Damage accumulation in quasibrittle fracture

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The strength of quasibrittle materials depends on the ensemble of defects inside the sample and on the way damage accumulates before failure. Using large-scale numerical simulations of the random fuse model, we investigate the evolution of the microcrack distribution as the applied load approaches the fracture point. We find that the distribution broadens mostly due to a tendency of cracks to coalesce in a way that increases with system size. We study how the observed behavior depends on the disorder present in the sample and relate the results with fracture size effects.

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I. INTRODUCTION

The fracture strength of materials is a problem whose general understanding is still based on empirical knowledge that would benefit from a more solid fundamental theory (for a review see [1,2]). In the simplest case without time-dependent rheology or memory effects due to plastic deformation, the only trace left in the sample by the loading history is the accumulated damage. In these conditions, the size dependence of the fracture (peak) strength can be understood by extreme value theory and renormalization group methods [3,4]. The main issue can be summarized as follows: Does one of the limiting extreme value distributions (i.e., the Gumbel, Weibull, or Frechet distribution) [5] describe fracture and why?

Engineers have used the Weibull distribution for decades on the basis of empirical considerations [6], although a different answer is found, in the limit of dilute disorder, considering the stresses induced by (micro)cracks randomly arranged in the sample [7,8]. Fracture is ruled by the subvolume with the largest defect that induces failure at a stress that can be estimated by linear elastic fracture mechanics [7,8]. If we consider an initial exponential defect distribution and neglect subsequent damage accumulation, the problem is directly solvable in terms of a modified Gumbel distribution [7,8], which can be shown to flow asymptotically to the limiting Gumbel case [3]. When stress enhancements turn out to be negligible, the Gaussian (normal) distribution arises instead (e.g., in the fiber bundle model with global load sharing [9]). As recently shown in Ref. [3], damage accumulation cannot be neglected as originally believed [7,8] and should therefore be studied to understand fracture size effects.

Damage accumulation in a disordered sample has also important implications for damage mechanics, a widely used method to treat the effect of microcracks by continuous variables defined over representative volume elements [10–12]. Simplified statistical physics models provide some key answers on the role of accumulated damage [13,14], which effectively changes the defect distribution. Recent studies have shown that damage accumulation and microcrack long-range interactions do not change the general form of the

distribution [4], but affect their tails in a significant way [3]. It is still unclear, however, if the change of the distribution tail due to the accumulated damage is due to the growth and coalescence of existing cracks or to the nucleation of new cracks.

Here we study the development of microcracks under loading for various disorders and as a function of sample size by numerical simulations of the random fuse model [15]. In this discretized scalar fracture model, one can keep track of the additional element failures and analyze in detail the microcrack geometries and densities. The main result we obtain is that the development of a broad damage distribution tail, of large defects, is a rare-event phenomenon: In each sample, the largest crack contributing to that tail is a unique case and it typically arises from the coalescence of two smaller microcracks. This phenomenon exhibits scaling with disorder and sample size and indicates how the details of the damage mechanics will influence the stress in a quantitative way. This is because of the general form of the defect population, which merely dictates the form of the extremal distribution and the scaling of failure stress with sample size.

The structure of the rest of this paper is as follows. In Sec. II we present briefly the numerical model, concentrating more on the analysis of damage and crack evolution. Section III presents the results. First we discuss the development of damage as a function of various parameters up to the maximum stress (current) and connect it to the known scaling of the peak stress. Then we analyze the microcrack distributions at maximum and pay particular attention to where the largest cracks (in each sample) come from. We present a scaling analysis of their properties with sample size and disorder. Finally, we look more closely at the creation of the largest ones to examine what the role of damage is in crack coalescence.

II. MODEL

We perform numerical simulations of the two-dimensional random fuse model (RFM) [15]. In the RFM we study a set of conducting fuses, with unit conductivity $\sigma_0 = 1$, that are arranged on a $45°$-tilted square lattice of size $L \times L$. A voltage
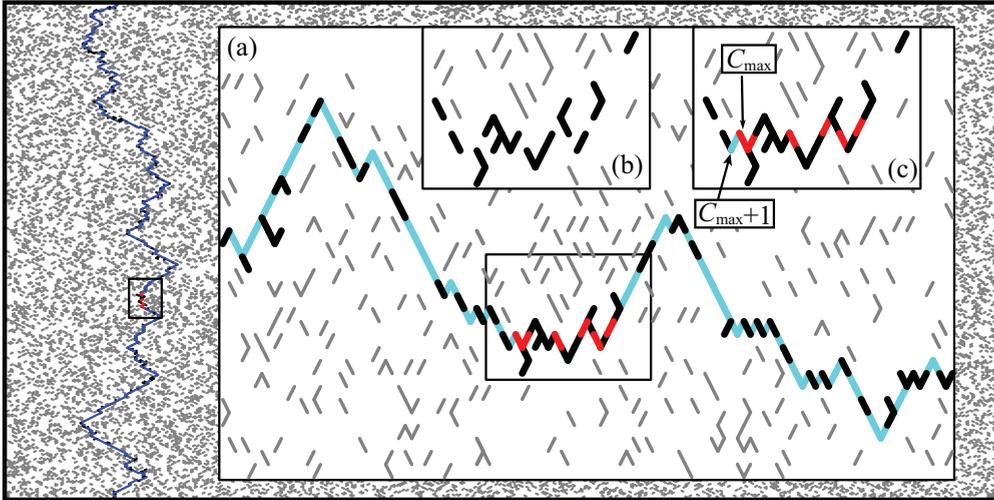
FIG. 1. (Color online) Sample snapshot from a 512×512 system. A fraction $n_r = 0.10$ of bonds has been removed at the beginning (in gray narrow segments, or black if belonging to the final spanning crack). In red (dark gray) the bonds broken during the prepeak damage and in cyan (light gray) or blue the ones that belong to the final avalanche. The inset in (a) shows a section of neighborhood for the final crack containing the red (dark gray) bonds in particular. More detail is shown in the two smaller insets: (b) the initial configuration and (c) the configuration at the peak. The bonds broken from the beginning contribute by bridging to increase the length of the cracks that will fail the entire sample. Here $C_{\max}$ indicates the maximum current and $C_{\max} + 1$ the next current value, as the current has discrete values for each of the bonds failed.

drop is applied along two parallel edges of the lattice while periodic boundary conditions are imposed along the other two edges. Initially, a fraction n_r of the bonds are removed at random to result in a statistically homogeneous damage field. It is easy to show, as already noted in the early extensive studies of this model [7], that an exponential distribution describes the initial microcrack distribution $P_{0,L}(w)$, where w is the crack width, defined as the length of the crack projected on the horizontal direction. The external voltage is then increased adiabatically and the current in each fuse is obtained by a solution of the Kirchhoff equations. When the current i_j in fuse j overcomes a unit threshold $t = 1$, the fuse burns irreversibly. In practice, fuses are broken one by one starting from the one with the largest current until the lattice is disconnected in two parts [1]. To quantify the damage, we consider the connected crack clusters for each L and show results for their number $N_{\text{clu}}(w)$ in what follows. We then analyze the evolution of the microcrack distribution $P_{N,L}(w)$, where, as indicated above in the case of the initial distribution, we denote by N the number of bonds (fuses) failed in each sample as a result of loading.

The following analyses are conditioned by the percolation threshold ($n_{r,c} = 0.5$ for the current geometry). One expects to observe different damage development in both the weak disorder and percolation limits. Larger values of n_r make the numerical study of crack geometries cumbersome, thus most of the results are confined to $n_r \leqslant 0.35$. This also excludes the crossovers from percolation-dominated to bulklike fracture that happens with increasing L in the proximity of the percolation value $n_{r,c}$ at which other phenomena would happen [16]. We first look in detail at the relevant damage, before concentrating on its role in the formation of the critical microcracks and the crack population dynamics.

III. DAMAGE ACCUMULATION AND CRACKS

Figure 1 shows an initial damage state and the system at peak current $C_{\max}$, with the damage at peak being $N_{\max} \equiv N(C_{\max}) = 5$. This quantity $N_{\max}$ is a fundamental one to characterize damage: How does its average scale with disorder and sample size and how does it vary from sample to sample? We notice the following details in the damage development. The prepeak damage is small (that is, the values of N up to the maximum current $C_{\max}$, $N_{\max}$). Part of that damage, if not all of it, is concentrated locally around some preexisting crack, creating (right small inset in the figure) a microcrack that turns out to be the critical one. This crack is created by the coalescence of microcracks by damage and bridging bonds in a process that we analyze separately below. Finally, an unstable avalanche creates the final fracture surface, which is indicated by the blue bonds in the figure. Thus, in addition to N and $N_{\max}$, also the role of the particular failed fuses is important. One can *a posteriori* study if the last failure event increasing the damage from $N_{\max} - 1$ to $N_{\max}$ is a bridging bond or not.

A. Scaling of damage

Figure 2 shows the peak damage sample-averaged $N_{\max}$ in a plot where L and $N_{\max}$ are depicted on logarithmic scales. Note that the z axis starts from unity, since one needs always to break at least one fuse to bring the system to failure. Across the variety of disorder strengths n_r and L values present here it is clear that the typical damage is small, while the strength distribution itself is of the modified Gumbel-type. In both limits of n_r we see that the damage approaches the minimum value. In between, there is a peak in the damage for an n_r value, which shifts with L and might seem to saturate (i.e., not approach $n_{r,c}$ as would be the opposite case).
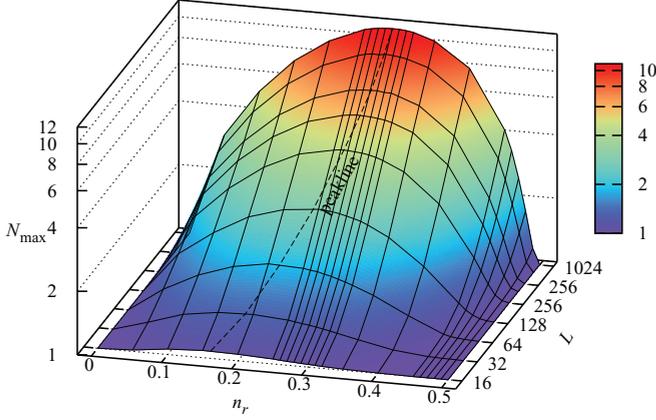
FIG. 2. (Color online) The amount of peak damage (number of broken bonds at maximum current) varies as a function of both the disorder n_r and the size of the system L (N_{max} and L on logarithmic scales). For every size a peak develops for some n_r, whose location (dashed line) increases with both the size and the initial disorder, but that seems to reach an asymptotic value for large sizes.

A more detailed look at the damage scaling in Fig. 3 indicates actually the same peak effect, as the largest disorder case (i.e., $n_r = 0.35$) illustrates. All the finite-size effects seem to adhere to a power-law increase of $N_{max}(L)$, which is subextensive $N_{max}/L^2 \propto L^b$, with b negative. Note that the exponent b changes monotonically with n_r and the peak damage behavior with n_r arises thus from a decrease in the prefactor of the power law. This would imply that the apparent asymptotic saturation of the peak damage in Fig. 2 is just illusory and extending the simulations to larger L should confirm this.

B. Crack populations at peak strength

The main question to address is how the damage influences the cracks at peak strength. Figure 4 shows both a typical
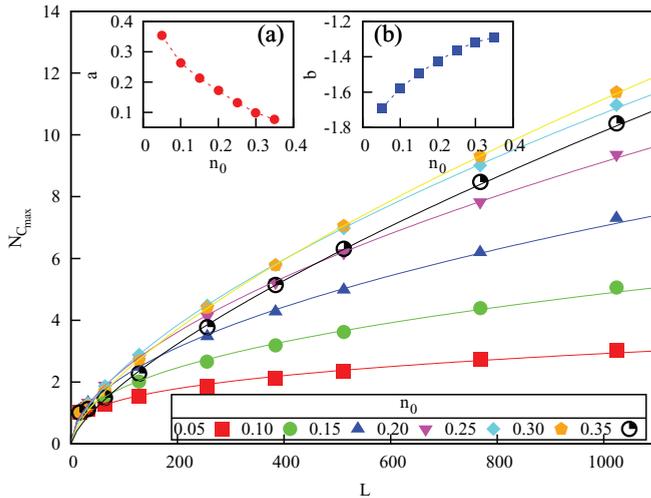

FIG. 3. (Color online) Peak damage (cf. Fig. 2) N_{max} as a function of the size L for various disorders n_r. For every fixed disorder n_r, $N_{max}(L)$ is well described by a power law $N_{max}(L)/L^2 = aL^b$. The fitting parameters for different disorders are showed in the insets: (a) prefactor and (b) exponent.
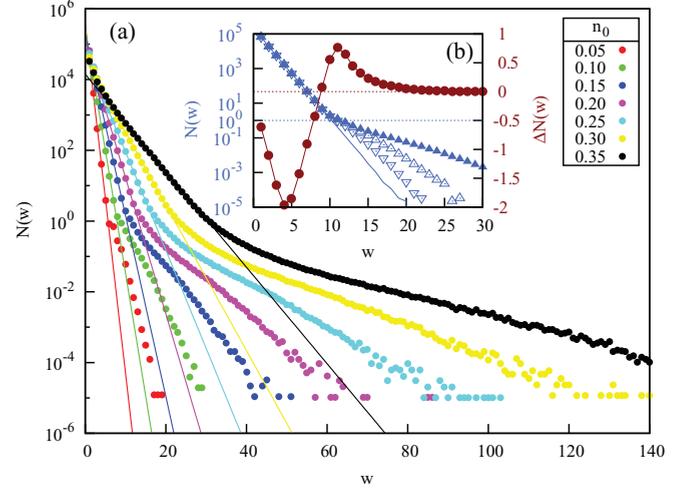

FIG. 4. (Color online) (a) Average number of clusters $N_{clu}(w)$ of a given width w per sample. The data corresponding to various disorder values, labeled from top to bottom in the legend, are displayed in the figure from left to right. The distribution at the beginning is exponential (solid line), while at the peak it develops a tail with a different slope (data points only). (b) Two distributions for a given disorder ($n_r = 0.15$) and size ($L = 1024$) [blue (light gray) triangles, y axis on the left] and their difference [red (dark gray) circles, y axis on the right]. The open-triangle data points present the two distributions at the peak obtained by subtracting the largest and the second-largest crack.

result and the main concepts of a detailed analysis to this end. One can compare at the peak current C_{max} among several microcrack populations: the original at zero damage ($N = 0$), the actual one at peak strength, and the one obtained by subtracting for each sample from the peak strength one the largest and second-largest microcracks.

As was already pointed out in Ref. [3], a wider exponential tail develops at the peak. This, by looking at the difference of the peak and original distributions, arises from the coalescence of original microcracks. By direct observation as such, but also by looking at the distributions with the largest microcrack (or second largest as well) removed, it becomes clear that the tail is indeed averaged over the largest cracks in each sample.

Interestingly, Fig. 5 shows that the mass transport (change of probability distribution from the initial distribution) of the size distributions of cracks exhibits for all n_r a power-law scaling with L. The inset of Fig. 4 allows us to identify three particular values for each L and n_r: the negative minimum, the positive maximum, and where the difference of the distributions is zero. All three follow for each n_r a power-law scaling with the same exponent. The prefactors and power-law exponents show monotonic trends with disorder.

Given that one can find the largest crack in each system, the question remains how w and N_{max} correlate with the strength (C_{max}) of each sample and with each other. Obviously, there might be a slight correlation between the two geometrical quantities, since the largest cracks should develop if the system undergoes more failures N. The initial step, with $N = 1$, is related to the initial strength C_1 (or current C at which the
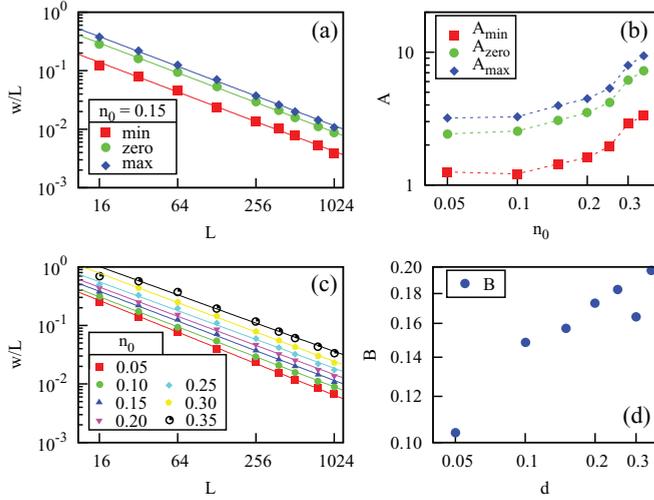
FIG. 5. (Color online) Evolution of crack distributions up to the peak. (a) For $n_r = 0.15$ the points (measured in w units) of the maximum and minimum of the differences of distributions as well as the point of zero difference (showed as red circles in the inset of Fig. 4) all scale following a power law $w(L) = AL^B$ with the same slope. This is true for every disorder, as shown in (c) (just the case of the maximum here). Also shown are the parameters resulting from the fitting: (b) The prefactor and (d) the exponent increase with the disorder.
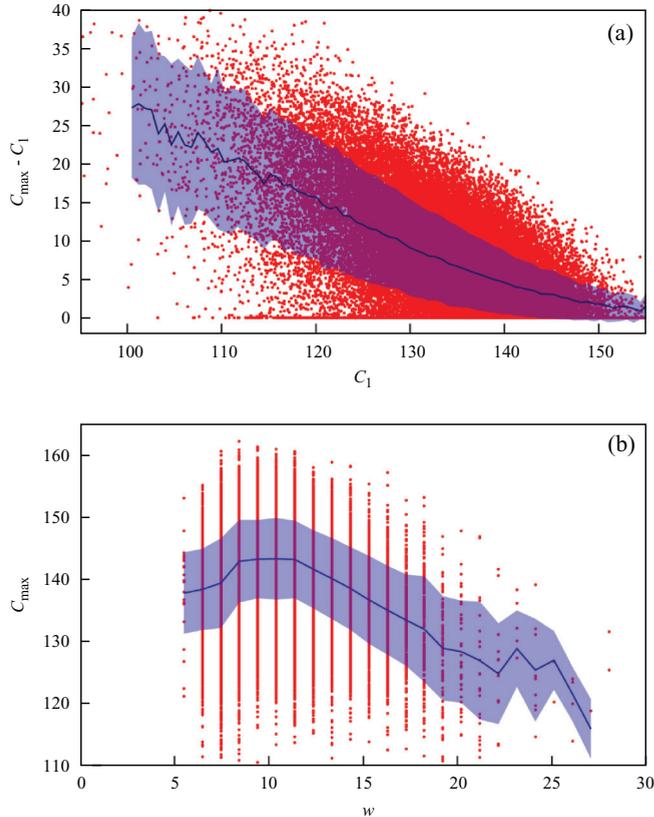

FIG. 6. (Color online) Scatter plot of peak strength values C_{max} vs the sample maximum defect size w. The figure includes a 50-point running average over the w values (black line). The data represented here are obtained from 10^5 realizations of a sample of size $L = 512$ with initial disorder $n_r = 0.10$.

first bond fails). There seems to be a slight anticorrelation in that a large C_1 implies a smaller final w. If the first bond breaks late, then relatively speaking there is less crack growth. Figure 6 shows the correlation of C_{max} and w: Obviously, this does not follow a naive linear elastic fracture mechanics prediction ($C_{max} \propto 1/\sqrt{w}$) one to one. It moreover illustrates that the damage that develops is able to decorrelate C_{max} from C_1. This is not a novelty, since it is known that in the RFM C_1 has a similar modified Gumbel scaling [8] as C_{max} but with different parameters and that there is no one-to-one correspondence between C_{max} and C_1, which would otherwise render fracture prediction for any sample rather trivial [8]. However, the comparison of w with C_{max} and C_1 allows us to conclude that such a decorrelation must be due to the fact that the final, critical defect experiences a stress that has a random component in addition to the one present at C_1.

C. Crack coalescence

The results presented above indicate that a small amount of damage is enough to have a profound influence on the kind of dominating microcracks at peak and thus also on the sample strength. Theoretically, the question could be formulated by a Smoluschowsky-like system of rate equations for defects of size w to a degree of some generality indeed. As we have already indicated, in the current case of dilutionlike random disorder the most important mechanism seems to be crack coalescence of fusion.

The defining rates or processes for $P_{N+1,L}(w)$ when $N \rightarrow N+1$—in other words a bond is broken—are (i) joining two cracks, (ii) crack growth ($w \rightarrow w+1$), and (iii) nucleation of a crack of size one ($w = 1$). One can now check what the effect at N_{max} is like and Fig. 7 shows the likelihood of the first of these three processes averaged over last failures to reach N_{max}. Three major features emerge. First, the microfailures have a large probability to contribute to crack coalescence as $P_{bridging} \gg 0$. Second, this probability increases with L.
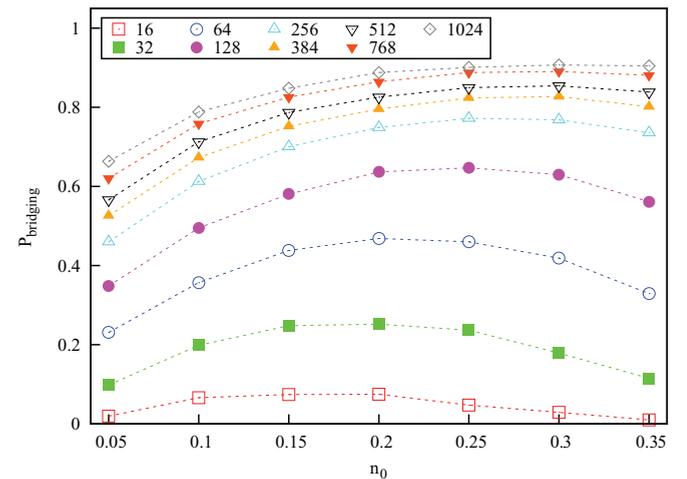

FIG. 7. (Color online) Bridging probability as a function of the disorder. The probability $P_{bridging}$ that a broken bond joins two preexisting cracks increases with the system size and, while the behavior for the large systems is not clear, the probability presents a maximum at a given disorder value for each L.

Third, there appears again to be a maximum efficiency for this process to contribute or happen, at a certain disorder, which shifts slowly with L. Attempts to find a scaling form for coalescence due to bridging events (bond failures) or the probability $P_{bridging}$ with n_r and L were not successful, leaving an important unanswered question: Does it saturate for a given disorder to a value smaller than unity?

IV. CONCLUSION

In this paper we have performed a systematic analysis of damage mechanics in brittle fracture, using the random fuse model. The approach is interesting for a number of reasons, including the fact that the problem is extremely hard to study experimentally to even a limited extent. What transpires from our results is that the peak damage and the microcrack populations in our samples at peak stress are determined by the microscopic dynamics of crack growth, which occurs mostly by coalescence. This dynamics is quantitatively dependent to a large extent on the disorder strength and the system size. While the general questions of size effects and fracture strength in these test systems are now finally well understood [3,4], the microscopic details turn out to be quite important: It is from them that the parameters of the coarse-grained (modified) Gumbel distributions ensue. Thus the present damage study highlights a connection between the microscopic and macroscopic, beyond the link that ensues during a renormalization or upscaling of sample-size-dependent strength distributions.

A number of open avenues for future research must be listed. The detailed connection of the damage scaling(s) and those of strength distributions should be analyzed. The particular case at hand is characterized by a very small damage degree at peak load. Scenarios where the crack population undergoes more a complicated development up to C_{max} would be of great interest for further studies, such as where the initial one has a power-law form [4,16]. In a more general sense, our results highlight the old engineering ideas of improving fracture resistance by inhibiting crack growth: Minute effects in the fracture resistance may influence the strength quantitatively.

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