrm{E}_\xi\{\mathrm{E}_\eta^2[g(\xi+\eta)]\}$ tends to a finite constant at large σ_η. Moreover, as the array size N increases, its contribution to the whole variance is enhanced by the proportional factor of $(N-1)/N$. Thus, the output-input SNR gain presents the double extrema.

Particularly, for the array size $N = 100$, the roots of uniform noise RMS amplitudes $\sigma_\eta^{(k)}$ in Eq. (21) are illustrated in Fig. 3 versus the exponent α of generalized Gaussian noise $\xi(t)$ of Eq. (22). For $\alpha = 2$, Eq. (22) corresponds to the Gaussian noise type. It is observed in Fig. 3 that, only for the tails of noise densities decaying at a higher rate $\alpha > 2$ than Gaussian noise, the array stochastic multiresonance effect occurs. The reason is, for the exponent $\alpha > 2$, the tails of these non-Gaussian noise densities decay at rates faster than the rate of decay of the Gaussian density tail. For the same noise RMS amplitude σ_ξ, this special non-Gaussian noise will produce less large-magnitude observations than would be predicted by a noise model with $\alpha \leqslant 2$. Therefore, in such noise types with $\alpha > 2$, the positive role of the internal noise will be manifested more clearly in assisting the input signal to cross the threshold, resulting in this multiple resonance effect.

It is also noted in Fig. 1(a) that the maximum of SNR gain $G_\infty = 2.36$ at the optimal noise level $\sigma_\eta^{(1)} = 0.12$. For $\sigma_\xi = 1/\sqrt{3}$ and $N = \infty$, the maximal SNR gains G_∞, at the corresponding optimal noise levels $\sigma_\eta^{(1)}$, are plotted as a function of the generalized Gaussian noise decay exponent α in Fig. 4. Interestingly, it is seen in Fig. 4 that the maximal SNR gains G_∞ can greatly exceed unity for the non-Gaussian

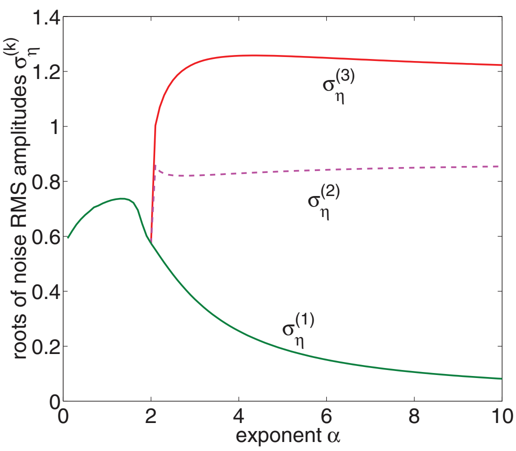

FIG. 3. (Color online) Roots of uniform noise RMS amplitudes $\sigma_\eta^{(k)}$ of the denominator in Eq. (21) versus the decay exponent α of generalized Gaussian noise $\xi(t)$. The array size $N = 100$ and the noise RMS amplitude $\sigma_\xi = 1/\sqrt{3}$. Solid lines represent the optimal noise levels $\sigma_\eta^{(1,3)}$ that locally maximize the SNR gain G_{100}, while the dashed line indicates the corresponding noise level $\sigma_\eta^{(2)}$ that locally minimizes G_{100}.

noise types with the exponents $\alpha \neq 2$. This is because, using the Cauchy-Schwarz inequality and as $N = \infty$, we find the SNR gain G_∞ in Eq. (14) bounded by

$$G_\infty = \frac{\sigma_\xi^2 \mathrm{E}_z^2[g'(z)]}{\mathrm{E}_\xi\left\{\mathrm{E}_\eta^2[g(\xi+\eta)]\right\}} = \frac{\sigma_\xi^2 \mathrm{E}_\xi^2\{d\mathrm{E}_\eta[g(\xi+\eta)]/d\xi\}}{\mathrm{E}_\xi\left\{\mathrm{E}_\eta^2[g(\xi+\eta)]\right\}}$$
$$\leqslant \sigma_\xi^2 \mathrm{E}_\xi\left[\frac{f_\xi'^2(\xi)}{f_\xi^2(\xi)}\right] = \sigma_\xi^2 I(f_\xi) = I(f_{\xi_0}), \tag{24}$$

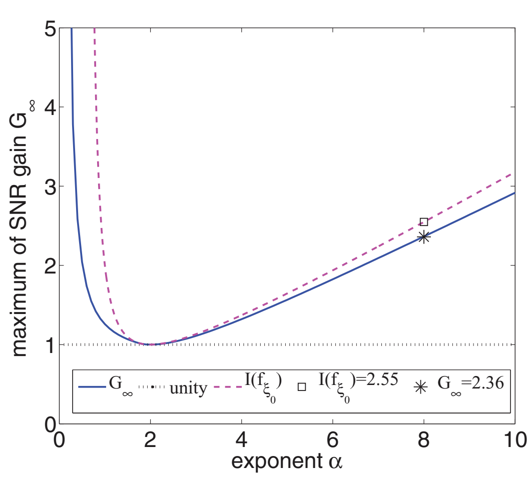

FIG. 4. (Color online) The maximal SNR gains G_∞ of Eq. (20) versus the generalized Gaussian noise decay exponent α at the corresponding optimal noise levels $\sigma_\eta^{(1)}$. The other parameters are the same as in Fig. 1. The solid line represents G_∞, the black dashed line indicates the unity line and the magenta dashed line is the Fisher information of $I(f_{\xi_0})$. At the exponent $\alpha = 8$, the the corresponding Fisher information $I(f_{\xi_0}) = 2.55$ marked by a square, and the asterisk mark indicates the maximum of SNR gain $G_\infty = 2.36$ that obtained by the optimal noise level and the infinity of array size N by the array of nonlinearities of Eq. (17).

where the equality occurs as the nonlinearity $\mathrm{E}_\eta[g_{\rm opt}(\xi+\eta)] = Cf_\xi'(\xi)/f_\xi(\xi)$, i.e., the locally optimum nonlinearity $g_{\rm opt}(\xi) = Cf_\xi'(\xi)/f_\xi(\xi)$ and the PDF $f_\eta(\eta) = \delta(\eta)$. This means there is no internal noise in each nonlinearity of $g_{\rm opt}$. Furthermore, we assume that the scaled noise $\xi(t) = \sigma_\xi \xi_0(t)$ has PDF $f_\xi(\xi) = f_{\xi_0}(\xi/\sigma_\xi)/\sigma_\xi$ and $\xi_0(t)$ has a standardized PDF f_{ξ_0} with unity variance $\sigma_{\xi_0}^2 = 1$. Then, the Fisher information $I(f_{\xi_0}) = \mathrm{E}_{\xi_0}[f_{\xi_0}'^2/f_{\xi_0}^2]$ and $I(f_\xi) = \mathrm{E}_\xi[f_\xi'^2/f_\xi^2] = I(f_{\xi_0})/\sigma_\xi^2$. It is known that the standardized generalized Gaussian noise distribution in Eq. (22) has the Fisher information $I(f_{\xi_0}) = \alpha^2\Gamma(3\alpha^{-1})\Gamma(2-\alpha^{-1})/\Gamma^2(\alpha^{-1})$ [43].

Our finding agrees with the conclusion in Ref. [4]: The addition of noise to a nonlinear system first degrades, and then improves the output SNR, resulting in a resonance-like behavior. However, the output SNR can never exceed the input SNR for a weak signal corrupted by Gaussian noise, i.e., an output-input SNR gain less than unity. Here, Eq. (24) generalizes this conclusion to *arbitrary* noise types of $\xi(t)$ for processing a weak periodic signal. We note that the maximum SNR gain, that is Fisher information $I(f_{\xi_0})$, can be only achieved by the locally optimum nonlinearity $g_{\rm opt}(\xi) = Cf_\xi'(\xi)/f_\xi(\xi)$ indicated in Eq. (24). The standardized Gaussian noise distribution is with $\alpha = 2$ and a minimal Fisher information $I(f_{\xi_0})$ of unity [42,43]. For processing a weak signal in Gaussian white noise, the optimum processor is the matched filter. For other suboptimal nonlinear systems that might show the stochastic resonance effect, the SNR gain can never exceed the bound of $I(f_{\xi_0}) = 1$.

Since the locally optimum nonlinearity $g_{\rm opt}(\xi)$ might be unavailable for unknown noise PDF, the suboptimal nonlinear systems are frequently employed [5,10,18,19,43]. For a weak signal buried in generalized Gaussian noise, the considered nonlinear system of Eq. (17) is suboptimal. We note that the maximum of SNR gain of G_∞ will never catch up, but it can come close to the upper limit of $I(f_{\xi_0})$. Thus, the possibility of the SNR gain G_∞ beyond unity can be expected for the non-Gaussian noise distribution with a larger Fisher information $I(f_{\xi_0}) > 1$. For instance, as the exponent $\alpha = 8$, the corresponding Fisher information $I(f_{\xi_0}) = 2.55$. In the considered suboptimal system of Eq. (17), the maximum of SNR gain $G_\infty = 2.36 \leqslant I(f_{\xi_0})$, even with the help of the optimal noise level and the array size N approaching infinite, as shown in Fig. 4 with the square $[I(f_{\xi_0})]$ and asterisk (G_∞). This results accord with the opinion of stochastic resonance in Ref. [4].

In each nonlinear element, the internal uniform noise plays a part to enhance the SNR gain, resulting in an explicit expression of G_N in Eq. (20). It is proved that for the addition of internal noise to a given weak signal, the output SNR of an arbitrary memoryless nonlinearity is bounded by the Fisher information $I(f_z)$ of the composed noise distribution [43,44]. Besides, it is known that the Fisher information quantities satisfy the inequality $I(f_z) \leqslant \min[I(f_\xi), I(f_\eta)]$ [45,46]. Thus, the internal noise type with a larger Fisher information $I(f_\eta) > I(f_\xi)$ is preferable. Here, we assume that the external noise has a fixed RMS amplitude of σ_ξ, but a varying $I(f_\xi)$ for various distributions. In line with this point, the uniform noise with an infinite Fisher information of $I(f_\eta) = \sigma_\eta^{-2} I(f_{\eta_0}) = \sigma_\eta^{-2}\alpha^2\Gamma(3\alpha^{-1})\Gamma(2-\alpha^{-1})/\Gamma^2(\alpha^{-1}) = \infty$ $(\sigma_\eta > 0$ and

$\alpha = \infty$) is a potential option [20,43,44]. Furthermore, for such an array of threshold-based nonlinearities, finding the optimal distribution of internal noise to maximize the output SNR deserves to be further studied.

B. Array vibration multiresonance by high-frequency sinusoidal vibrations

In practice, the internal noise type or level might be difficult to control. We hereby replace the internal noise terms by the high-frequency sinusoidal vibrations

$$\eta_n(t) = A_\eta \sin(2\pi f_n t), \tag{25}$$

in each element for the improvement of the output-input SNR gain of an array. Here, the vibrations $\eta_n(t)$ have the common amplitude A_η, but different frequencies f_n for $n = 1,2,\ldots,N$. In this case, the nonstationary expectation can be expressed as

$$\mathrm{E}[y(t)] = \frac{1}{N}\sum_{n=1}^{N}\mathrm{E}[y_n(t)], \tag{26}$$

and

$$\mathrm{E}[y_n(t)] = \int_{-\infty}^{\infty} g[x + s(t) + \eta_n(t)] f_\xi(x) dx. \tag{27}$$

For index $n,m = 1,2,\ldots,N$, the second moment

$$\mathrm{E}[y_n(t)y_m(t)] = \int_{-\infty}^{\infty} g(x + s + \eta_n) g(x + s + \eta_m) f_\xi(x) dx. \tag{28}$$

Therefore, the nonstationary variance

$$\begin{aligned}\mathrm{var}[y(t)] &= \mathrm{E}[y^2(t)] - \mathrm{E}^2[y(t)] \\ &= \frac{1}{N^2}\sum_{n=1}^{N}\sum_{m=1}^{N}\{\mathrm{E}[y_n(t)y_m(t)] - \mathrm{E}[y_n(t)]\mathrm{E}[y_m(t)]\}.\end{aligned} \tag{29}$$

Substituting Eqs. (26)–(29) into Eq. (10), the output-input SNR gain G_N can be also calculated.

The output-input SNR gain G_N is illustrated in Fig. 5 as a function of the sinusoidal vibration amplitude A_η for different array sizes $N = 1,2,5,10$, and ∞. Here, the input signal $s(t) = 0.01\sin(2\pi t/T)$, and the external generalized Gaussian noise $\xi(t)$ is with RMS amplitude $\sigma_\xi = 1/\sqrt{3}$ and exponent $\alpha = 8$. In each element, the sinusoidal vibration frequencies take $f_n = (20 + n)/T$ for $n = 1,2,\ldots,N$. It is seen in Fig. 5 that as the vibration amplitude A_η increases, the multiple vibrational resonance effect occurs for $N \geqslant 1$. This method of injecting high-frequency sinusoidal vibrations can also obtain the maximal SNR gain G_N that exceeds unity. For instance, the maximal $G_\infty = 2.32$ at $A_\eta = 0.15$, as shown in Fig. 5 by an asterisk mark. Moreover, it is noted in Fig. 5 that the locally maximal $G_\infty = 1.17$ (marked by a downward triangle) is still larger than unity at $A_\eta = 2.06$. While, in Fig. 1(a) by the addition of uniform noise, the second resonant peak value of G_N are always less than unity.

The mechanism of vibrational multiresonance is similar to the stochastic multiresonance. Actually, the sinusoidal vibration modulation can be viewed as a variable with the

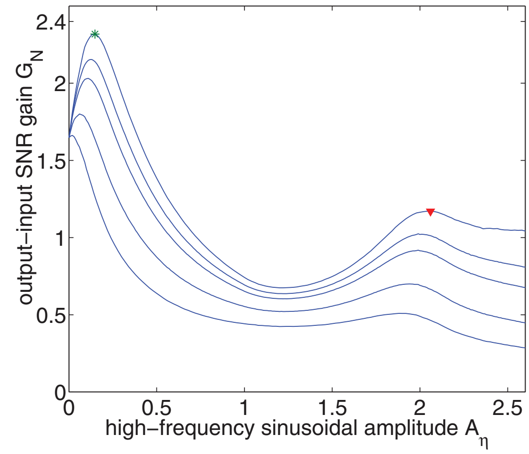

FIG. 5. (Color online) Output-input SNR gain G_N of Eq. (10) versus the sinusoidal vibration amplitude A_η for different array sizes $N = 1,2,5,10$, and ∞ (from the bottom up). For $N = \infty$, the stationary variance $\langle \mathrm{var}[y(t)]\rangle$ is calculated by Eq. (31). Here, the external generalized Gaussian noise $\xi(t)$ is with RMS amplitude $\sigma_\xi = 1/\sqrt{3}$ and exponent $\alpha = 8$. The sinusoidal vibration frequencies $f_n = (20 + n)/T$ for $n = 1,2,\ldots,N$. The input signal $s(t) = 0.01\sin(2\pi t/T)$.

distribution $f_\eta(u) = 1/(\pi\sqrt{A_\eta^2 - u^2})$ over the interval $u \in [-A_\eta, A_\eta]$. The roles of internal noise components are filled by the sinusoidal vibrations. Therefore, the output-input SNR gain also presents the multiple peaks as the sinusoidal vibration amplitude A_η increases.

For the method of adding noise and array size $N = \infty$, Eq. (12) becomes

$$\begin{aligned}\lim_{N\to\infty}\mathrm{E}[y^2(t)] &= \frac{1}{N}\mathrm{E}_\xi\left\{\mathrm{E}_\eta\left[y_n^2\right] + (N-1)\mathrm{E}_\eta[y_n y_m]\right\}\ (\forall n \neq m) \\ &= \mathrm{E}_\xi\{\mathrm{E}_\eta[y_n y_m]\}.\end{aligned} \tag{30}$$

It is noted that the indices n and m are different, but arbitrary in Eq. (30), thus we can adopt two elements, each embedded with independent internal noise components η_n and η_m, to evaluate the SNR gain of a parallel array with size $N = \infty$, as shown in Fig. 1(a). This method is tractable and effective. Similarly, for the approach of injecting sinusoidal vibrations and based on Eq. (29), the stationary variance can be calculated as

$$\begin{aligned}&\lim_{N\to\infty}\langle \mathrm{var}[y(t)]\rangle \\ &= \frac{1}{N^2}\left\{\sum_{n=1}^{N}\langle \mathrm{E}\left[y_n^2(t)\right] - \mathrm{E}^2[y_n(t)]\rangle + \sum_{n=1}^{N}\sum_{m=1}^{N}\langle \mathrm{E}[y_n(t)y_m(t)]\right. \\ &\quad \left. - \mathrm{E}[y_n(t)]\mathrm{E}[y_m(t)]\rangle\right\}(\forall n \neq m) \\ &= \langle \mathrm{E}[y_n(t)y_m(t)]\rangle - \langle \mathrm{E}[y_n(t)]\mathrm{E}[y_m(t)]\rangle.\end{aligned} \tag{31}$$

Therefore, via two arbitrary sinusoidal vibrations η_n and η_m, we can also evaluate the array SNR gain in Eq. (10) with size $N = \infty$, as shown in Fig. 5.

IV. CONCLUSION

In this paper, we report the double-peak resonance effect in an array of threshold elements, which can be achieved by two methods of injecting internal noise components or high-frequency sinusoidal vibrations. In the limit of weak signals, we analytically show the occurrence condition of array stochastic multiresonance, and obtain the maximal SNR gain greatly larger than unity for a small signal buried in non-Gaussian noise. Since the method of adding noise in an array might be not always practical, we inject high-frequency sinusoidal vibrations into arrays to enhance the output-input SNR gain. The vibrational resonance effect is also observed. The output-input SNR gain not only can be enhanced by the increase of the sinusoidal vibration amplitude and the array size, but also exhibits two maxima at multiple values of the vibrational modulation amplitude. We argue that this easily implemented method will be valuable for nonlinear signal processing, and deserves to be further studied extensively. For instance, besides the considered threshold-based characteristics, it is also interesting to further explore the possibility of multiple resonance in other electronics devices or a bundle of sensory excitable neurons, wherein the sinusoidal vibrations are exploited to optimize the output SNR of an array of nonlinear elements.

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