PHYSICAL REVIEW E **90**, 022121 (2014)

Asymptotic properties of a bold random walk

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(Received 13 June 2014; published 19 August 2014)

In a recent paper we proposed a non-Markovian random walk model with memory of the maximum distance ever reached from the starting point (home). The behavior of the walker is different from the simple symmetric random walk only when she is at this maximum distance, where, having the choice to move either farther or closer, she decides with different probabilities. If the probability of a forward step is higher than the probability of a backward step, the walker is bold and her behavior turns out to be superdiffusive; otherwise she is timorous and her behavior turns out to be subdiffusive. The scaling behavior varies continuously from subdiffusive (timorous) to superdiffusive (bold) according to a single parameter $\gamma \in R$. We investigate here the asymptotic properties of the bold case in the nonballistic region $\gamma \in [0,1/2]$, a problem which was left partially unsolved previously. The exact results proved in this paper require new probabilistic tools which rely on the construction of appropriate martingales of the random walk and its hitting times.

DOI: 10.1103/PhysRevE.90.022121 PACS number(s): 05.40.−a, 02.50.Ey, 89.75.Da

I. INTRODUCTION

The appellative anomalous diffusion is associated to a scaling relation $E[x^2(t)] \sim t^{2\nu}$ with $\nu \neq 1/2$. It may arise in random walks via diverging steps length, as in Lévy flights [1] or via long-range memory as in self-avoiding random walks [2,3]. Diverging steps length and long-range memory are two different ways of violating the necessary conditions for the central limit theorem when applied to random walks.

In some cases, the mechanism which gives origin to anomalous scaling can be different, special deterministic or random environments [4,5] or multiparticle interactions [6]. Moreover, diffusion can be strongly anomalous ($E[|x(t)|^q] \sim t^{q\nu}$ with ν depending on q) in complex systems [7–9].

There is a very large number of phenomena which exhibit anomalous diffusion as well a variety of models which have been used to describe them (for a review of both see Refs. [10–14]). Nevertheless, exact solutions of nontrivial models with memory are quite scarce [15–21]. Motivated by this lack of exact solutions, we presented in Ref. [22] a model which is exactly treatable although genuinely non-Markovian. The model shows anomalous scaling which can be subdiffusive, superdiffusive, and also ballistic according to a single parameter $\gamma \in R$.

It is the aim of this work to investigate the asymptotic properties in the range $\gamma \in [0,1/2]$, a problem which was left partially unsolved in Ref. [22]. This range corresponds to a nonballistic superdiffusive behavior except at the two extremes were it is ordinary simple symmetric random walk (SSRW) ($\gamma = 0$) and ballistic ($\gamma = 1/2$).

II. MODEL AND RESULTS

The model, as defined in Ref. [22], is one-dimensional, steps all have the same unitary length, time is discrete, and the walker can only move left or right at any time step. The behavior of the random walker is modified with respect to SSRW only when she is at the maximum distance ever reached from her starting point (home). In this case, she decides with different probabilities to make a step forward (going farther from home) or a step backward (going closer to home).

More precisely, the model is the following: the walker starts from home ($x(0) = 0$), then, at any time she can make a (unitary length) step to the right or to the left, i.e.,

$$x(t+1) = x(t) + \sigma(t) \tag{1}$$

with $\sigma(t) = \pm 1$. Then let us define

$$z(t) = \max_{0 \leqslant s \leqslant t} |x(s)|, \tag{2}$$

which is the maximum distance from home she ever attained.

We assume that the walker has no preference for the direction of the first step ($\sigma(0) = \pm 1$ with equal probability), and as well that she has no preference when she is not at the maximum distance ($\sigma(t) = \pm 1$ with equal probability if $|x(t)| < z(t)$). On the contrary, when she is at the maximum distance ($|x(t)| = z(t)$), she can choose to step away from the origin with probability $p[z(t)]$ or toward the origin with probability $1 - p[z(t)]$, i.e., $\sigma(t) = \text{sign}[x(t)]$ with probability $p[z(t)]$ and $\sigma(t) = -\text{sign}[x(t)]$ with probability $1 - p[z(t)]$. It is assumed that the probability $p(z)$ depends on z according to

$$p(z) = \frac{z^\gamma}{1 + z^\gamma}, \tag{3}$$

where $\gamma \in R$.

Therefore, the walker performs a SSRW when $|x(t)| < z(t)$; however, when she is at the maximum distance from home ($|x(t)| = z(t)$), she boldly prefers to move farther if $\gamma > 0$ or timorously prefers to move closer if $\gamma < 0$. Moreover, if her attitude is neutral ($\gamma = 0$), the model is globally SSRW since $p(z) = 1/2$ and all steps are always equally probable.

The main results of both the present paper and Ref. [22], concerning the asymptotic behavior of $z(t)$, are summarized by the following limits which hold in probability for large times ($t \to \infty$):

(i) $\gamma \in (-\infty,0)$: $z(t)/t^\nu \to (1/2\nu)^\nu$ where the scaling exponent is $\nu = 1/(2-\gamma)$,

(ii) $\gamma = 0$: $z(t)/t^{1/2} \to 1/T^{1/2}$ where T is a random variable described below,

(iii) $\gamma \in (0,1/2)$: $z(t)/t^\nu \to (1/2\nu)^{2\nu}/L^\nu$ where $\nu = 1/(2-2\gamma)$ and L is a Lévy variable,

(iv) $\gamma = 1/2$: $z(t)/t \to 1/(4L+1)$ where L is the same Lévy variable,

(v) $\gamma \in (1/2,\infty)$: $z(t)/t \to 1$.

We will show in the next that the Laplace transforms of probability densities of L and T are

$$E[e^{-\lambda L}] = e^{-\sqrt{2\lambda}}, \quad E[e^{-\lambda T}] = \frac{1}{\cosh(\sqrt{2\lambda})}. \tag{4}$$

Accordingly, L is a Lévy variable with parameter 1, and it corresponds to the first hitting time of a barrier in 1 by a continuous Brownian motion with unitary diffusion constant, starting in 0. The variable T is the first exit time from the interval $[-1,1]$ by the same Brownian motion. Explicitly, the probability density of L is

$$\rho(L) = \sqrt{\frac{1}{2\pi L^3}}\,\exp\left(-\frac{1}{2L}\right). \tag{5}$$

Thus, in the range $\gamma \in (0,1/2)$, one has

$$\lim_{t\to\infty} E\left[\left(\frac{z(t)}{t^\nu}\right)^q\right] = \left(\frac{1}{2\nu^2}\right)^{q\nu} \frac{\Gamma(q\nu+1/2)}{\Gamma(1/2)}, \tag{6}$$

where Γ is the gamma function, q is a real positive constant, and $\nu = 1/(2-2\gamma)$.

The subdiffusive range $\gamma \in (-\infty,0)$ and the ballistic range $\gamma \in (1/2,\infty)$ were already fully solved in Ref. [22] where both ν and the exact (constant) values of the limits were determined.

In this paper we focus on the remaining range $\gamma \in [0,1/2]$, which in Ref. [22] was solved only for what concerns the scaling exponent ν.

For $\gamma = 0$ the model reduces to SSRW, all of which is already known. Nevertheless, for sake of comparison, we obtain again here already known results by our method. The range $\gamma \in (0,1/2)$ corresponds to a nonballistic superdiffusive behavior $(1/2 < \nu < 1)$. We prove here that the ratio $z(t)/t^\nu$ is distributed as $(1/2\nu)^{2\nu}/L^\nu$ for large times. For $\gamma = 1/2$ one has a ballistic behavior, but the large time limit of $z(t)/t$ is not 1 as in the range $\gamma \in (1/2,\infty)$. We show here that it is distributed as $1/(4L+1)$, meaning that the walker spends only a finite but random fraction of her time moving linearly away from home.

III. MATHEMATICAL METHODS

Let us now outline our mathematical approach. Trajectories are decomposed in active journeys and lazy journeys.

The lazy journey starts at time t when a walker, which is on a maximum $x(t) = z(t)$ or $x(t) = -z(t)$, leaves it (first step) and continues for m time steps until she reaches again one of the two maxima. The total number of time steps of this journey is $1+m$ where m is the random time necessary to hit the frontier of the interval $[-z(t),z(t)]$ starting from one of the two positions $z(t)-1$ or $-z(t)+1$. During all the lazy journey the maximum remains the same $(z(t+m+1) = z(t))$, and, very importantly, the m steps of the walk necessary for hitting the frontier are those of a SSRW. Notice, in fact, that all the m steps are made choosing the direction with equal probability.

The active journey starts at the time $t+m+1$ when the walker arrives on a maximum, and it has a duration of n time steps which she makes remaining on a maximal position. The n steps are all made in the opposite direction with respect to the origin. In numbers, $|x(t+m+1+s)| = z(t+m+1+s) = z(t)+s$ for $0 \leqslant s \leqslant n$ while $|x(t+m+n+2)| < z(t+m+1+n)$, this last being the first step of a new lazy journey. The active journey has a minimum duration of zero time steps ($n = 0$ when the walker immediately leaves the maximum after having arrived). During the active journey the maximum increases by n.

A cycle journey, starting in a position $|x(t)| = z(t)$, is composed by a lazy journey followed by an active journey; its duration is $1+m+n$, and the maximum increases by n.

Notice that $m = m(z)$ is a random variable whose distribution only depends on z. In fact, $m(z)$ is the SSRW first hitting time of one of the barriers z or $-z$ starting from position $x = z-1$ or $x = -z+1$. On the contrary, the distribution of $n = n(z)$ depends on both z and γ through $p(z)$.

Let us indicate by k (not to be confused with time t) the progressive integer number identifying cycle journeys, each composed by a lazy journey followed by an active journey. Also, let us indicate by $z(k)$ the value of the maximum when the cycle journey number $k+1$ starts.

Then the time t is linked to the progressive number k by the stochastic relation

$$t(k+1) = t(k) + 1 + m[z(k)] + n[z(k)], \tag{7}$$

while the value of the maximum is given by

$$z(k+1) = z(k) + n[z(k)], \tag{8}$$

where $m[z(k)]$ and $n[z(k)]$ are all independent random variables.

For the sake of completeness let us also write the initial condition. At the start $(x(0) = z(0) = 0)$ the walker moves left or right so that $x(0) = \pm 1$ and $z(1) = 1$. Then, starting from the maximum $z(1) = 1$, she begins an active journey (which can also be of $n(1) = 0$ steps if she immediately steps back to the origin) so that

$$t(1) = z(1) = 1 + n(1). \tag{9}$$

In principle one should simply solve the two equations (7) and (8) with initial condition (9) in order to obtain the scaling behavior of $z(t)$. Obviously, this asks for some work since we need to characterize probabilistically $m(z)$ and $n(z)$.

Let us start with $m(z)$ which by definition is the SSRW exit time from the interval $[-z,z]$ starting in $z-1$ or $-z+1$. By translational invariance, $m(z)$ can be also considered as the SSRW exit time from the interval $[-2z+1,1]$ starting in 0.

The third Wald identity [23], when applied to SSRW trajectories $w(s)$ starting in $w(0) = 0$, states that for any stopping time τ

$$E\left[\frac{e^{\theta w(\tau)}}{[\cosh(\theta)]^\tau}\right] = 1. \tag{10}$$

Considered that $e^{\theta w(s)}/[\cosh(\theta)]^s$ is a martingale, this equality is a simple consequence of the strong Markov property.

Equation (10) also holds if θ is replaced by $-\theta$ so that

$$E\left[\frac{A\,e^{\theta w(\tau)}+(1-A)\,e^{-\theta w(\tau)}}{[\cosh(\theta)]^{\tau}}\right]=1 \tag{11}$$

for any real A.

Suppose $a<0<b$ and assume that that $\tau=\tau(a,b)$ is the first exit time of $w(s)$ from the interval $[a,b]$ so that $w(\tau)=a$ or $w(\tau)=b$. One can choose the real constant A in order that the numerator in (11) has the same value in $w(\tau)=a$ and $w(\tau)=b$ obtaining

$$E[[\cosh(\theta)]^{-\tau}]=\frac{\cosh(\theta c)}{\cosh(\theta d)}, \tag{12}$$

where $c=(a+b)/2$ and $d=(b-a)/2$.

Then, having defined $\lambda=\ln[\cosh(\theta)]$, one can rewrite the above equality as a Laplace transform of the distribution of the stopping time $\tau=\tau(a,b)$

$$E[e^{-\lambda\tau}]=\frac{\cosh[\theta(\lambda)\,c]}{\cosh[\theta(\lambda)\,d]}, \tag{13}$$

where $\theta(\lambda)=\ln(e^{\lambda}+\sqrt{e^{2\lambda}-1})$.

We simply use $a=-2z+1$ and $b=1$ so that $c=1-z$ and $d=z$. Thus, the Laplace transform of the probability density of the exit time $m(z)$ is

$$E[e^{-\lambda m(z)}]=\frac{\cosh[\theta(\lambda)\,(z-1)]}{\cosh[\theta(\lambda)\,z]}. \tag{14}$$

From (14) one can derive the expected values of all powers of $m(z)$. For large values of z one finds $E[m(z)]\simeq 2z$ and $E[m^2(z)]\simeq(8/3)z^3$, which implies that the standard deviation is $\sigma_{m(z)}\simeq(8/3)^{1/2}z^{3/2}$. All these quantities diverge for large values of z.

In the limit of large z, nevertheless, the Laplace transform remains finite and well defined; one has, in fact, $E[e^{-\lambda m(z)}]\to e^{-\theta(\lambda)}$. Moreover, for small values of λ one has that $\theta(\lambda)\simeq\sqrt{2\lambda}$, meaning that the probability density of $m(z)$ is substantially a truncated Lévy density.

Let us now evaluate the probability $\pi_\gamma\,(n|z)$ that the walker makes at least $n(z)=n$ steps during the active journey, i.e., $\pi_\gamma\,(n|z)=\text{prob}\,[n(z)\geqslant n]$. Straightforwardly,

$$\pi_\gamma\,(n|z)=\prod_{s=0}^{n-1}p(z+s), \tag{15}$$

where $p(z+s)=(z+s)^\gamma/[1+(z+s)^\gamma]$.

At variance with $m(z)$, the variable $n(z)$ depends on γ. In this paper we focus on the range $\gamma\in[0,1/2]$ and, in order to describe the probabilistic behavior of $n(z)$, we have to distinguish two different subranges.

The first is $\gamma=0$, for this value (ordinary SSRW) one has $p(z)=1/2$ and $\pi_\gamma\,(n|z)=(1/2)^n$. Accordingly, $E[n(z)]=1$ and all averages $E[n(z)^\delta]$ are finite, and they are independent from z for any positive δ.

The second case corresponds to the range $\gamma\in(0,1/2]$, included in $(0,1)$, which, in turn, can be treated at once. We directly obtain from (15)

$$[p(z)]^n\leqslant\pi_\gamma(n|z)\leqslant[p(z+n-1)]^n. \tag{16}$$

In fact, ν being positive, $p(z)$ is the smallest among the elements of the product and $p(z+n-1)$ the largest.

Then assume $n=I[\beta z^\gamma]$ (the integer part) where β is real and strictly positive. The inequality (16) can be rewritten as

$$[p(z)]^{I[\beta z^\gamma]}\leqslant\pi_\gamma(n|z)\leqslant\{p(z+I[\beta z^\gamma])\}^{I[\beta z^\gamma]}. \tag{17}$$

Then, since $\gamma\in(0,1)$, one gets that the limit for $z\to\infty$ of both bounds is $e^{-\beta}$. Given that $\pi_\gamma(n|z)=\text{prob}\,[n(z)\geqslant I[\beta z^\gamma]]$, one finally has

$$\text{prob}\,[n(z)\geqslant\beta z^\gamma]\simeq e^{-\beta}. \tag{18}$$

The approximated equality (18) means that for large values of z the limit $n(z)/z^\gamma\to\xi$ holds where ξ is a random variable distributed according to an unitary exponential probability density. Since $n(z)\simeq\xi z^\gamma$, one can easily compute $E[n(z)]\simeq z^\gamma$ and $E[n(z)^\delta]\sim z^{\delta\gamma}$ for any positive δ.

Summarizing, the relation $E[n(z)^\delta]\sim z^{\delta\gamma}$ holds for all $\gamma\in[0,1)$ and, thus, in the range $[0,1/2]$.

Let us consider again equation (8), one has for any positive α

$$E[z(k+1)^\alpha]\simeq E[z(k)^\alpha]+\alpha E[z(k)^{\alpha-1+\gamma}], \tag{19}$$

where the omitted terms are of lower order in $z(k)$ since the conditional expectation of $n[z(k)]^\delta$ given $z(k)$ satisfies $E[n[z(k)]^\delta]\sim z(k)^{\delta\gamma}\ll z(k)^\delta$. Choosing $\alpha=1-\gamma$ in (19), we immediately obtain $E[z(k)^{1-\gamma}]\simeq(1-\gamma)k$. Then, choosing $\alpha=l(1-\gamma)$, we get by iteration $E[z(k)^{l(1-\gamma)}]\simeq(1-\gamma)^l k^l$ where l is any positive integer number.

Thus, $E[z(k)^{l(1-\gamma)}]\simeq E[z(k)]^{l(1-\gamma)}\simeq(1-\gamma)^l k^l$, which implies that the relation

$$z(k)\simeq(1-\gamma)^{1/(1-\gamma)}\,k^{1/(1-\gamma)} \tag{20}$$

holds deterministically in the range $\gamma\in[0,1)$, i.e., the large k limit of the ratio of the two sides of (20) is one.

On the other hand, from Eq. (7) we have by a direct sum

$$t(k)=z(k)+k^2L(k)+k-1, \tag{21}$$

where we have defined

$$L(k)=\frac{1}{k^2}\sum_{i=1}^{k-1}m[z(i)]. \tag{22}$$

Then we can use (14) and straightforwardly obtain

$$E[e^{-\lambda L(k)}]=\prod_{i=1}^{k-1}\frac{\cosh[\theta(\lambda/k^2)\,(z(i)-1)]}{\cosh[\theta(\lambda/k^2)\,z(i)]}, \tag{23}$$

where $z(i)$ is given by (20). This expression can be rewritten as

$$E[e^{-\lambda L(k)}]=[e^{-\theta(\lambda/k^2)}]^{k-1}R(k), \tag{24}$$

where

$$R(k)=\prod_{i=1}^{k-1}\frac{1+e^{-2\theta(\lambda/k^2)\,(z(i)-1)}}{1+e^{-2\theta(\lambda/k^2)\,z(i)}}. \tag{25}$$

It is easy to check that in the limit of large k one has $[e^{-\theta(\lambda/k^2)}]^{k-1}\to e^{-\sqrt{2\lambda}}$ Moreover, some lengthy but straightforward calculations lead to $R(k)\to 1$ for $\gamma\in(0,1)$, while for $\gamma=0$ they lead to $R(k)\to 2/(1+e^{-2\sqrt{2\lambda}})$.

In conclusion, for $\gamma\in(0,1)$,

$$\lim_{k\to\infty}E[e^{-\lambda L(k)}]=e^{-\sqrt{2\lambda}}, \tag{26}$$

which implies that $L = \lim_{k\to\infty} L(k)$ is a (parameter = 1) Lévy variable which has an infinite expectation. Another way to see this result is to consider it as a direct consequence of the generalized central limit for leptokurtic variables [1].

On the other hand, for $\gamma = 0$,

$$\lim_{k\to\infty} E[e^{-\lambda L(k)}] = \frac{1}{\cosh(\sqrt{2\lambda})}, \tag{27}$$

which corresponds to a variable $T = \lim_{k\to\infty} L(k)$ with finite expectation and standard deviation (1 and $\sqrt{2/3}$, respectively). This variable is the exit time from the interval $[-1,1]$ of a continuous Brownian motion, with unitary variance, starting in 0 (see, for example, Ref. [24], p. 212). Since $\lim_{k\to\infty} L(k)$ equals L for $\gamma \in (0,1/2]$ and it equals T for $\gamma = 0$, the above two relations (26) and (27) coincide with (4).

Now, consider Eq. (21) for large values of k and take into account that $\lim_{k\to\infty} L(k) = T$ for $\gamma = 0$ and that $\lim_{k\to\infty} L(k) = L$ for $\gamma \in (0,1/2]$. Also taking into account (20), one has the asymptotic relations: $t(k) \simeq k^2 T$ for $\gamma = 0$, $t(k) \simeq k^2 L$ for $\gamma \in (0,1/2)$ and $t(k) \simeq z(k) + k^2 L = k^2/4 + k^2 L$ for $\gamma = 1/2$. These relations are obtained neglecting terms of lower order with respect to k^2.

Finally, taking again into account (20), one obtains for large times (which imply large k): $z(t)/t^{1/2} \to 1/T^{1/2}$ for $\gamma = 0$, $z(t)/t^{\nu} \to (1/2\nu)^{2\nu}/L^{\nu}$ where $\nu = 1/(2-2\gamma)$ for $\gamma \in (0,1/2)$ and $z(t)/t \to 1/(4L+1)$ for $\gamma = 1/2$. These results complete the characterization of the asymptotic behavior of $z(t)$ initiated in Ref. [22].

ACKNOWLEDGMENTS

The author warmly thanks Michele Pasquini, Eudenilson Lins de Albuquerque, Angelo Vulpiani, and Umberto Laino Fulco, who have contributed with many discussions and suggestions.

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