Loop-erased random walk on a percolation cluster is compatible with Schramm-Loewner evolution

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We study the scaling limit of a planar loop-erased random walk (LERW) on the percolation cluster, with occupation probability $p \geqslant p_c$. We numerically demonstrate that the scaling limit of planar LERW_p curves, for all $p > p_c$, can be described by Schramm-Loewner evolution (SLE) with a single parameter κ that is close to the normal LERW in a Euclidean lattice. However, our results reveal that the LERW on critical incipient percolation clusters is compatible with SLE, but with another diffusivity coefficient κ. Several geometrical tests are applied to ascertain this. All calculations are consistent with SLE_κ, where $\kappa = 1.732 \pm 0.016$. This value of the diffusivity coefficient is outside the well-known duality range $2 \leqslant \kappa \leqslant 8$. We also investigate how the winding angle of the LERW_p crosses over from Euclidean to fractal geometry by gradually decreasing the value of the parameter p from 1 to p_c. For finite systems, two crossover exponents and a scaling relation can be derived. This finding should, to some degree, help us understand and predict the existence of conformal invariance in disordered and fractal landscapes.

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I. INTRODUCTION

Anomalous diffusion in disordered media is a ubiquitous phenomenon in nature, ranging from physics and chemistry to biology and medicine [1,2]. The main feature of anomalous diffusion in disordered media is the fact that the mean square displacement of the diffusing species has a nonlinear relationship with time [3]. Such disordered media are typically simulated through percolation systems; diffusion on percolation clusters has been studied in great detail [4,5]. One could restrict the diffusion of a simple random walk (RW) to the incipient infinite cluster. It is known that, above criticality $p > p_c$, diffusion is anomalous over short distances and normal over long distances [5]. However, diffusion on critical incipient percolation clusters is anomalous on all length scales. On the other hand, one could erase the loops from the trajectory of the RW; chronologically this operation results in the loop-erased random walk (LERW) [6]. This model is equivalent to the uniform spanning trees [7], the q-state Potts model in the limit $q \to 0$ [8], and the avalanche frontier in the Abelian sandpile model [9]. It is known that the fractal dimension of the LERW in $D = 2$ is $5/4$. Although the scaling and universality class of the LERW in an integer lattice are known, the universality class of this model in the fractal landscape and especially in critical percolation has not been hitherto studied.

In addition to scale invariance and, consequently, fractal properties, it is well known that the two-dimensional (2D) LERW is conformally invariant. This property causes the measure of such 2D random curves to remain unchanged under transformations that preserve angles. A recent breakthrough of the complex analysis has created a powerful tool for statistical characterization of conformal invariance of many discrete models in the scaling limit (see, e.g., [10], and references therein). In this approach, now called Schramm-Loewner evolution (SLE), each random non-self-crossing curve, which possesses conformal invariance and the domain Markov property, is mapped to a 1D Brownian motion on the real axis. Such Brownian motion has zero mean value and its variance grows linearly in time with a real positive coefficient κ known as diffusivity [11]. In this approach, all statistical properties of such 2D random curves (such as critical exponents and fractal dimension) can be obtained as functions of κ [10,12]. Also due to a well-established relation between SLE and conformal field theory (CFT), a relationship between such 2D random curves and CFT models is possible [10,13,14]. So far, the SLE approach has been identified and studied theoretically and numerically in different statistical models such as in critical percolation [15], self-avoiding walks [16], the Ising model [17], spin glasses [18,19], the watershed [20], and turbulence [21], as well as some other disordered models [22–25]. In particular, it has been proved that the scaling limit of the LERW in a simply connected domain converges to SLE_2 [11,26,27]. Establishing SLE for such systems has provided valuable information on the underlying symmetries and paved the way to some exact results [15,17,28,29]. In fact, SLE is not a general property of non-self-crossing walks since many curves have been shown not to be SLE (see, for example, [30]).

Recently, the scaling behavior of the LERW on percolation cluster was investigated [31]. As it has been rigorously proven, the scaling behavior of planar LERW_p, for all $p > p_c$, is the same as the LERW on Euclidean lattices [32]. However, the LERW on critical percolation clusters scales with a fractal dimension $d_f = 1.217 \pm 0.002$ [31]. This fractal dimension clearly shows that this model is related to a family of curves appearing in different contexts such as the watershed of random landscapes [33–35], polymers in strongly disordered media [36], invasion percolation [37], bridge percolation [33], and optimal path cracks [38].

By assuming translation, rotation, and scaling invariance for two-dimensional LERW_p on the percolation cluster, the question arises as to whether SLE can be identified in such random curves. In the continuum limit of a two-dimensional LERW_p on the percolation cluster one can check consistency with the SLE process. The fractal dimension of the SLE_κ curves is related to the diffusivity by the relation $d_f = \min\{2, 1 + \frac{\kappa}{8}\}$ [10,12,39]. If the LERW_p is described

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by the SLE process, then the diffusivity of them is given with the same relationship. Although it has been reported that the scaling limit of watersheds can be described by SLE [20], it does not directly imply that the LERW on critical percolation is compatible with SLE because of the need for further conformal invariance and a domain Markov property.

In this paper we study the LERW on the percolation cluster, with occupation probability above and equal to the critical value $p \geqslant p_c$. Our results show that for all $p > p_c$, the scaling limit of obtained LERW$_p$ curves is close to the exact results for the LERW on Euclidean lattices first proposed by Schramm [11]. To study the scaling limit of LERW$_p$ in two dimensions and compare it with SLE$_\kappa$, we carried out three different statistical evaluations, namely, the variance of the winding angle (quantifying the angular distribution of the curves) [40,41], the left-passage probability [11,42], and the characterization of the driving function (direct SLE) [19]. We find that above the percolation threshold, i.e., $p > p_c$, all statistical evaluations are consist with $\kappa = 2$. However, the LERWs on critical percolation are SLE curves of diffusivity $\kappa = 1.732 \pm 0.016$. We simulate the LERW$_p$ on the percolation cluster as described in [31]. Then we show that the values of κ independently obtained for each test are numerically consistent and in line with the fractal dimension of the LERW on the critical percolation cluster. Hereafter, we discuss each analysis separately.

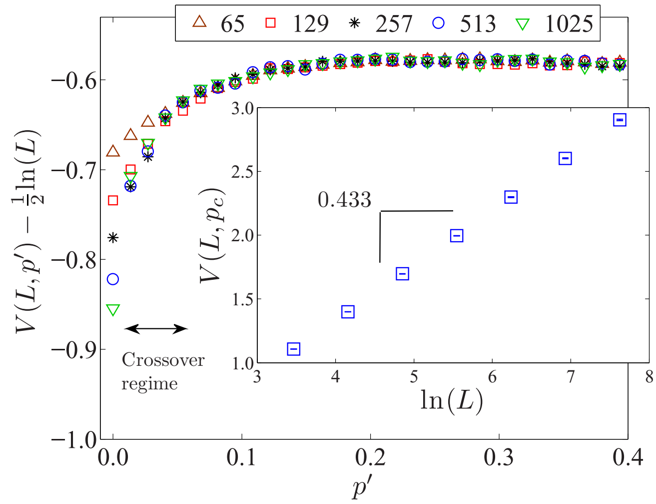

FIG. 1. (Color online) Deviation of the winding angle variance of the LERW$_p$ on the percolation cluster from the normal LERW variance, i.e., $V(L,p') - (\kappa/4)\ln L$ as a function of p'. The inset shows the dependence of the winding angle variance on the lateral size of the lattice L for the LERW on the critical percolation cluster (the statistical error bars are shown, but are quite shorter and appear as horizontal lines). The slope in the linear-logarithmic plot corresponds to $\kappa/4 = 0.433 \pm 0.004$.

II. WINDING ANGLE STATISTICS

It is known that the winding angle distribution around a point for a 2D conformally invariant random curve can be related to the Coulomb-gas parameter g (which is directly related to the central charge $c = 1 - \frac{g(1-g)^2}{g}$ [43]) and the system size L [40]. The correspondence of the Coulomb-gas parameter g to the relation for the winding angle variance can be extended to SLE [41]. So we can test conformal invariance of the LERW$_p$ on the percolation cluster and consistency with the SLE description by measuring the winding angle variance, as defined in [20,41]. The variance of the winding angle over all edges in the curve $V(L,p)$ increases with the system size as $V(L,p) = b(p) + (\kappa/4)\ln L$, where $b(p)$ is a constant that depends on the details of the definition [41]. To measure the winding angle variance, we performed simulations for different lattice sizes $L = 2^{4+n}$ for $n = 1,2,\ldots,6$. We generated 10^6 LERW curves for small systems and more than 2×10^4 for the largest one. In the case of a normal LERW ($p = 1$), the winding angle variance of the curves logarithmically increases with system size as $V(L) \sim \frac{1}{2}\ln L$, consistent with the $\kappa = 2$ of the Euclidean LERW. By decreasing the occupation probability p, the diffusivity coefficient of these random curves remains unchanged. At percolation threshold, these curves are smoother than normal LERWs and the winding angle variance increases logarithmically with the system size with different slope $V(L) \sim \frac{\kappa}{4}\ln L$, with $\kappa \approx 1.7$. Figure 1 shows the dependence of the $V(L) - \frac{1}{2}\ln L$ on p' for different system sizes, where p' is $p - p_c$. The overlap of the different curves confirms that the diffusion coefficient of the LERW$_p$ above p_c is 2. A small deviation is observed due to finite-size effects. There is a crossover between two different regimes near the critical point $p \gtrsim p_c$, which can be observed in Fig. 1.

At critical percolation, to obtain a more precise numerical estimation of κ, we increased the system size to 2^{11}. The winding angle variance of the LERW on critical percolation for different lattice size L is shown in the inset of Fig. 1. We observe a slope of 0.433 ± 0.004 in a linear-logarithmic plot, which means that the diffusivity is $\kappa = 1.732 \pm 0.016$. This is in good agreement with the fractal dimension formula for SLE, i.e., $d_f = 1 + \kappa/8$ [44].

III. CROSSOVER SCALING FUNCTION

As shown in Fig. 1, the winding angle variance of LERW$_p$ increases with increasing occupation probability. For large systems, the winding angle variance of LERW$_p$ grows with p' such that $V(L,p') \sim \beta\ln(p')$, where $\beta \approx 0.09$ is a different coefficient, which we call variance-growth coefficient. There is a crossover behavior from Euclidean to fractal geometry [31]. Here we try to investigate how the winding angle variance of the LERW$_p$ crosses over between these two universality classes by decreasing the value of the parameter p from 1 to p_c. For the complete crossover scaling of the winding angle variance, V can be considered as a logarithm of a homogeneous function on the relevant scaling fields $V(bL, b^{y_p}p') = y_v\ln b + V(L,p')$, where b is a scaling parameter and y_v and y_p are relevant exponents for V and p scaling parameters, respectively. One could restrict attention to the $p \to p_c$ regime; then for a finite size of L, it is expected that V increases with $\frac{\kappa}{4}$ slope, so in this regime $y_v = \frac{\kappa}{4}$. The next exponent can be found by trying to collapse the data (setting $b = L^{-1}$). The scaling ansatz for the winding angle variance is given by

$$V(L,p') = \ln(L^{\kappa/4}\mathcal{G}[p'L^{\theta}]), \quad (1)$$

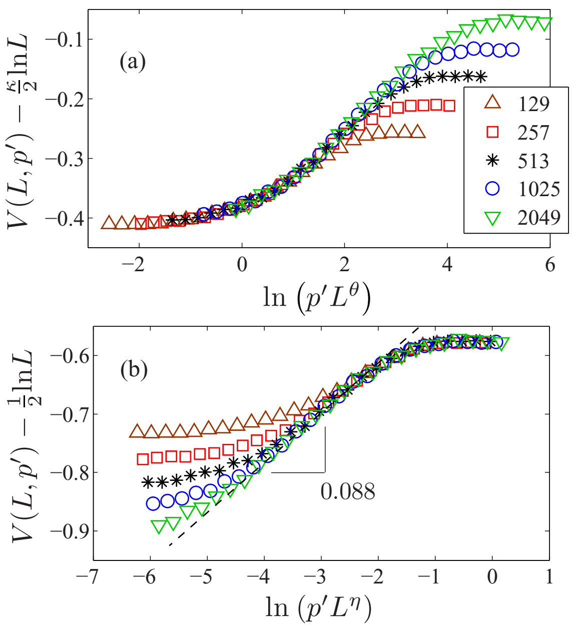

FIG. 2. (Color online) Crossover scaling and data collapse for LERW_p for different system sizes. (a) Deviation of the winding angle variance of the LERW_p on the percolation cluster from the variance of the LERW on the critical percolation cluster, i.e., $V(L,p') - (0.433)\ln L$ versus $\ln(p'L^\theta)$ for different system sizes. The scaling function given by Eq. (1) is applied, with $\theta = 0.89 \pm 0.05$. (b) Deviation of the winding angle variance of the LERW_p on the percolation cluster from the normal LERW variance, i.e., $V(L,p') - \frac{1}{2}\ln L$ versus $\ln(p'L^\eta)$, with $\eta = 0.14 \pm 0.03$, for different system sizes. For each finite lattice size L, three regimes can be obtained with two crossover exponents, i.e., θ and η. A more precise estimate for β can be obtained by the data collapsing for different lattice sizes in the intermediate regime, which is $\beta = 0.088 \pm 0.004$. All results have been averaged over 4×10^4 samples.

where $\mathcal{G}[u]$ is a scaling function such that $\mathcal{G}[u] \sim u^\beta$ for small values of u and is nonzero at $u \to 0$. The exponent $\theta = y_p$ is the crossover exponent in the $p \to p_c$ regime. Figure 2(a) shows crossover scaling for different lattice sizes, close to the critical point. As shown, we have a good data collapse for small values of u with $\theta = 0.89 \pm 0.05$. For each finite lattice size L, there is a crossover point such that $p'_{\times_1}$ scales like $L^{-\theta}$, which for $u \ll 1$ we have a saturation regime, and for $u \gg 1$ results are consistent with $\beta\ln(u)$ for all lattice sizes L. However, for large values of $p'L^\theta$, we do not observe data collapse and the winding angle variance behaves as $\frac{2-\kappa}{4}\ln(L)$. On the other hand, for large values of p, it is expected that the winding angle variance behaves like Euclidean geometry, so $y_v = \frac{1}{2}$ in this regime. If we follow the same strategy as above, we could find another scaling function

$$V(L,p') = \ln(L^{1/2}\mathcal{F}[p'L^\eta]), \tag{2}$$

where the scaling function $\mathcal{F}[x]$ has a saturation regime for large values of x and the exponent $\eta = y_p$ is the corresponding crossover exponent in this regime. In fact, we could find another crossover point $p'_{\times_2}$ scaling with $L^{-\eta}$ for which the winding angle variance behaves like $\ln(\mathcal{F}[x]) \sim \beta\ln(x)$ for $x \ll 1$ and is a constant value for $x \gg 1$. Figure 2(b) shows the scaling behaviors for different lattice sizes L. As shown, we have a good data collapse with $\eta = 0.14 \pm 0.03$, which clearly shows that the argument of $p'L^\eta$ at the crossover point should be independent of lattice size, so the crossing probability $p'_{\times_2}$ scales like $L^{-\eta}$ with system size. The overlap of the different curves confirms that the diffusion coefficient of the LERW_p above p_c is 2. Three different regimes, as shown in Fig. 2, are clearly identified: For $p' < p'_{\times_1}$ the winding angle variance behaves like $V \sim (0.433)\ln(L)$; for $p'_{\times_1} < p' < p'_{\times_2}$, S has a logarithmic behavior as $\beta\ln(p')$; and for $p'_{\times_2} < p'$, S behaves with a Euclidean exponent, i.e., $\sim\frac{1}{2}\ln(L)$. Therefore, the following relation can be derived:

$$\beta(\theta - \eta) = \tfrac{1}{4}(2 - \kappa), \tag{3}$$

which is in good agreement with our numerical values obtained for the exponents. Interestingly, by considering $\kappa = 8(d_f - 1)$ for SLE curves, this relation is consistent with the reported scaling relation for a mean total length of LERW_p [31].

IV. LEFT-PASSAGE PROBABILITY

By considering the scale invariance of SLE_κ curves in the upper half plane $\mathbb{H}$, one can determine the probability that a point $Re^{i\phi}$ is on the right side of the curve [see Fig. 3(b)]. This probability only depends on ϕ and is given by Schramm's

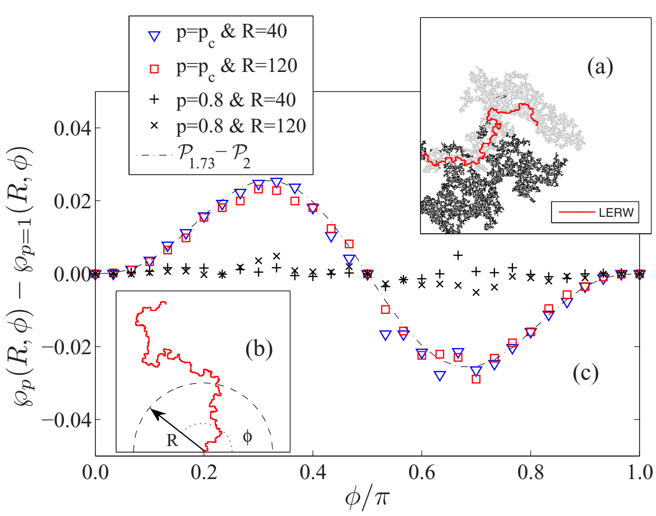

FIG. 3. (Color online) Left-passage probability for LERW_p on percolation clusters as a function of polar angle. (a) The LERW on the critical percolation cluster (shown in black) on a 513×513 lattice (the visited sites are shown in gray). (b) Schematic representation of the left-passage definition (details in the text) on the obtained curve after rotation and translation. (c) Plot of $\wp_p(\phi,R) - \wp_{p=1}(\phi,R)$ for the LERW in the upper half plane where $\wp_p(\phi,R)$ is the probability for the LERW on the percolation cluster to pass to the left of a point with polar coordinates (R,ϕ). At $p = p_c$ the results are in good agreement with $\mathcal{P}_{1.73} - \mathcal{P}_2$, where $\mathcal{P}_{1.73}(\phi)$ is the left-passage probability for $\text{SLE}_{1.73}$ given by Schramm's formula (4). The magnitude of statistical errors (not shown) is consistent with the apparent fluctuations of the data lines. The results are averages over 5×10^4 curves on a square lattice with $L = 513$.

formula [11]

$$\mathcal{P}_\kappa(\phi) = \frac{1}{2} + \frac{\Gamma\left(\frac{4}{\kappa}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{8-\kappa}{2\kappa}\right)}\cot(\phi)\,{}_2F_1\left(\frac{1}{2};\frac{4}{\kappa},\frac{3}{2};-\cot^2(\phi)\right), \tag{4}$$

where ${}_2F_1$ is the hypergeometric function and Γ is the Gamma function. This formula is valid only for the upper half plane domain where the SLE curve starts from the origin and goes to infinity, i.e., chordal SLE. In order to simulate a LERW on the upper half plane, it can be obtained by some rotations and translations of the curves described above on the whole plane to the upper half plane (for details see [45]). In Figs. 3(a) and 3(b) a random LERW curve before and after transformations is shown. It is important to note that these curves are not chordal completely due to restriction of going to the middle of the lattice; however, they behave chordally near their starting points [46]. We measure the left passage probability $\wp_p(\phi,R)$ for the LERW$_p$ curves for three different occupation probabilities $p = 1$, 0.8, and p_c. We can reduce both the finite-size effects and other effects that are related to not being of chordal type by comparing them with a normal LERW on the same lattice size, i.e., $\Delta\wp(\phi,R,p) = \wp_p(\phi,R) - \wp_{p=1}(\phi,R)$. Our results for $p = 0.8$ (as an example in the Euclidean regime) and $p = p_c$ are shown in Fig. 3. The comparison of the left-passage probability of the LERW curves on critical percolation, i.e., $\Delta\wp(\phi,R,p_c)$, is in good agreement with $\mathcal{P}_{1.73}(\phi) - \mathcal{P}_2(\phi)$. As shown in Fig. 3, this quantity is independent of the R values and consequently our results for the LERW on critical percolation are consistent with the SLE$_{1.73}$.

V. DIRECT SLE TEST

Consider a random non-self-crossing SLE curve $\gamma(t)$, which starts at a point on the real axis and grows to infinity inside a region of the upper half plane $\mathbb{H}$. We parametrize the curve with the dimensionless parameter t, typically called Loewner time. At each time t, the $\mathbb{H}$ minus the curve $\gamma(t)$ can be mapped back to the $\mathbb{H}$ by a unique function $g_t(z)$, where z is a point on $\mathbb{H}$ (its real and imaginary parts are denoted by Rez and Imz, respectively). This function satisfies the Loewner equation [47]

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \xi_t}, \tag{5}$$

where the initial condition is $g_{t=0}(z) = z$ and ξ_t is a continuous real-valued function called the driving function. The driving function is proportional to the Brownian motion B_t, i.e., $\xi_t = \sqrt{\kappa}B_t$, if and only if the probability measure of $\gamma(t)$ satisfies conformal invariance and the domain Markov property [11,42]. This type of conformal curve is known as chordal SLE$_\kappa$. In addition to the chordal SLE there is another type of SLE known as dipolar that joins the origin to a point on the line Im$z = \pi$ (in the strip geometry), which is described by a Loewner-type equation [48]. As discussed before, the transformed LERW$_p$ are dipolar curves, which start at one point on the lower boundary and end when they touch a point on $y = L/2$, for the first time. We scale the LERW$_p$ curves by a factor $2\pi/L$ to be in a strip $0 < \mathrm{Im}z < \pi$. To more carefully inspect the correspondence with SLE, we use a discrete version of the Loewner equation for dipolar random curves to map the transformed LERW$_p$, represented by sequences of points z_i, $i = 1,2,\ldots,N$, onto a real-valued sequence $\xi(t_i)$ defined at discrete t_i. Initially, we set $t = 0$ and $\xi(0) = 0$. We then apply a sequence of slit maps obtained by considering of a piecewise constant for the driving function at each step. At each iteration i, we map the point z_i to the real axis at ξ_i defined at discrete $t_i = t_{i-1} + \delta_i$ and we transform other points z_j (for $j > i$) of the curve using the map appropriate for dipolar SLE [19,48],

$$\delta t_i = -2\ln\left[\cos\left(\frac{\mathrm{Im}z_i}{2}\right)\right], \quad \xi_i = \mathrm{Re}z_i,$$
$$z_j = \xi_i + 2\cosh^{-1}\left\{\cosh\left[\frac{(z_j-\xi_i)}{2}\right]\Big/\exp(-\delta t_i/2)\right\}. \tag{6}$$

This map converges to the exact one for vanishing δt [14]. Here we restrict our attention to the LERW on the critical percolation cluster. For comparison, we also study the LERW on a Euclidean lattice (i.e., $p = 1$). We take 4×10^4 disorder realizations of the LERW in a lattice with $L = 1025$ for $p = 1$ and $p = p_c$. The average over realizations of the disorder of $\langle\xi^2(t)\rangle$ versus Loewner time t in a dipolar LERW is plotted. The obtained diffusion coefficients are $\kappa = 1.68 \pm 0.07$ and $\kappa = 1.94 \pm 0.07$ for $p = p_c$ and $p = 1$, respectively. To confirm the Gaussianity of the driving function ξ_t, the probability distribution for the rescaled driving function $X = \xi(t)/\sqrt{\kappa t}$ for two different times for the LERW is plotted in the inset of Fig. 4. This result indicates that the statistics

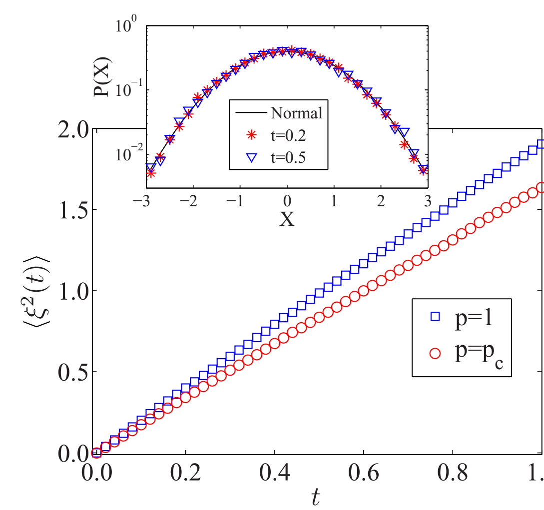

FIG. 4. (Color online) Statistics of the obtained driving function for LERW$_p$: the second moment of the driving function $\langle\xi^2(t)\rangle$ versus Loewner time t for a normal LERW (i.e., $p = 1$) and for a LERW on the critical percolation cluster. The obtained diffusion coefficients are $\kappa = 1.68 \pm 0.07$ and $\kappa = 1.94 \pm 0.07$ for $p = p_c$ and $p = 1$, respectively. The inset shows the probability distribution of the driving function at two different Loewner times for LERWs on the critical percolation cluster. The rescaled parameter X is defined as $X = \xi(t)/\sqrt{\kappa t}$, where we have taken $\kappa = 1.68$. The solid line is the normal distribution of the vanishing mean value and unit dispersion. Results are averages over 4×10^4 realizations on a square lattice with $L = 1025$ for both cases.

of $\xi(t)$ converges to a Gaussian process with zero mean and 1.68 ± 0.07 variance, in good agreement with the results discussed above. We also study the correlation function $C(n) = \langle[\xi(t_{i+n+1}) - \xi(t_{i+n})][\xi(t_{i+1}) - \xi(t_i)]\rangle$ at intermediate times to test the Markovian property for $\xi(t)$; it decays rapidly for both cases. It is also important to remark that the slit map goes to real mapping only in the continuum limit and converges for sufficiently small δt_i [14]. Due to these strong discretization effects, the numerical results obtained with the direct SLE method are less precise than the other two methods (winding angle and left-passage probability), as is well known in the literature [19–21]. For both the fractal ($p = p_c$) and Euclidean geometries ($p > p_c$), within the error bars, the results we have obtained for κ are in agreement with the ones obtained with the fractal dimension, winding angle, and left-passage probability.

VI. SUMMARY AND DISCUSSION

In this paper we mainly studied the scaling limit of LERW_p on a percolation cluster, with occupation probability above and equal to the critical value $p \geqslant p_c$. The SLE tests indicated that the scaling limit of this model for $p > p_c$ is SLE_2. Although our study in this regime was restricted to a few points, it was recently shown in Ref. [32] that if the scaling limit of the RW on a planar graph is planar Brownian motion, such as a percolation cluster for $p > p_c$, then the scaling limit of its loop erasure is SLE_2, which confirms our results in this regime. However, LERWs on critical percolation are likely SLE curves with $\kappa = 1.732 \pm 0.016$. This value, similar to our recent finding in a watershed model [20], is outside the well-known duality conjecture range $2 \leqslant \kappa \leqslant 8$.

Near the percolation threshold p_c there is a crossover regime, shown in Fig. 1, from Euclidean to fractal geometry. To achieve a better understanding of this regime, we have also investigated how the winding angle of the LERW_p crosses over between these two universality classes by gradually decreasing the value of the parameter p from 1 to p_c. Our findings for the crossover regime, shown in Fig. 2, clearly demonstrate that for finite systems, two crossover exponents and a scaling relation can be derived.

A well-known relation between the central charge of conformal models that possess a second-level null vector in their Verma module and the diffusivity κ is $c = (3\kappa - 8)(6 - \kappa)/2\kappa$ [14]. If the LERW on the critical percolation cluster is conformally invariant it likely corresponds to a logarithmic CFT with central charge $c = -3.45 \pm 0.10$. In particular, the LERW on a Euclidean lattice is believed to have $c = -2$. It is also noteworthy that negative central charges have been reported in different contexts, e.g., stochastic growth models, $2D$ turbulence, and quantum gravity [49]. However, the conformal invariance of the LERW_p on the percolation cluster cannot be comprehended as strong proof. Nevertheless, if such invariance is established, it becomes possible to develop a field theory for this universality class. Moreover, due to the connection between the LERW and other important statistical models, and also some mathematical constructions for this model, it is possible to find exact results regarding the existence of conformal invariance and scaling properties. In addition to the conformal symmetry, the LERW_p curves must possess a domain Markov property in the scaling limit to be SLE. However, the direct numerical examination of the domain Markov property is an extremely challenging task and only a few numerical studies have tested it nonrigorously [19]. Here we did not attempt to check the domain Markov property of the LERW_p curves. Instead, we simply tested the Markovian property of $\xi(t)$; since at each time t there is a unique conformal map that takes the LERW curve to a real function $\xi(t)$ on the real axis, it is expected that the Markovian property of $\xi(t)$ is as a result of the domain Markov property of LERW_p.

The connection between SLE and statistical properties of LERW_p provides an alternative perspective to look at such a random path and to build bridges between connectivity in disordered media and other research areas in mathematics, percolation, and quantum field theory. This work opens up several challenges. Besides the need to examine directly both the conformal invariance and the domain Markov property, it would be interesting to formulate a CFT scheme in a fractal geometry. Finally, the scaling limit of the LERW obtained from an anomalous diffusion on the fractal landscape is still an important open question.

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[46] The transformed curve is dipolar, which starts from the origin and goes to a random point on $y = L/2$, so our measurement should be done only for a small radius for dipolar curves as discussed in [19]. Our results are independent of R values for $R < L/4$.

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