Statistical symmetries of the Lundgren-Monin-Novikov hierarchy

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It was shown by Oberlack and Rosteck [Discr. Cont. Dyn. Sys. S, **3**, 451 2010] that the infinite set of multipoint correlation (MPC) equations of turbulence admits a considerable extended set of Lie point symmetries compared to the Galilean group, which is implied by the original set of equations of fluid mechanics. Specifically, a new scaling group and an infinite set of translational groups of all multipoint correlation tensors have been discovered. These new statistical groups have important consequences for our understanding of turbulent scaling laws as they are essential ingredients of, e.g., the logarithmic law of the wall and other scaling laws, which in turn are exact solutions of the MPC equations. In this paper we first show that the infinite set of translational groups of all multipoint correlation tensors corresponds to an infinite dimensional set of translations under which the Lundgren-Monin-Novikov (LMN) hierarchy of equations for the probability density functions (PDF) are left invariant. Second, we derive a symmetry for the LMN hierarchy which is analogous to the scaling group of the MPC equations. Most importantly, we show that this symmetry is a measure of the intermittency of the velocity signal and the transformed functions represent PDFs of an intermittent (i.e., turbulent or nonturbulent) flow. Interesting enough, the positivity of the PDF puts a constraint on the group parameters of both shape and intermittency symmetry, leading to two conclusions. First, the latter symmetries may no longer be Lie group as under certain conditions group properties are violated, but still they are symmetries of the LMN equations. Second, as the latter two symmetries in its MPC versions are ingredients of many scaling laws such as the log law, the above constraints implicitly put weak conditions on the scaling parameter such as von Karman constant κ as they are functions of the group parameters. Finally, let us note that these kind of statistical symmetries are of much more general type, i.e., not limited to MPC or PDF equations emerging from Navier-Stokes, but instead they are admitted by other nonlinear partial differential equations like, for example, the Burgers equation when in conservative form and if the nonlinearity is quadratic.

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I. INTRODUCTION

The Lie group theory acts as the foremost guiding principle to understand and mathematically model new physical laws to be discovered (see, e.g., Refs. [1,2]). This paper refers to its application in hydrodynamics. With respect to turbulence research we are in the convenient position to know three complete descriptions of turbulence, i.e., the multipoint correlation (MPC) approach [3], the Lundgren-Monin-Novikov probability density functions (PDF) approach [4], and the Hopf functional approach [5]. All of them are in rather distinct form, still allowing for an exhaustive description of statistical turbulence, at least in principle. Indeed, even though the fact that a full mathematical treatment is mostly hindered by the infinite dimensional nature of all of them, it is still possible to compute the underlying symmetry structure and in turn to construct special solutions.

One of the authors and his coworkers followed this route mainly based on the MPC approach, extracting symmetries of the underlying equations and constructing, mostly for turbulent wall-bounded shear flows, invariant solutions for the mean velocity and higher-order correlations [3,6–10]. Only recently, in Ref. [3], was it discovered that the MPC hierarchy admits an infinite number of statistical groups, which was further extended in Ref. [10] using Lie algebra methods. In Ref. [3] it was recognized that these groups play an indispensable role in the construction of various turbulent scaling laws; most prominently, the first of this infinite row constitutes the basis for the logarithmic law of the wall. Moreover, in Ref. [11], the Lie group analysis was extended towards the functional differential equations in order to extract the symmetries of the Hopf functional equation. It has been done so far for the case of the Hopf formulation of the Burgers equation [12].

Most likely R. H. Kraichan should be given the credit for discovering the first of the infinite many statistical symmetry in Refs. [13–15], giving it the name *random Galilean invariance*. In fact, he recognized this symmetry to be absolutely crucial to be obeyed by turbulence models and reformulated his direct interaction approximation (DIA) according to this important constraint. Finally, it should be noted that the latter symmetry has been employed by turbulence modelers for decades, though rather implicitly. Even the earliest statistical turbulence models such as the eddy-viscosity approach by Prandtl [16] are consistent with it and one will hardly find any model not obeying this constraint.

In this paper we present a set of transformations for the PDFs under which the Lundgren-Monin-Novikov (LMN) hierarchy is left invariant. However, since these

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transformations correspond to the statistical symmetries found for the MPC equation in Ref. [3], we first reconsider in Sec. II the infinite set of multipoint correlation equations (MPCE) and show that there exist an infinite number of discrete Lie point symmetries. Then it is in Sec. III that we revisit the LMN equations for the PDFs of turbulence and, on the basis of this and the infinite symmetries of the MPCE, we derive two sets of transformations for the PDFs. We argue that the derived translation symmetry, modifies the contribution of laminar solutions in the PDF. The second statistical symmetry transforms the PDFs into the functions describing intermittent flow, i.e., a flow where irregular alternations between turbulent and nonturbulent fluid are observed.

II. MULTIPOINT CORRELATION EQUATIONS

The idea of two- and multipoint correlation equations in turbulence was presumably first established by Friedmann and Keller [17]. In the beginning it was assumed that all correlation equations of orders higher than two may be neglected. Theoretical considerations led to the result that all higher correlations have to be taken into account. Consequently, all multipoint correlation equations have to be considered in the symmetry analysis to follow.

In order to write the MPC equations in a very compact form, we introduce the following notation. The multipoint velocity correlation tensor of order $n+1$ (also referred to as multipoint moment) is defined as follows:

$$H_{i_{\{n+1\}}} = H_{i_{(1)}i_{(2)}\dots i_{(n+1)}} = \langle U_{i_{(1)}}(\boldsymbol{x}_{(1)},t)\cdots U_{i_{(n+1)}}(\boldsymbol{x}_{(n+1)},t)\rangle, \quad (1)$$

where the first index of the $\mathbf{H}$ tensor defines the tensor character of the term and the second index in braces denotes the order of the tensor. The bracket $\langle\cdot\rangle$ refers to a statistical averaging of Reynolds type and in Sec. III will be defined more precisely employing the multipoint probability density function. The connection between $\mathbf{H}$ and the mean velocity is simply given according to $H_{i_{\{1\}}} = H_{i_{(1)}} = \langle U_{i_{(1)}}(\mathrm{x}_{(1)},t)\rangle$.

In some cases the list of indices is interrupted by one or more other indices; this is pointed out by attaching the replaced value in brackets to the following index:

$$H_{i_{\{n+1\}}[i_{(l)}\mapsto k_{(l)}]} = \langle U_{i_{(1)}}(\boldsymbol{x}_{(1)},t)\cdots U_{i_{(l-1)}}(\boldsymbol{x}_{(l-1)},t)U_{k_{(l)}}(\boldsymbol{x}_{(l)},t) \times U_{i_{(l+1)}}(\boldsymbol{x}_{(l+1)},t)\cdots U_{i_{(n+1)}}(\boldsymbol{x}_{(n+1)},t)\rangle; \quad (2)$$

this is further extended by

$$H_{i_{\{n+2\}}[i_{(n+2)}\mapsto k_{(l)}]}[\boldsymbol{x}_{(n+2)}\mapsto \boldsymbol{x}_{(l)}] = \langle U_{i_{(1)}}(\boldsymbol{x}_{(1)},t)\cdots U_{i_{(n+1)}}(\boldsymbol{x}_{(n+1)},t)U_{k_{(l)}}(\boldsymbol{x}_{(l)},t)\rangle, \quad (3)$$

where not only the index $i_{(n+1)}$ is replaced by $k_{(l)}$, but also the independent variable $\boldsymbol{x}_{(n+1)}$ is replaced by $\boldsymbol{x}_{(l)}$. Finally, if the pressure P is involved, then we write

$$I_{i_{\{n\}}[l]} = \langle U_{i_{(1)}}(\boldsymbol{x}_{(1)},t)\cdots U_{i_{(l-1)}}(\boldsymbol{x}_{(l-1)},t)P(\boldsymbol{x}_{(l)},t) \times U_{i_{(l+1)}}(\boldsymbol{x}_{(l+1)},t)\cdots U_{i_{(n+1)}}(\boldsymbol{x}_{(n+1)},t)\rangle, \quad (4)$$

which is, together with all the other definitions, sufficient to derive the MPCE from the Navier-Stokes (NS) equation. The details of this derivation may be taken from Refs. [3,7]; in the end the result is

$$\frac{\partial H_{i_{\{n+1\}}}}{\partial t} + \sum_{l=1}^{n+1}\left[\frac{\partial H_{i_{\{n+2\}}[i_{(n+2)}\mapsto k_{(l)}]}[\boldsymbol{x}_{(n+2)}\mapsto \boldsymbol{x}_{(l)}]}{\partial x_{k_{(l)}}} + \frac{\partial I_{i_{\{n\}}[l]}}{\partial x_{i_{(l)}}} - \nu\frac{\partial^2 H_{i_{\{n+1\}}}}{\partial x_{k_{(l)}}\partial x_{k_{(l)}}}\right] = 0 \quad (5)$$

for $n = 0,\dots,\infty$. Loosely speaking, Eq. (5) implies the statistical information of the NS equations at the expense to deal with an infinite dimensional chain of differential equations starting with order 1, i.e., $n = 0$. The rather remarkable consequence of the derivation is that (5) is a linear equation, which considerably simplifies the finding of Lie symmetries to be pointed out below.

From the continuity equation we may derive related equations for $H_{i_{\{n+1\}}}$ and $I_{i_{\{n\}}[l]}$, i.e.,

$$\frac{\partial H_{i_{\{n+1\}}[i_{(l)}\mapsto k_{(l)}]}}{\partial x_{k_{(l)}}} = 0, \quad \text{for} \quad l = 1,\dots,n+1 \quad (6)$$

and

$$\frac{\partial I_{i_{\{n\}}[k][i_{(l)}\mapsto m_{(l)}]}}{\partial x_{m_{(l)}}} = 0, \quad \text{for} \quad k,l = 1,\dots,n+1 \quad \text{and} \quad k \neq l. \quad (7)$$

A. Classical symmetries of the MPCE

Here we first revisit the Lie symmetries of the MPCE, which have their roots in the Euler and NS equations. In Sec. II B we show that the MPCE admit even more Lie symmetries, called statistical symmetries, which are not reflected in the original equations.

Presently, we adopt the symmetry notation most common in the mathematical literature where the new variables are denoted by an asterisk; as an example, the translation group is defined as $x^* = x + a$, where $a \in \mathbb{R}$ is the group parameter. The range of validity for all group parameters to follow will be omitted. Moreover, the full form of the symmetry will be given, meaning that all the variables of a given system will be presented, even if part of them undergoes the identity transformation.

Adopting the notation given above based on the instantaneous quantities, symmetries of classical mechanics rewritten for the statistical variables have the following form:

$$\bar{T}_1: \ t^* = t + a_1, \quad \boldsymbol{x}^*_{(l)} = \boldsymbol{x}_{(l)}, \quad (8)$$

$$\mathbf{H}^*_{\{n\}} = \mathbf{H}_{\{n\}}, \quad \mathbf{I}^*_{\{n\}} = \mathbf{I}_{\{n\}};$$
$$\bar{T}_2: \ t^* = t, \quad \boldsymbol{x}^*_{(l)} = e^{a_2}\boldsymbol{x}_{(l)}, \quad (9)$$

$$\mathbf{H}^*_{\{n\}} = e^{na_2}\mathbf{H}_{\{n\}}, \quad \mathbf{I}^*_{\{n\}} = e^{(n+2)a_2}\mathbf{I}_{\{n\}};$$
$$\bar{T}_3: \ t^* = e^{a_3}t, \quad \boldsymbol{x}^*_{(l)} = \boldsymbol{x}_{(l)}, \quad (10)$$

$$\mathbf{H}^*_{\{n\}} = e^{-na_3}\mathbf{H}_{\{n\}}, \quad \mathbf{I}^*_{\{n\}} = e^{-(n+2)a_3}\mathbf{I}_{\{n\}};$$
$$\bar{T}_4\text{--}\bar{T}_6: \ t^* = t, \quad \boldsymbol{x}^*_{(l)} = \mathbf{a}\cdot\boldsymbol{x}_{(l)}, \quad (11)$$

$$\mathbf{H}^*_{\{n\}} = \mathbf{A}_{\{n\}}\otimes\mathbf{H}_{\{n\}}, \quad \mathbf{I}^*_{\{n\}} = \mathbf{A}_{\{n\}}\otimes\mathbf{I}_{\{n\}};$$
$$\bar{T}_7\text{--}\bar{T}_9: \ t^* = t, \quad \boldsymbol{x}^*_{(l)} = \boldsymbol{x}_{(l)} + \boldsymbol{f}(t), \quad (12)$$

$$\mathbf{H}^*_{\{n\}} = \mathbf{H}_{\{n\}} + \sum_{b=1}^{n} f'_{i_{(b)}}(t)\mathbf{H}_{\{n-1\}[i_{(b)}\to\emptyset]},$$
$$\mathbf{I}^*_{\{n\}[l]} = \mathbf{I}_{\{n\}[l]} - f''_\beta x_{\beta_{(l)}} \mathbf{H}_{\{n-1\}[l\to\emptyset]} + \sum_{c=1,c\neq l}^{n+1} f'_{i(b)} \mathbf{I}_{\{n-1\}[l][c\to\emptyset]};$$
$$\bar{T}_{10}: \ t^* = t, \ \ \boldsymbol{x}^*_{(l)} = \boldsymbol{x}_{(l)},$$
$$\mathbf{H}^*_{\{n\}} = \mathbf{H}_{\{n\}},$$
$$\mathbf{I}^*_{\{n\}[l]} = \mathbf{I}_{\{n\}[l]} + f_4(t)\mathbf{H}_{\{n\}[i_{(l)}\to\emptyset]}, \tag{13}$$

where f's are free functions and **A** is a concatenation of rotation matrices as $A_{i_{(1)}j_{(1)}i_{(2)}j_{(2)}\dots i_{(n)}j_{(n)}} = a_{i_{(1)}j_{(1)}}a_{i_{(2)}j_{(2)}}\cdots a_{i_{(n)}j_{(n)}}$ and each $a_{i_\alpha j_\beta}$ is a rotating matrix.

B. Statistical symmetries of the MPCE

Actually finding the symmetries of the MPCE is rather difficult since an infinite system of equations has to be analyzed. For this task, however, the linearity of the **H**-**I** system (5)–(7) makes the investigation considerably easier.

New symmetries have been identified in Ref. [3] and slightly extended, using Lie algebra methods, in Ref. [10]. In its most general form it reads

$$\bar{T}'_{2_{\{n\}}}: \ t^* = t, \ \ \boldsymbol{x}^*_{(l)} = \boldsymbol{x}_{(l)},$$
$$\mathbf{H}^*_{\{n\}} = \mathbf{H}_{\{n\}} + \mathbf{C}_{\{n\}}, \quad \mathbf{I}^*_{\{n\}} = \mathbf{I}_{\{n\}} + \mathbf{D}_{\{n\}}, \tag{14}$$

with $n = 1,\dots,\infty$, where $\mathbf{C}_{\{n\}}$ and $\mathbf{D}_{\{n\}}$ refer to group parameters independent for each of the according tensor orders. The first one in the row of infinite symmetries (14) was in fact already identified in Refs. [13–15] and therein named the *random Galilean group*,

$$\bar{T}'_{2_{\{l\}}}: \quad t^* = t, \quad \boldsymbol{x}^*_{(l)} = \boldsymbol{x}_{(l)}, \quad \mathbf{H}^*_{\{1\}} = \mathbf{H}_{\{1\}} + \mathbf{C}_{\{1\}},$$
$$\mathbf{H}^*_{\{n\}} = \mathbf{H}_{\{n\}}, \quad \mathbf{I}^*_{\{m\}} = \mathbf{I}_{\{m\}} + \mathbf{D}_{\{m\}}, \tag{15}$$

with $n = 2,\dots,\infty$ and $m = 1,\dots,\infty$, where $\mathbf{H}_{\{1\}}$ refers to the mean velocity $\langle \boldsymbol{U}(\boldsymbol{x}_{(1)},t)\rangle = \mathbf{H}_{\{1\}}$. Clearly the notation in (14) implies an infinite but discrete number of groups since each tensor has its own independent corresponding group parameter.

The second statistical group that has been identified denotes simple scaling of all MPC tensors as may be directly read off from Eq. (5),

$$\bar{T}'_s: \quad t^* = t, \quad \boldsymbol{x}^*_{(l)} = \boldsymbol{x}_{(l)}, \quad \mathbf{H}^*_{\{n\}} = e^{a_s}\mathbf{H}_{\{n\}}, \quad \mathbf{I}^*_{\{n\}} = e^{a_s}\mathbf{I}_{\{n\}}. \tag{16}$$

It should be finally added that, due to the linearity of the MPC equation (5), another rather generic symmetry is admitted. This is in fact featured by all linear differential equations (see, e.g., Ref. [18]) and merely reflects the superposition property of linear differential equations; therefore it will not be further employed presently.

III. THE LUNDGREN-MONIN-NOVIKOV APPROACH

As already stated in the Introduction, the LMN approach is one of the known methods which provide full statistical description of turbulence. In this approach indeed one assumes that for the velocity field there exist PDFs which describe the joint probabilities of measuring contemporarily certain sets of velocities at multiple points in space. When calculating mean values thereof, these PDFs play the role of the weighting measure.

For this, we address the same problem that was introduced in the seminal paper by Lundgren [4]. Let us therefore take into consideration an ensemble of incompressible fluids occupying the entire infinite space $\mathbb{R}^3$ and having identical physical properties but different initial conditions. Let the velocity field of each member of the ensemble be denoted by $\boldsymbol{U}$, in agreement with the notation of the preceding section. It is assumed that $\boldsymbol{U}$ satisfies the Navier-Stokes and the continuity equations and that the statistical distribution of $\boldsymbol{U}$ over the ensemble at the initial time t_0 is given. The main goal is the statistical distribution of the velocity field as it evolves with time. In order to do so, let us define multipoint PDFs in the usual way: the one-point PDF $f_1(\boldsymbol{x}_{(1)},\boldsymbol{v}_{(1)};t)$ is such that $f_1(\boldsymbol{x}_{(1)},\boldsymbol{v}_{(1)};t)d\boldsymbol{v}_{(1)}$ expresses the probability to measure a velocity in an infinitesimal interval $d\boldsymbol{v}_{(1)}$ around $\boldsymbol{v}_{(1)}$ at position $\boldsymbol{x}_{(1)}$ (or, equivalently, the fraction of systems in the esemble such that the given condition is satisfied). The one-point PDF can be written as follows:

$$f_1(\boldsymbol{x}_{(1)};\boldsymbol{v}_{(1)},t) = \langle \delta(\boldsymbol{U}(\boldsymbol{x}_{(1)},t) - \boldsymbol{v}_{(1)})\rangle. \tag{17}$$

Analogously to (17) the two-point PDF, which denotes the joint probability to measure two given velocities at two defined points in space at the same time t, can be expressed as follows:

$$f_2(\boldsymbol{v}_{(1)},\boldsymbol{v}_{(2)};\boldsymbol{x}_{(1)},\boldsymbol{x}_{(2)},t) = \langle \delta(\boldsymbol{U}(\boldsymbol{x}_{(1)},t) - \boldsymbol{v}_{(1)})\delta(\boldsymbol{U}(\boldsymbol{x}_{(2)},t) - \boldsymbol{v}_{(2)})\rangle$$

and so forth. Sometimes the following abbreviation will be used in this paper [4]:

$$f_n \equiv f_n(1,\dots,n) \equiv f_n(\boldsymbol{v}_{(1)},\dots,\boldsymbol{v}_{(n)};\boldsymbol{x}_{(1)},\dots,\boldsymbol{x}_{(n)},t).$$

In the LMN language it is the moments associated to the PDFs that correspond to the multipoint moments (1) of the MPC approach, i.e., to the components of the tensor **H**,

$$H_{i_{\{n+1\}}} \equiv \langle U_{i_{(1)}}(\boldsymbol{x}_{(1)},t)\cdots U_{i_{(n+1)}}(\boldsymbol{x}_{(n+1)},t)\rangle = \int d\boldsymbol{v}_{(1)}\cdots d\boldsymbol{v}_{(n+1)}\ f_{n+1}\ v_{(1)i_{(1)}}\cdots v_{(n+1)i_{(n+1)}}. \tag{18}$$

A. Properties of the PDFs

In order for the previously defined PDFs to be well defined from a physical point of view, they are required to satisfy four conditions as follows [4]:

(1) the reduction or normalization property imposed by the concept of probability:

$$\int d\boldsymbol{v}_{(1)} f_1(\boldsymbol{v}_{(1)};\boldsymbol{x}_{(1)},t) = 1;$$
$$\int d\boldsymbol{v}_{(1)} f_2(\boldsymbol{v}_{(1)},\boldsymbol{v}_{(2)};\boldsymbol{x}_{(1)},\boldsymbol{x}_{(2)},t) = f_1(\boldsymbol{v}_{(1)};\boldsymbol{x}_{(1)},t); \tag{19}$$
$$\vdots$$

(2) an infinite number of "continuity" conditions dictated by the incompressibility of the fluid:

$$\nabla_{(i)} \cdot \int d\boldsymbol{v}_{(i)}\, \boldsymbol{v}_{(i)}\, f_n = 0, \quad \forall i \in \{1,\ldots,n\}; \tag{20}$$

(3) the "coincidence" property, required by the condition for the velocity field to be well defined:

$$\lim_{|\boldsymbol{x}_{(1)}-\boldsymbol{x}_{(2)}|\to 0} f_2(1,2) = f_1(1)\,\delta(\boldsymbol{v}_{(2)} - \boldsymbol{v}_{(1)}),$$
$$\lim_{|\boldsymbol{x}_{(1)}-\boldsymbol{x}_{(3)}|\to 0} f_3(1,2,3) = f_2(1,2)\,\delta(\boldsymbol{v}_{(3)} - \boldsymbol{v}_{(1)}); \tag{21}$$
$$\vdots$$

(4) the "separation" property (here shown only for the two-point PDF) which expresses the fact that the velocities of two fluid elements tends to become independent if the two points are set far apart from each other:

$$\lim_{|\boldsymbol{x}_{(1)}-\boldsymbol{x}_{(2)}|\to \infty} f_2(1,2) = f_1(1) f_1(2). \tag{22}$$

B. The LMN hierarchy

From the NS equations $[\partial_t + \boldsymbol{U}(\boldsymbol{x},t)\cdot\nabla]\boldsymbol{U}(\boldsymbol{x},t) = -\nabla P(\boldsymbol{x},t) + \nu\triangle\boldsymbol{U}(\boldsymbol{x},t)$, where the density has been absorbed into the definition of the pressure P, the LMN equations follow [4,19,20]. The first equation of the infinite hierarchy reads

$$\begin{aligned}
&[\partial_t + \boldsymbol{v}_{(1)}\cdot\nabla_{(1)}] f_1 \\
&= \nabla_{\boldsymbol{v}_{(1)}} \cdot \left[\int d\boldsymbol{x}_{(2)} \left(\nabla_1 \frac{1}{4\pi|\boldsymbol{x}_{(1)} - \boldsymbol{x}_{(2)}|} \right) \right. \\
&\quad \left. \times \int d\boldsymbol{v}_{(2)} (\boldsymbol{v}_{(2)}\cdot\nabla_{(2)})^2 f_2 \right] \\
&\quad - \nabla_{\boldsymbol{v}_{(1)}} \cdot \left[\lim_{|\boldsymbol{x}_{(2)}-\boldsymbol{x}_{(1)}|\to 0} \nu\triangle_2 \int d\boldsymbol{v}_{(2)} \boldsymbol{v}_{(2)} f_2 \right],
\end{aligned} \tag{23}$$

and the n-th equation of this hierarchy is given by the following:

$$\begin{aligned}
&\left[\partial_t + \sum_{i=1}^{n} \boldsymbol{v}_{(i)}\cdot\nabla_{(i)}\right] f_n \\
&= \sum_{j=1}^{n} \nabla_{\boldsymbol{v}_{(j)}} \cdot \left[\int d\boldsymbol{x}_{(n+1)} \left(\nabla_j \frac{1}{4\pi|\boldsymbol{x}_{(j)} - \boldsymbol{x}_{(n+1)}|} \right) \right. \\
&\quad \left. \times \int d\boldsymbol{v}_{(n+1)} (\boldsymbol{v}_{(n+1)}\cdot\nabla_{(n+1)})^2 f_{n+1} \right] \\
&\quad - \sum_{j=1}^{n} \nabla_{\boldsymbol{v}_{(j)}} \cdot \left[\lim_{|\boldsymbol{x}_{(n+1)}-\boldsymbol{x}_{(j)}|\to 0} \nu\triangle_{n+1} \int d\boldsymbol{v}_{(n+1)} \boldsymbol{v}_{(n+1)} f_{n+1} \right],
\end{aligned} \tag{24}$$

where $\nabla_{\boldsymbol{v}_{(j)}}$ denotes the differential operator with respect to $\boldsymbol{v}_{(j)}$, while ∇_j and $\triangle_j$, operators with respect to $\boldsymbol{x}_{(j)}$. Hence, the LMN hierarchy constitutes an infinite chain of equations where, on the n-th level, the unknown $n+1$-point PDF is present. As it was pointed in [21] the chain can be formally truncated at the n-th level by replacing the terms with f_{n+1} by conditional averages. For the one-point PDF equations and in the case of homogeneous, isotropic turbulence these conditionally averaged quantities were estimated based on the DNS data in Refs. [22,23] in order to study the deviations of the PDF from Gaussianity.

C. Classical symmetries of LMN hierarchy

In this subsection we discuss the invariance of the LMN hierarchy (24) under time and space translations, scaling, Galilean transformations, and extended Galilean transformations [24].

The invariance under time cf. (8) and space translations can be very easily inspected. Equation (24) for $\nu = 0$ is invariant under two scaling groups,

$$\bar{T}_2:\; t^* = t,\quad \boldsymbol{x}^*_{(l)} = e^{a_2}\boldsymbol{x}_{(l)},\quad \boldsymbol{v}^*_{(l)} = e^{a_2}\boldsymbol{v}_{(l)} \tag{25}$$
$$f^*_n = e^{-3na_2} f_n,\quad f^*_{n+1} = e^{-3(n+1)a_2} f_{n+1},$$
$$\bar{T}_3:\; t^* = e^{a_3}t,\quad \boldsymbol{x}_{(l)} = \boldsymbol{x}_{(l)},\quad \boldsymbol{v}^*_{(l)} = e^{-a_3}\boldsymbol{v}_{(l)} \tag{26}$$
$$f^*_n = e^{3na_3} f_n,\quad f^*_{n+1} = e^{3(n+1)a_3} f_{n+1},$$

which can be compared to the analogous symmetry of the MPCE, cf. (9) and (10). The scaling of the PDFs f_n and f_{n+1} assures that the normalization property (19) is satisfied for both transformed functions f^*_n and f^*_{n+1}. In the viscous case the two above symmetries reduce to one scaling group with $a_3 = 2a_2$.

As regards the Galilean transformations $t^* = t$, $\boldsymbol{x}^*_i = \boldsymbol{x}_i + \boldsymbol{v}_0\, t$, and $\boldsymbol{v}^*_i = \boldsymbol{v}_i + \boldsymbol{v}_0$, where $\boldsymbol{v}_0$ is a constant vector, the right-hand side of (24) is easily shown to be invariant. Let us start with the last term, the viscous one. The extra term that we get reads as follows:

$$\begin{aligned}
&\lim_{|\boldsymbol{x}_{(n+1)}-\boldsymbol{x}_{(i)}|\to 0} \triangle_{n+1} \int d\boldsymbol{v}_{(n+1)}\boldsymbol{v}_0\, f_{n+1} \\
&= \boldsymbol{v}_0 \lim_{|\boldsymbol{x}_{(n+1)}-\boldsymbol{x}_{(i)}|\to 0} \triangle_{n+1} f_n = 0,
\end{aligned} \tag{27}$$

where we have used the "reduction" condition (19).

As regards the pressure-gradient term, the extra terms that we obtain are two, namely

$$\begin{aligned}
&2\sum_{i=1}^{n} \frac{\partial}{\partial \boldsymbol{v}_i} \cdot \int d\boldsymbol{x}_{n+1} \left(\nabla_i \frac{1}{|\boldsymbol{x}_i - \boldsymbol{x}_{n+1}|} \right) \\
&\quad \times \int d\boldsymbol{v}_{(n+1)} [(\boldsymbol{v}_0\cdot\nabla_{n+1})(\boldsymbol{v}_{(n+1)}\cdot\nabla_{n+1})] f_{n+1}
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{\partial}{\partial \boldsymbol{v}_i} \cdot \int d\boldsymbol{x}_{n+1} \left(\nabla_i \frac{1}{|\boldsymbol{x}_i - \boldsymbol{x}_{n+1}|} \right) \\
&\quad \times \int d\boldsymbol{v}_{(n+1)} (\boldsymbol{v}_0\cdot\nabla_{n+1})^2 f_{n+1}.
\end{aligned} \tag{29}$$

In both cases, performing the integration over the sample space velocity variable $\boldsymbol{v}_{(n+1)}$, we end up with a null contribution: in (28) by exploiting the continuity property (20), while in (29) the reduction one (19).

At the left-hand side of (24) the translation of the velocities yields the following extra term:

$$\boldsymbol{v}_0 \cdot \sum_{i=1}^{n} \nabla_i f_n. \tag{30}$$

However, the transformation of the time and the spatial coordinates imply $\partial_{t^*} = \partial_t - \boldsymbol{v}_0 \cdot \sum_{i=1}^n \nabla_i$, where we have considered summations up to the index n because all further terms give null contributions. Therefore this extra term eliminates (30) and the Galilean invariance is restored.

The extended Galilean transformations [24] read $t^* = t$, $\boldsymbol{x}^* = \boldsymbol{x} + \boldsymbol{y}(t)$ and $\boldsymbol{u}^* = \boldsymbol{u} + \boldsymbol{y}'(t)$, where the vector $\boldsymbol{y}$ is time dependent, but again always along the same direction. Substituting the transformed variables back into the NS equation, we find that the pressure must undergo the transformation

$$p^* = p - \boldsymbol{x} \cdot \boldsymbol{y}''(t), \tag{31}$$

in order for the NS equation to be invariant. However, the transformation (31), being not bounded in $\mathbb{R}^3$, is not compatible with the integral representation of the pressure-gradient term that is present in the LMN hierarchy; the extended Galilean invariance is therefore broken.

D. Statistical symmetries of the LMN hierarchy

The content of this section represents the key contribution of this paper, namely we discuss a set of transformations for the PDFs under which the LMN equations (24) turn out to be invariant and which corresponds to the set of statistical symmetries (14) and (16) found for the MPCE, where the moments $\mathbf{H}$ are, respectively, shifted by a constant and scaled. For simplicity we will consider the first equation in the LMN hierarchy, i.e., Eq. (23) where only the one- and two-point PDFs are present. The generalization to the n-point PDF will be given in the Appendix.

1. *Shape symmetry*

In order to introduce these transformations, let us recall the relation (18) between the MPC tensors and the moments associated to the PDFs. The statistical symmetry (14) for the one-point moments can be written down in terms of the one-point PDF f_1 as follows:

$$\begin{aligned}\left[\int d\boldsymbol{v}_{(1)} f_1\, v_{(1)i_{(1)}} \cdots v_{(1)i_{(n)}}\right]^* &= \int d\boldsymbol{v}_{(1)}\, f_1\, v_{(1)i_{(1)}} \cdots v_{(1)i_{(n)}} + C_{i_{(1)}\ldots i_{(n)}} \\ &= \left\langle U_{i_{(1)}}(\boldsymbol{x}_{(1)},t) \cdots U_{i_{(n)}}(\boldsymbol{x}_{(1)},t)\right\rangle + C_{i_{(1)}\ldots i_{(n)}}.\end{aligned} \tag{32}$$

As it will be shown further below such transformation of statistics follows from the transformation (translation) of the PDF. As the following considerations concern the one-point PDF function as an example we will skip the index (1) and write, e.g., $\boldsymbol{x}$ instead of $\boldsymbol{x}_{(1)}$ The PDF can be represented as a Fourier transform of the characteristic function Φ_1 [24],

$$f_1(\boldsymbol{v};\boldsymbol{x},t) = \frac{1}{(2\pi)^3}\int ds e^{-i\boldsymbol{v}\cdot\boldsymbol{s}}\Phi_1(\boldsymbol{s};\boldsymbol{x},t) \tag{33}$$

or, in other words,

$$\Phi_1(\boldsymbol{s};\boldsymbol{x},t) = \int f_1(\boldsymbol{v};\boldsymbol{x},t)e^{i\boldsymbol{v}\cdot\boldsymbol{s}}d\boldsymbol{v} = \langle e^{i\boldsymbol{U}(\boldsymbol{x},t)\cdot\boldsymbol{s}}\rangle. \tag{34}$$

The Taylor-series expansion of this function is

$$\begin{aligned}\Phi_1(\boldsymbol{s};\boldsymbol{x},t) = 1 &+ \left.\frac{\partial\Phi_1}{\partial s_{i_{(1)}}}\right|_{s=0} s_{i_{(1)}} + \frac{1}{2!}\left.\frac{\partial^2\Phi_1}{\partial s_{i_{(1)}}\partial s_{i_{(2)}}}\right|_{s=0} s_{i_{(1)}}s_{i_{(2)}} \\ &+ \frac{1}{3!}\left.\frac{\partial^3\Phi_1}{\partial s_{i_{(1)}}\partial s_{i_{(2)}}\partial s_{i_{(3)}}}\right|_{s=0} s_{i_{(1)}}s_{i_{(2)}}s_{i_{(3)}} + \cdots\end{aligned} \tag{35}$$

with summation over repeating indices $i_{(1)},i_{(2)},i_{(3)},\ldots, = 1,\ldots,3$. On the other hand, the one-point velocity statistics can be calculated as the n-th order derivative of Φ_1 at the origin

$$\begin{aligned}\left.\frac{\partial\Phi_1}{\partial s_{i_{(1)}}}\right|_{s=0} &= i\left\langle U_{i_{(1)}}(\boldsymbol{x},t)\right\rangle, \\ \left.\frac{\partial^2\Phi_1}{\partial s_{i_{(1)}}\partial s_{i_{(2)}}}\right|_{s=0} &= (i^2)\left\langle U_{i_{(1)}}(\boldsymbol{x},t)U_{i_{(2)}}(\boldsymbol{x},t)\right\rangle,\end{aligned} \tag{36}$$

$$\left.\frac{\partial^3\Phi_1}{\partial s_{i_{(1)}}\partial s_{i_{(2)}}\partial s_{i_{(3)}}}\right|_{s=0} = (i^3)\left\langle U_{i_{(1)}}(\boldsymbol{x},t)U_{i_{(2)}}(\boldsymbol{x},t)U_{i_{(3)}}(\boldsymbol{x},t)\right\rangle, \ldots. \tag{37}$$

Hence,

$$\begin{aligned}\Phi_1 &= 1 + i\left\langle U_{i_{(1)}}(\boldsymbol{x},t)\right\rangle s_{i_{(1)}} \\ &\quad - \frac{1}{2!}\left\langle U_{i_{(1)}}(\boldsymbol{x},t)U_{i_{(2)}}(\boldsymbol{x},t)\right\rangle s_{i_{(1)}}s_{i_{(2)}} + \cdots.\end{aligned} \tag{38}$$

which is also the Taylor series expansion of the exponent $\langle\exp(i\boldsymbol{s}\cdot\boldsymbol{U}(\boldsymbol{x},t))\rangle$ from Eq. (34). If we write this Taylor series for the transformed characteristic function Φ_1^* and substitute the translation symmetry of moments (14) we obtain

$$\Phi_1^*(\boldsymbol{s};\boldsymbol{x},t) = \Phi_1(\boldsymbol{s};\boldsymbol{x},t) + \underbrace{iC_{i_{(1)}}s_{i_{(1)}} - \frac{1}{2!}C_{i_{(1)}i_{(2)}}s_{i_{(1)}}s_{i_{(2)}} - \frac{1}{3!}iC_{i_{(1)}i_{(2)}i_{(3)}}s_{i_{(1)}}s_{i_{(2)}}s_{i_{(3)}}\cdots}_{\phi(\boldsymbol{s})}. \tag{39}$$

The underbraced sum is the Taylor series expansion of a function $\phi(\boldsymbol{s})$ which equals 0 at the origin and its derivatives at the origin equals, respectively, $iC_{i_{(1)}}, -C_{i_{(1)}i_{(2)}}, -iC_{i_{(1)}i_{(2)}i_{(3)}}$, and so on. If Eq. (39) is substituted into Eq. (33), the transformed PDF reads

$$\begin{aligned}f_1^*(\boldsymbol{v};\boldsymbol{x},t) &= f(\boldsymbol{v};\boldsymbol{x},t) + \frac{1}{(2\pi)^3}\int ds e^{-i\boldsymbol{v}\cdot\boldsymbol{s}}\phi(\boldsymbol{s}) \\ &= f_1(\boldsymbol{v};\boldsymbol{x},t) + \psi(\boldsymbol{v}),\end{aligned} \tag{40}$$

where $\psi(\boldsymbol{v})$ is an inverse Fourier transform of $\phi(\boldsymbol{s})$ and

$$\int d\boldsymbol{v}\psi(\boldsymbol{v}) = 0, \tag{41}$$

which follows from the fact that $\phi(0) = 0$. Note that neither $\phi(\boldsymbol{s})$ nor $\psi(\boldsymbol{v})$ depend on $\boldsymbol{x}$ or time t.

In the appendix the above considerations are generalized to the case of the n-point PDF leading to the following form of the transformed function:

$$f_n^* = f_n + \psi(\boldsymbol{v}_{(1)})\delta(\boldsymbol{v}_{(1)} - \boldsymbol{v}_{(2)})\cdots\delta(\boldsymbol{v}_{(1)} - \boldsymbol{v}_{(n)}). \tag{42}$$

It can be readily verified that the transformations (42) correspond to the statistical symmetry (14) where the MPC $\mathbf{H}$ tensors are translated. Indeed, under the transformations (42), the moments of the PDFs are mapped to the following:

$$\begin{aligned}\langle U_{i_{(1)}}(\boldsymbol{x}_{(1)},t)\cdots U_{i_{(n)}}(\boldsymbol{x}_{(n)},t)\rangle^* &\equiv \left(\int d\boldsymbol{v}_{(1)}\cdots d\boldsymbol{v}_{(n)} f_n v_{(1)i_{(1)}}\cdots v_{(n)i_{(n)}}\right)^* = \langle U_{i_{(1)}}(\boldsymbol{x}_{(1)},t)\cdots U_{i_{(n)}}(\boldsymbol{x}_{(n)},t)\rangle \\ &\quad + \int d\boldsymbol{v}_{(1)}\cdots d\boldsymbol{v}_{(n)}\psi(\boldsymbol{v}_{(1)})\,\delta(\boldsymbol{v}_{(1)}-\boldsymbol{v}_{(2)})\cdots\delta(\boldsymbol{v}_{(n)}-\boldsymbol{v}_{(n-1)})\,v_{(1)i_{(1)}}\cdots v_{(n)i_{(n)}} \\ &= \langle U_{i_{(1)}}(\boldsymbol{x}_{(1)},t)\cdots U_{i_{(n)}}(\boldsymbol{x}_{(n)},t)\rangle + \int d\boldsymbol{v}_{(1)}\,\psi(\boldsymbol{v}_{(1)})\,v_{(1)i_{(1)}}\cdots v_{(1)i_{(n)}} = \mathbf{H}_{\{n\}} + \mathbf{C}_{\{n\}}.\end{aligned} \tag{43}$$

Let us first examine as follows which conditions the above transformed functions have to satisfy in order to be an acceptable PDF:

(1) The coincidence property (21) is satisfied by the transformed PDFs (42).

(2) Because of the probabilistic interpretation of the PDFs, it is locally required that

$$f_n + \psi(\boldsymbol{v}_{(1)})\delta(\boldsymbol{v}_{(1)} - \boldsymbol{v}_{(2)})\cdots\delta(\boldsymbol{v}_{(1)} - \boldsymbol{v}_{(n)}) \geqslant 0, \quad \forall \boldsymbol{v} \in \mathbb{R}^3, \forall \boldsymbol{x} \in \mathbb{R}^3, \tag{44}$$

while the normalization condition globally imposes

$$\int d\boldsymbol{v}\,[f_1(\boldsymbol{v};\boldsymbol{x},t) + \psi(\boldsymbol{v})] = 1, \ \ \forall \boldsymbol{x} \in \mathbb{R}^3, \tag{45}$$

from which, recalling the normalization condition (19) for f_1, we get

$$\int d\boldsymbol{v}\ \psi(\boldsymbol{v}) = 0, \tag{46}$$

which agrees with Eq. (41).

(3) The divergence or continuity condition (20) is satisfied, since ψ does not depend on the space variable.

(4) The separation property (22) is not satisfied, as the transformation of the PDF is independent of the spatial variable and thus cannot satisfy this limiting behavior. On the other hand, let us note that, while it is reasonable to require this property, this is never used in the derivation of the equations of the LMN hierarchy. Moreover, this property is not satisfied by the corresponding symmetries of the MPC equations, either.

Finally, let us show that the LMN equations (24) are invariant under the transformations (42). Let us begin with the left-hand side: since ψ is independent of the space variables and of t, we get no extra term. On the right-hand side, we have to take into account two terms, the pressure-gradient one and the viscous one.

As regards the pressure-gradient term, again exploiting the fact that we add a function to the transformed PDF f_{n+1}, which does not depend on the space variable, namely $\psi(\boldsymbol{v}_{(1)})\delta(\boldsymbol{v}_{(1)} - \boldsymbol{v}_{(2)})\cdots\delta(\boldsymbol{v}_{(n)} - \boldsymbol{v}_{(n+1)})$, we get a null contribution when the differential operator ∇_{n+1} is applied to it.

As regards the viscous term, using the function $\delta(\boldsymbol{v}_{(n+1)} - \boldsymbol{v}_{(n)})$ the integral over $\boldsymbol{v}_{(n+1)}$ can be carried out, leaving us with something which does not depend on $\boldsymbol{x}_{n+1}$; the application therefore of the derivatives with respect to $\boldsymbol{x}_{n+1}$ yields a null contribution. The equations of the hierarchy are therefore invariant with respect to the transformations (42).

Hence it follows from (42) and (43) that the function ψ which satisfies the conditions (44) and (46) uniquely determines the set of constants $C_{\{1\}},C_{\{2\}},\ldots,$ for the translation of the $\mathbf{H}$ tensors as in (14). The function ψ, constant in space and time modifies the shape of the PDF and for this reason we will further call it the "shape" symmetry. We also note in passing that the analog of the statistical symmetries (14) for the MPC equations and symmetries (42) for the LMN hierarchy exists also in the Hopf functional formulation [12]. This creates a remarkable symmetry between the three different approaches to the full description of turbulence.

An important question to be asked here is if the transformation (42) together with the conditions (44) and (46) is a Lie group (see, e.g., Ref. [18]), i.e., if it satisfies the four group axioms, namely closure, associativity, identity, and invertibility. The function f_n^* transformed according to (42) is always a solution of Eq. (24) also for, e.g., a transformation with the inverse element of $\psi(\boldsymbol{v}_{(1)})$. However, such f_n^* may not be a PDF any longer, as it may have negative values. For the same reason the associativity axiom may not be satisfied. Hence, the transformation (42) is a Lie group if arbitrary solutions of Eq. (24) are considered. However, if we take into account only such solutions which are PDFs, the group properties are not satisfied due to the condition (44) for the transformed PDF. Then (42) may not necessary be a Lie group but still is a symmetry of the LMN equation Eq. (24).

With this, we also note that the constants $C_{\{1\}},C_{\{2\}}\ldots$ obtained in Eq. (43) are not arbitrary but due to the condition (44) on the functions f_n^* and ψ we expect that they might be contained within a certain range. We conclude that considering the symmetries of the LMN hierarchy provides additional restrictions on the group parameters which were not observed in the MPC approach.

In order to derive these restrictions we attempt to find a physical interpretation of the shape symmetry. We start with the observation that the n-point PDF of a laminar field constant in space and time $\boldsymbol{U}_0^{\{\omega\}}$ where ω is an element from the probability space (i.e., $\boldsymbol{U}_0^{\{\omega\}}$ can differ in different flow realizations) reads

$$\begin{aligned} &f_L(\boldsymbol{v}_{(1)},\ldots,\boldsymbol{v}_{(n)};\boldsymbol{x}_{(1)},\ldots,\boldsymbol{x}_{(n)},t)\\ &=\left\langle\delta\left(\boldsymbol{v}_{(1)}-\boldsymbol{U}_0^{\{\omega\}}\right)\cdots\delta\left(\boldsymbol{v}_{(n)}-\boldsymbol{U}_0^{\{\omega\}}\right)\right\rangle\\ &=\left\langle\delta\left(\boldsymbol{v}_{(1)}-\boldsymbol{U}_0^{\{\omega\}}\right)\right\rangle\delta(\boldsymbol{v}_{(2)}-\boldsymbol{v}_{(1)})\cdots\delta(\boldsymbol{v}_{(n)}-\boldsymbol{v}_{(1)})\\ &=f(\boldsymbol{v}_{(1)})\delta(\boldsymbol{v}_{(2)}-\boldsymbol{v}_{(1)})\cdots\delta(\boldsymbol{v}_{(n)}-\boldsymbol{v}_{(1)}). \end{aligned} \tag{47}$$

If both laminar and turbulent solutions are elements of the ensemble, then the PDF can be written as a sum of a turbulent and laminar part $f=g_T+g_L$, where

$$g_L=g(\boldsymbol{v}_{(1)})\delta(\boldsymbol{v}_{(2)}-\boldsymbol{v}_{(1)})\cdots\delta(\boldsymbol{v}_{(n)}-\boldsymbol{v}_{(1)}), \tag{48}$$

$\int d\boldsymbol{v}_{(1)}g_L=a\leqslant 1$. Let us compare Eq. (48) with the shape symmetry (42). Due to the condition (41) the translation function in (42) has, at a certain range of $\boldsymbol{v}_{(1)}$, negative and infinite values at the diagonal $\boldsymbol{v}_{(2)}=\boldsymbol{v}_{(1)}$, $\boldsymbol{v}_{(3)}=\boldsymbol{v}_{(1)}$, and so on. Hence, we come to the conclusion that for a nonzero function ψ the transformed PDF f_n^* could be non-negative only if the function f_n would contain a laminar part (48) which is also infinite at $\boldsymbol{v}_{(2)}=\boldsymbol{v}_{(1)}$, $\boldsymbol{v}_{(3)}=\boldsymbol{v}_{(1)}$ for arbitrary separations $\boldsymbol{x}_{(2)}-\boldsymbol{x}_{(1)}$, $\boldsymbol{x}_{(3)}-\boldsymbol{x}_{(1)}$, and so on. Equation (42) would then transform the laminar part of the PDF such that we would obtain $f^*=g_T+g_L^*=g_T+g_L+\psi$, where $g_L^*\geqslant 0$. This also explains why the separation property (22) is not satisfied by the shape symmetry, namely, this property refers to the turbulent part of the PDF only. We argue that the restrictions on the constants $C_{\{1\}},C_{\{2\}},\ldots,$ will follow from the range of laminar solutions for velocity realizable in the given flow configuration. We address here the particular case of the fully developed plane Poiseuille channel flow. Due to the presence of the nonmoving boundaries the laminar part of the PDF takes the form

$$\begin{aligned} &f_L(\boldsymbol{v}_{(1)},\ldots,\boldsymbol{v}_{(n)};\boldsymbol{x}_{(1)},\ldots,\boldsymbol{x}_{(n)},t)\\ &=\left\langle\delta\left[\boldsymbol{v}_{(1)}-\boldsymbol{U}_0^{\{\omega\}}\left(1-\frac{y_{(1)}^2}{H^2}\right)\right]\right.\\ &\qquad\left.\cdots\delta\left[\boldsymbol{v}_{(n)}-\boldsymbol{U}_0^{\{\omega\}}\left(1-\frac{y_{(n)}^2}{H^2}\right)\right]\right\rangle, \end{aligned} \tag{49}$$

where $\boldsymbol{U}_0^{\{\omega\}}=[U_0^{\{\omega\}},0,0]$ is the streamwise velocity in the centerline and $y_{(k)}=x_{(k)2}$ is the wall-normal coordinate. Using properties of the δ function the above equation can be rewritten as

$$f_L(\boldsymbol{v}_{(1)},\ldots,\boldsymbol{v}_{(n)};\boldsymbol{x}_{(1)},\ldots,\boldsymbol{x}_{(n)},t) \tag{50}$$

$$\begin{aligned} &=F(y_{(1)},\ldots,y_{(n)})\left\langle\delta\left(\boldsymbol{v}'_{(1)}-\boldsymbol{U}_0^{\{\omega\}}\right)\right\rangle\delta(\boldsymbol{v}'_{(2)}-\boldsymbol{v}'_{(1)})\cdots\delta(\boldsymbol{v}'_{(n)}-\boldsymbol{v}'_{(1)})\\ &=F(y_{(1)},\ldots,y_{(n)})f(\boldsymbol{v}'_{(1)})\delta(\boldsymbol{v}'_{(2)}-\boldsymbol{v}'_{(1)})\cdots\delta(\boldsymbol{v}'_{(n)}-\boldsymbol{v}'_{(1)}), \end{aligned} \tag{51}$$

where

$$F(y_{(1)},\ldots,y_{(n)})=\left(1-\frac{y_{(1)}^2}{H^2}\right)^{-1}\cdots\left(1-\frac{y_{(n)}^2}{H^2}\right)^{-1} \tag{52}$$

and for each k

$$v'_{(k)1}=v_{(k)1}\left(1-\frac{y_{(k)}^2}{H^2}\right)^{-1},\quad v'_{(k)2}=v_{(k)2},\quad v'_{(k)3}=v_{(k)3}. \tag{53}$$

If we compare Eq. (50) with Eqs. (47) and (42) it follows that the shape symmetry in the channel flow has the following form:

$$\begin{aligned} f_n^*=f_n+F(y_{(1)},\ldots,y_{(n)})\psi(\boldsymbol{v}'_{(1)})\delta(\boldsymbol{v}'_{(1)}-\boldsymbol{v}'_{(2)})\\ \cdots\delta(\boldsymbol{v}'_{(1)}-\boldsymbol{v}'_{(n)}). \end{aligned} \tag{54}$$

Such a function gives rise to the following translations of the moments:

$$\begin{aligned} &\langle U_1(\boldsymbol{x}_{(1)},t)\cdots U_1(\boldsymbol{x}_{(n)},t)\rangle^*\\ &=\langle U_1(\boldsymbol{x}_{(1)},t)\cdots U_1(\boldsymbol{x}_{(n)},t)\rangle\\ &\quad+C_{1\ldots1}\left(1-\frac{y_{(1)}^2}{H^2}\right)\cdots\left(1-\frac{y_{(n)}^2}{H^2}\right). \end{aligned} \tag{55}$$

In this case only the $C_{1\ldots1}$ component of the translation tensor $\mathbf{C}_{\{n\}}$ is nonzero. Moreover, the translation tensor depends now on wall-normal coordinates $y_{(1)},\ldots,y_{(n)}$. In order to assess the limits for the translation coefficients we assume that laminar solutions in the channel are realizable up to a certain critical Reynolds number $\mathrm{Re}_{\mathrm{cr}}$ and hence up to a certain maximal centerline velocity $U_{0\mathrm{cr}}$. In such a case the support of the function $\psi(\boldsymbol{v}'_{(1)})$ from Eq. (54) is restricted to $-U_{0\mathrm{cr}}\leqslant v'_{1(1)}\leqslant U_{0\mathrm{cr}}$, hence $-U_{0\mathrm{cr}}(1-\frac{y_{(1)}^2}{H^2})\leqslant v_{1(1)}\leqslant U_{0\mathrm{cr}}(1-\frac{y_{(1)}^2}{H^2})$. Such a transformation of the PDF is sketched in a schematic in Fig. 1. We note here that in order to satisfy the property (46) the ψ function must have a null integral.

The ranges of possible values of coefficients $C_{1\ldots1}$ from (55) can be found by considering the extremum case where the one-point PDF $f_1=f_L=F(y_{(1)})\delta(\boldsymbol{v}'_{(1)}\pm U_{0\mathrm{cr}})$ before the transformation and $f_1^*=f_L^*=F(y_{(1)})\delta(\boldsymbol{v}'_{(1)}\mp U_{0\mathrm{cr}})$, after the transformation. Such a case corresponds to the extremum velocity at the centerline $\mp U_{0\mathrm{cr}}$ before and $\pm U_{0\mathrm{cr}}$ after the transformation. Hence, we find that the coefficients $C_{1\ldots1}$ are contained within the following ranges:

$$-2U_{0\mathrm{cr}}\leqslant C_1\leqslant 2U_{0\mathrm{cr}}, \tag{56}$$

$$-U_{0\mathrm{cr}}^2\leqslant C_{11}\leqslant U_{0\mathrm{cr}}^2, \tag{57}$$

$$-2U_{0\mathrm{cr}}^3\leqslant C_{111}\leqslant 2U_{0\mathrm{cr}}^3, \tag{58}$$

$$\ldots \tag{59}$$

Moreover, additional restrictions on the parameters will follow from the fact that both f_1 and f_1^* must be realizable and satisfy the properties of the PDF, in particular being non-negative. If we knew explicitly the PDF describing the plane channel flow or, equivalently, all velocity statistics as, e.g. invariant solutions of the LMN or MPC hierarchy, then the restrictions on the translation function $F(y)\psi(\boldsymbol{v}')$ could be derived.

2. Intermittency symmetry

Next, we derive an analog of the scaling symmetry of the MPC (16) in the PDF approach. We also make use of the Taylor series representation of the characteristic function (38). The characteristic function Φ_1, transformed according to (16), reads

$$\begin{aligned} \Phi_1^*=1+ie^{a_s}\langle U_{i_{(1)}}(\boldsymbol{x},t)\rangle s_{i_{(1)}}\\ -\frac{1}{2!}e^{a_s}\langle U_{i_{(1)}}(\boldsymbol{x},t)U_{i_{(2)}}(\boldsymbol{x},t)\rangle s_{i_{(1)}}s_{i_{(2)}}+\cdots. \end{aligned} \tag{60}$$

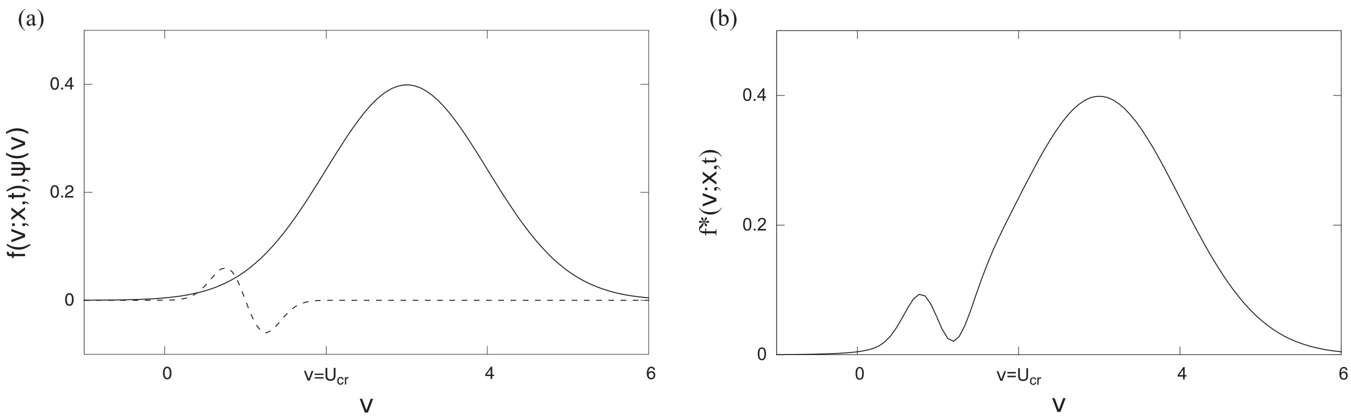

FIG. 1. (a) Initially symmetric PDF $f(\boldsymbol{v};\boldsymbol{x},t)$ (solid line), function $\psi(\boldsymbol{v})$ (dashed line). (b) Transformed, nonsymmetric PDF $f^*(\boldsymbol{v};\boldsymbol{x},t) = f(\boldsymbol{v};\boldsymbol{x},t) + \psi(\boldsymbol{v})$.

We note that the symmetry (16) transforms moments of the velocity, starting from the first-order moment, whereas the first term in the above Taylor series expansion is in fact the normalization of the PDF [$\Phi_n(0) = 1$, hence $\int d\boldsymbol{v}_{(1)}\ldots d\boldsymbol{v}_{(n)} f_n = 1$]. This term cannot be scaled in order not to violate the properties of the PDF. Substituting (60) into Eq. (33) we obtain the transformed PDF, which can be written in the following form:

$$f_1^*(\boldsymbol{v};\boldsymbol{x},t) = \delta(\boldsymbol{v}) + \frac{1}{(2\pi)^3}e^{a_s}\int ds e^{-\mathrm{i}\boldsymbol{v}\cdot\boldsymbol{s}}\Big[i\langle U_{i_{(1)}}(\boldsymbol{x},t)\rangle s_{i_{(1)}} - \frac{1}{2!}\langle U_{i_{(1)}}(\boldsymbol{x},t)U_{i_{(2)}}(\boldsymbol{x},t)\rangle s_{i_{(1)}}s_{i_{(2)}} + \cdots\Big],$$

$$f_1^*(\boldsymbol{v};\boldsymbol{x},t) = \delta(\boldsymbol{v}) + \frac{1}{(2\pi)^3}e^{a_s}\int ds e^{-\mathrm{i}\boldsymbol{v}\cdot\boldsymbol{s}}(\Phi_1 - 1), \tag{61}$$

$$f_1^*(\boldsymbol{v};\boldsymbol{x},t) = \delta(\boldsymbol{v}) + e^{a_s}(f_n - \delta(\boldsymbol{v})). \tag{62}$$

As it is shown in the Appendix, the symmetry (62) may be extended to the n-point PDF according to

$$f_n^* = \delta(\boldsymbol{v}_{(1)})\cdots\delta(\boldsymbol{v}_{(n)}) + e^{a_s}[f_n - \delta(\boldsymbol{v}_{(1)})\cdots\delta(\boldsymbol{v}_{(n)})]. \tag{63}$$

The function f_n^* must satisfy all the properties of the PDFs. Hence, we note that

(1)

$$\delta(\boldsymbol{v}_{(1)})\cdots\delta(\boldsymbol{v}_{(n)}) + e^{a_s}[f_n - \delta(\boldsymbol{v}_{(1)})\cdots\delta(\boldsymbol{v}_{(n)})] \geqslant 0, \quad \forall\boldsymbol{v}\in\mathbb{R}^3, \forall\boldsymbol{x}\in\mathbb{R}^3, \tag{64}$$

which, for a continuous function f_n implies that $e^{a_s} \leqslant 1$, hence, $a_s \leqslant 0$. However, if the initial PDF f_n has the form $f_n = g_n + c\delta(\boldsymbol{v}_{(1)})\cdots\delta(\boldsymbol{v}_{(n)})$), where $0 < c < 1$ is a constant and g_n is a continuous function such that $\int d\boldsymbol{v}_{(1)}\ldots d\boldsymbol{v}_{(n)}g_n = 1 - c$, then the condition (64) for such a function implies that $e^{a_s} \leqslant 1/(1-c)$. Apparently, such restrictions for the scaling parameter a_s means that the group axioms are not satisfied by the transformation (63).

(2) The normalization condition (19) is satisfied.

(3) The coincidence property (21) is satisfied as $\delta(\boldsymbol{v}_{(1)})\cdots\delta(\boldsymbol{v}_{(n)}) = \delta(\boldsymbol{v}_{(1)})\delta(\boldsymbol{v}_{(1)} - \boldsymbol{v}_{(2)})\cdots\delta(\boldsymbol{v}_{(n-1)} - \boldsymbol{v}_{(n)})$.

(4) The divergence or continuity condition (20) is satisfied, since $\delta(\boldsymbol{v}_{(1)})\cdots\delta(\boldsymbol{v}_{(n)})$ does not depend on the space variable.

(5) As far as the separation property (22) is concerned, we note that if the PDF is itself a δ function, then, according to Eq. (63), $f_n^*(1,\ldots,n) = f_n(1,\ldots,n) = \delta(\boldsymbol{v}_{(1)})\cdots\delta(\boldsymbol{v}_{(n)})$. Hence, if in the far field $|\boldsymbol{x}_{(1)} - \boldsymbol{x}_{(2)}| \to \infty$ the one-point PDF is a δ function $f_1(2) = f_1^*(2) = \delta(\boldsymbol{v}_{(2)})$, then the separation property is satisfied for the scaling invariance,

$$\lim_{|\boldsymbol{x}_{(1)}-\boldsymbol{x}_{(2)}|\to\infty} f_2^*(1,2) = e^{a_s}f_1(1)\delta(\boldsymbol{v}_{(2)}) + (1-e^{a_s})\delta(\boldsymbol{v}_{(1)})\delta(\boldsymbol{v}_{(2)}) = f_1^*(1)\delta(\boldsymbol{v}_{(2)}) = f_1^*(1)f_1^*(2). \tag{65}$$

If we note that $\delta(\boldsymbol{v}_{(1)})\cdots\delta(\boldsymbol{v}_{(n)})$ does not depend on the time and space variables, we see that the LMN equations (24) are in fact invariant under the transformation (63). The moments calculated from the transformed PDFs read

$$\langle U_{i_{(1)}}(\boldsymbol{x}_{(1)},t)\cdots U_{i_{(n)}}(\boldsymbol{x}_{(n)},t)\rangle^* \tag{66}$$

$$= \int d\boldsymbol{v}_{i_{(1)}}\cdots d\boldsymbol{v}_{i_{(n)}}[f_n e^{a_s} + (1-e^{a_s})\delta(\boldsymbol{v}_{(1)})\cdots\delta(\boldsymbol{v}_{(n)})] \times v_{(1)i_{(1)}}\cdots v_{(n)i_{(n)}}$$

$$= e^{a_s}\langle U_{i_{(1)}}(\boldsymbol{x}_{(1)},t)\cdots U_{i_{(n)}}(\boldsymbol{x}_{(n)},t)\rangle = e^{a_s}\mathbf{H}_{\{n\}}, \tag{67}$$

which is identical to Eq. (16). In an attempt to find a physical interpretation of the symmetry transformation (63) we consider an example of a continuous one-point distribution f_1, cf. Fig. 2. The function $f_1 - \delta(\boldsymbol{v})$ is scaled by $e^{a_s} < 1$. The transformed PDF $f_1^* = e^{a_s}f_1 + (1-e^{a_s})\delta(\boldsymbol{v})$ is a noncontinuous function with δ at $\boldsymbol{v} = 0$. Such PDF functions characterize the intermittent flows (where by "intermittency" we understand a flow with subsequently changing turbulent and nonturbulent regimes). With this at hand, we can justify that the presence of the same scaling exponent e^{a_s} for velocity statistics of any order in the MPC equations, cf. Eq. (16), follows from the averaging of an intermittent signal.

3. Invariant solution for the mean velocity in a plane channel flow

We derive here an invariant solution for the first-order statistics in the plane channel flow and discuss possible restrictions of parameters in this solution. With the physical interpretation of the statistical symmetries presented in this paper the derivation slightly differs from those presented in

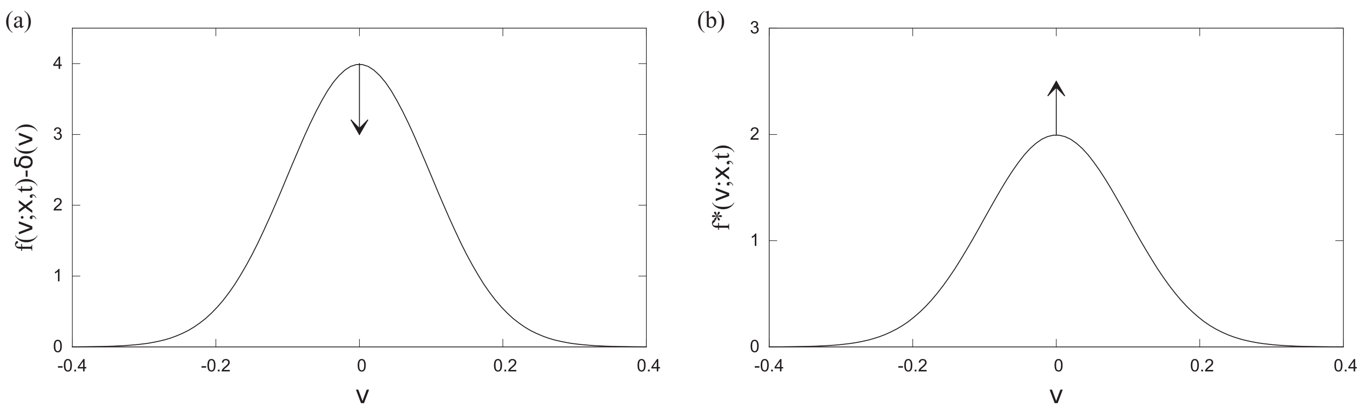

FIG. 2. (a) δ function subtracted from a PDF. (b) Transformed PDF $f^*(\boldsymbol{v}) = e^{a_s} f_1 + (1 - e^{a_s})\delta(\boldsymbol{v})$.

Refs. [3] and [25], as, finally, a solution of an intermittent flow, having both laminar and turbulent parts is derived.

We take into account two particular symmetries: classical scaling of the Navier-Stokes equations $y^* = e^{k_2} y$ and $\langle U\rangle^* = e^{-k_2}\langle U\rangle$, where k_2 is an arbitrary constant; new scaling symmetry $\langle U\rangle^* = e^{a_s}\langle U\rangle$; and translation of the mean velocity $\langle U\rangle^* = \langle U\rangle + C_1(1 - y^2/H^2)$, cf. Eq. (55), where C_1 is restricted by condition (56). The invariant solution can be found from the solution of the characteristic equation [25]

$$\frac{d\langle U\rangle}{(a_s - k_2)\langle U\rangle + C_1(1 - y^2/H^2)} = \frac{dy}{k_2 y}, \tag{68}$$

which for $a_s = k_2$ gives

$$\langle U\rangle = \frac{C_1}{k_2}\ln(y) + \frac{C_1}{2k_2}\left(1 - \frac{y^2}{H^2}\right) + \mathcal{C}, \tag{69}$$

where $\mathcal{C}$ is a constant. The formula above is, apparently, a sum of the turbulent and laminar velocity in the plane channel flow. As the laminar velocity in the centerline is contained within $\langle -U_{0\mathrm{cr}}, U_{0\mathrm{cr}}\rangle$,

$$-U_{0\mathrm{cr}} \leqslant \frac{C_1}{2k_2} \leqslant U_{0\mathrm{cr}}. \tag{70}$$

The coefficients k_2, C_1 as well as C_{11}, C_{111}, etc., from Eqs. (56)–(58) will be ingredients of scaling laws for higher-order velocity moments. We expect that more restrictions on these coefficients could be derived when considering higher-order moments and, mutually, the restrictions on coefficients would provide restrictions on parameters contained in the scaling laws. Here we only addressed a possible route of determining the ranges of realizable values of the coefficients; a complete derivation is, however, left for further work.

IV. SUMMARY AND CONCLUSIONS

In this paper we investigated the LMN hierarchy for the multipoint PDFs and derived the symmetries for this infinite set of equations, cf. (42) and (63). They correspond to the statistical symmetries of the hierarchy of the multipoint correlation equations, previously derived in Ref. [3], namely (42) correspond to shifting of the MPC tensors $\mathbf{H}$ by constants and (63) is an analog of the scaling of $\mathbf{H}$. Such correspondences were expected, with the MPC and the LMN being equivalent pictures of the same subject, turbulence. However, as already pointed out in Sec. III D, the properties of the PDFs provide additional constraints on the transformations, in particular, the translation coefficients in (42) cannot take arbitrary values and the exponent e^{a_s} in (63) should also satisfy certain condition depending on the shape of the function f_n to be transformed.

Two important conclusions follow from this observation. First, it is noted that the transformations (42) and (63), and, consequently, also the transformations of the MPC hierarchy (14) and (16), do not constitute Lie groups, as some of the group axioms are not satisfied, e.g., (63) has no inverse element and forms, instead, a semigroup. Second, with the above-mentioned constraints on the group parameters of the shape and intermittency symmetries obtain first-principles hints which may restrain the values of the scaling-law parameters such as κ in the classical lag law. Such parameters are simple functions of the group parameters, in particular depending on the above-mentioned statistical groups.

Moreover, consider that the LMN hierarchy gives us more insight into the meaning of the statistical symmetries and on how it is represented in the sample space of instantaneous velocity. We considered a PDF with laminar and turbulent parts for a one-point PDF $f = g_L + g_T$ such that $\int d\boldsymbol{v} g_L = a$ and $\int d\boldsymbol{v} g_T = 1 - a$. It was argued that the symmetry (42) applied to such a PDF transforms its laminar part, $f^* = g_L^* + g_T$, leaving the coefficient a unchanged. The second transformation of the PDFs (63) distorts them into noncontinuous functions describing the intermittent flow with changing turbulent and nonturbulent regimes. It changes in fact the coefficient a, i.e., increases or decreases the contribution of laminar velocities in the PDF. Hence, only by considering the LMN hierarchy did it become clear that both statistical symmetries are connected with intermittent turbulent or laminar flows.

Symmetry analysis provides a valuable link between the mathematics and engineering applications. It has been found that the prediction ability of engineering models depends on how many symmetries of the Navier-Stokes and equations derived thereoff are recovered, and each additional unphysical symmetry that does not exist in the Navier-Stokes equation deteriorated the model predictions. The set of statistical symmetries which should also be included in turbulence models allows for further investigation of existing turbulence closures. In particular, the influence of these symmetries on the prediction of laminar or turbulent flows, flows with laminar-turbulent transition, and relaminarization will be investigated.

Finally, we mention that, due to the fact that the LMN hierarchy is infinite, we cannot prove that our set of symmetries is complete. Hence, finding new statistical symmetries and/or proving the completeness of the set of symmetries is a next task a further study.

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APPENDIX

The n-point characteristic function Φ_n can be represented as a Fourier transform of the n-point PDF as follows [24]:

$$f_n = \frac{1}{(2\pi)^{3n}} \int d\boldsymbol{s}_{(1)} \cdots d\boldsymbol{s}_{(n)} e^{-i\boldsymbol{v}_{(1)}\cdot\boldsymbol{s}_{(1)}} \cdots e^{-i\boldsymbol{v}_{(n)}\cdot\boldsymbol{s}_{(n)}} \Phi_n(\boldsymbol{s}_{(1)},\ldots,\boldsymbol{s}_{(n)};\boldsymbol{x}_{(1)},\ldots,\boldsymbol{x}_{(n)},t) \tag{A1}$$

and the n-th order moments are calculated as n-th order derivative of Φ_n at the origin

$$\left.\frac{\partial \Phi_n}{\partial s_{i_{(1)}} \cdots \partial s_{i_{(n)}}}\right|_{s=0} = (i)^n \langle U_{i_{(1)}}(\boldsymbol{x}_{(1)},t) \cdots U_{i_{(n)}}(\boldsymbol{x}_{(n)},t)\rangle. \tag{A2}$$

Hence, the function Φ_n can be expanded in a Taylor series yielding

$$\Phi_n = 1 - i \sum_{k=1}^{n} \sum_{i_{(1)}=1}^{3} \langle U_{i_{(1)}}(\boldsymbol{x}_{(k)},t)\rangle s_{(k)i_{(1)}} \tag{A3}$$

$$-\frac{1}{2!} \sum_{k,l=1}^{n} \sum_{i_{(1)},i_{(2)}=1}^{3} \langle U_{i_{(1)}}(\boldsymbol{x}_{(k)},t) U_{i_{(2)}}(\boldsymbol{x}_{(l)},t)\rangle s_{(k)i_{(1)}} s_{(l)i_{(2)}} + \cdots . \tag{A4}$$

If the above formula is substituted into Eq. (A1), the translation symmetry of the moments (A4) is represented as the following translation of the PDF with the elements of the tensors $\mathbf{C}_{\{n\}}$ as group parameters,

$$\begin{aligned} f_n^* = f_n + \frac{1}{(2\pi)^{3n}} \int d\boldsymbol{s}_{(1)} \cdots d\boldsymbol{s}_{(n)} e^{-i\boldsymbol{v}_{(1)}\cdot\boldsymbol{s}_{(1)}} \cdots e^{-i\boldsymbol{v}_{(n)}\cdot\boldsymbol{s}_{(n)}} \cdot \\ \times \left(-i \sum_{k=1}^{n} \sum_{i_{(1)}=1}^{3} C_{i_{(l)}\{1\}} s_{(k)i_{(1)}} - \frac{1}{2!} \sum_{k,l=1}^{n} \sum_{i_{(1)},i_{(2)}=1}^{3} C_{i_{(1)}i_{(2)}\{2\}} s_{(k)i_{(1)}} s_{(l)i_{(2)}} + \cdots \right), \end{aligned} \tag{A5}$$

which can be rewritten as

$$\begin{aligned} f_n^* = f_n + \frac{1}{(2\pi)^{3n}} \int d\boldsymbol{s}_{(1)} \cdots d\boldsymbol{s}_{(n)} e^{-i\boldsymbol{v}_{(1)}\cdot\boldsymbol{s}_{(1)}} \cdots e^{-i\boldsymbol{v}_{(n)}\cdot\boldsymbol{s}_{(n)}} \cdot \\ \times \left(-i \sum_{i_{(1)}=1}^{3} C_{i_{(l)}\{1\}} \sum_{k=1}^{n} s_{(l)i_{(1)}} - \frac{1}{2!} \sum_{i_{(1)},i_{(2)}=1}^{3} C_{i_{(1)}i_{(2)}\{2\}} \sum_{k,l=1}^{n} s_{(k)i_{(1)}} s_{(l)i_{(2)}} + \cdots \right). \end{aligned} \tag{A6}$$

The term in brackets is a Taylor-series expansion of a function (let us call it ϕ) of a sum $\sum_{k=1}^{n} \boldsymbol{s}_{(k)}$ which equals zero at the origin [$\phi(0) = 0$] and is uniform in space and time

$$f_n^* = f_n + \frac{1}{(2\pi)^{3n}} \int d\boldsymbol{s}_{(1)} \cdots d\boldsymbol{s}_{(n)} e^{-i\boldsymbol{v}_{(1)}\cdot\boldsymbol{s}_{(1)}} \cdots e^{-i\boldsymbol{v}_{(n)}\cdot\boldsymbol{s}_{(n)}} \phi(\boldsymbol{s}_{(1)} + \boldsymbol{s}_{(2)} + \cdots + \boldsymbol{s}_{(n)}), \tag{A7}$$

which, after the change of the integration variables $\boldsymbol{s} = \boldsymbol{s}_{(1)} + \boldsymbol{s}_{(2)} + \cdots + \boldsymbol{s}_{(n)}$ gives

$$f_n^* = f_n + \frac{1}{(2\pi)^{3n}} \int d\boldsymbol{s} d\boldsymbol{s}_{(2)} \cdots d\boldsymbol{s}_{(n)} e^{-i\boldsymbol{v}_{(1)}\cdot\boldsymbol{s}} e^{-i\boldsymbol{s}_{(2)}\cdot(\boldsymbol{v}_{(2)}-\boldsymbol{v}_{(1)})} \cdots e^{-i\boldsymbol{s}_{(n)}\cdot(\boldsymbol{v}_{(n)}-\boldsymbol{v}_{(1)})} \phi(\boldsymbol{s}), \tag{A8}$$

$$f_n^* = f_n + \psi(\boldsymbol{v}_{(1)}) \delta(\boldsymbol{v}_{(1)} - \boldsymbol{v}_{(2)}) \cdots \delta(\boldsymbol{v}_{(1)} - \boldsymbol{v}_{(n)}). \tag{A9}$$

where $\psi(\boldsymbol{v}_{(1)})$ is an inverse Fourier transform of $\phi(\boldsymbol{s})$ and

$$\int d\boldsymbol{v}_{(1)}\psi = 0. \tag{A10}$$

The scaling symmetry of the MPC (16) can be calculated by substituting the transformed moments to Eq. (A3)

$$\Phi_n^* = 1 + i e^{a_s} \sum_{k=1}^{n} \sum_{i_{(1)}=1}^{3} \langle U_{i_{(1)}}(\boldsymbol{x}_{(k)},t)\rangle s_{(k)i_{(1)}} \tag{A11}$$

$$- \frac{1}{2!} e^{a_s} \sum_{k,l=1}^{n} \sum_{i_{(1)},i_{(2)}=1}^{3} \langle U_{i_{(1)}}(\boldsymbol{x}_{(k)},t) U_{i_{(2)}}(\boldsymbol{x}_{(l)},t)\rangle s_{(k)i_{(1)}} s_{(l)i_{(2)}} + \cdots . \tag{A12}$$

$$\Phi_n^* = 1 + e^{a_s}(\Phi - 1). \tag{A13}$$

Substituting the above result to (A1) we obtain the transformed, n-point PDF,

$$f_n^* = \delta(\boldsymbol{v}_{(1)}) \cdots \delta(\boldsymbol{v}_{(n)}) + e^{a_s}[f_n - \delta(\boldsymbol{v}_{(1)}) \cdots \delta(\boldsymbol{v}_{(n)})]. \tag{A14}$$

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