

Condensation of fluctuations in and out of equilibriumMarco Zannetti,^{1,2,*} Federico Corberi,^{1,†} and Giuseppe Gonnella^{3,‡}¹*Dipartimento di Fisica “E.R. Caianiello”**and CNISM Unità di Salerno, Università di Salerno, via Ponte don Melillo, 84084 Fisciano (SA), Italy*²*Kavli Institute for Theoretical Physics China, CAS, Beijing 100190, China*³*Dipartimento di Fisica, Università di Bari**and INFN Sezione di Bari, via Amendola 173, 70126 Bari, Italy*

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Condensation of fluctuations is an interesting phenomenon conceptually distinct from condensation on average. One striking feature is that, contrary to what happens on average, condensation of fluctuations may occur even in the absence of interaction. The explanation emerges from the duality between large deviation events in the given system and typical events in a new and appropriately biased system. This phenomenon is investigated in the context of the Gaussian model, chosen as a paradigmatic noninteracting system, before and after an instantaneous temperature quench. It is shown that the bias induces a mean-field-like effective interaction responsible for the condensation on average. Phase diagrams, covering both the equilibrium and the off-equilibrium regimes, are derived for observables representative of generic behaviors.

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I. INTRODUCTION

Condensation is a ubiquitous phenomenon in nature. It may take place in equilibrium, off-equilibrium, in real space, or in momentum space. Starting from the most familiar condensation of supersaturated vapor, there is a great abundance of examples which includes, among others, the Bose-Einstein condensation (BEC) [1] and the related transition in mean-field systems, such as the spherical [2] or the large- N model [3]. More recently there has been much interest in condensation transitions arising out of equilibrium, both in classical [4] and in quantum systems [5]. In the nonequilibrium context the phenomenology of condensation turns out to be very rich with a variety of manifestations in fields as diverse as economics, information theory, traffic models, granular materials, networks, and mass transport [4,6]. Much progress in the understanding of the basic features common to most of these different instances of condensation has been achieved through the study of driven diffusive systems and, in particular, of the zero-range process or variants of it [4].

In this paper we will focus on a yet another manifestation of condensation, which is somewhat conceptually different. In the usual contexts mentioned above, condensation is a phenomenon observed in the average behavior of the system. Instead, we will be concerned with condensation occurring in the *fluctuations*, namely, with condensation as a *rare* event [7–11]. The conceptual and substantial difference is that condensation of fluctuations may occur even in systems which cannot sustain condensation on average, such as noninteracting systems. In order to emphasize this point, we will work with the Gaussian model, which is the paradigmatic noninteracting system in the theory of phase transitions [12]. Although the average properties of this system are well known to be trivial

in and out of equilibrium, we will find that fluctuations of extensive quantities may condense.

Most of the work quoted above on condensation, both on average and in the fluctuations, has been carried out in the context of nonequilibrium steady states, obtained by driving an externally generated current into the system. Here, instead, we will carry further the program initiated in Ref. [9] of exploring fluctuations in the largely unknown area of the processes without time translation invariance [13]. Specifically, we will consider the relaxation following the instantaneous quench from an initial temperature T_I to a lower temperature T_F . With such a choice, we can overview the entire evolution from the equilibrium behavior before the quench to the off-equilibrium relaxation after the quench. We will see that, depending on the nature of the observable, fluctuations may condense either in and out of equilibrium or just as an out-of-equilibrium phenomenon. We will analyze in detail the mechanism of condensation, and we will derive phase diagrams, extending into the time direction. These diagrams show that during relaxation condensation is enhanced by the dynamics if occurring also in equilibrium or dynamically generated if absent in equilibrium.

The paper is organized as follows: In Sec. II we set up the ensemble theory apparatus needed in the rest of the paper. The Gaussian model is introduced in Sec. III. Section IV is the central section of the paper where the notions of condensation on average and condensation of the fluctuations are discussed in general. The example of a macrovariable condensing both in equilibrium and off-equilibrium is treated in Sec. V, whereas the example of condensation as an out-of-equilibrium phenomenon is discussed in Sec. VI. Concluding remarks are made in Sec. VII.

II. ENSEMBLES

The apparently puzzling feature of condensation appearing in the fluctuations of a noninteracting system finds explanation in the framework of large deviation theory [14], through the mapping of rare fluctuations in the given system (in our case the

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Gaussian model) into typical events in a new system, obtained by the application of an appropriate bias. The key point, as we will see, is that the bias produces an effective interaction, which is responsible for the condensation on average in the biased system. The basic idea amounts to an extension of ensemble theory beyond the realm of equilibrium statistical mechanics, according to a scheme which has been recently used in a variety of different contexts, classical [15–17] and quantum [18].

In order to give a general presentation of the method, let us consider a generic probability distribution $P(\varphi, J)$, referred to as the *prior* and describing the state of a system of volume V with microstates consisting of sets of degrees of freedom $\varphi = \{\varphi_i\}$, where i is a generic label, and control parameters J . In this paper i is the position vector \vec{x} in real space or the wave vector \vec{k} in Fourier space, and J stands for temperature T in equilibrium or for time t off-equilibrium. The probability of a fluctuation M of a random variable $\mathcal{M}(\varphi)$ is given by

$$P(M, J) = \int_{\Omega} d\varphi P(\varphi, J) \delta[M - \mathcal{M}(\varphi)], \quad (1)$$

where Ω is the phase space. Introducing the integral representation of the δ function $\delta(x) = \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{dz}{2\pi i} e^{-zx}$ this becomes

$$P(M, J) = \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{dz}{2\pi i} e^{-zM} K_{\mathcal{M}}(z, J), \quad (2)$$

where

$$K_{\mathcal{M}}(z, J) = \langle e^{z\mathcal{M}(\varphi)} \rangle \quad (3)$$

is the moment generating function of \mathcal{M} and the brackets $\langle \cdot \rangle$ denote the average in the prior ensemble. If the system is extended and $\mathcal{M}(\varphi)$ is an extensive macrovariable, for large volumes Eq. (2) can be rewritten as

$$P(M, J, V) = \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{dz}{2\pi i} e^{-V[z m + \lambda_{\mathcal{M}}(z, J)]}, \quad (4)$$

where m is the density M/V and

$$-\lambda_{\mathcal{M}}(z, J) = \frac{1}{V} \ln K_{\mathcal{M}}(z, J, V) \quad (5)$$

is the volume-independent scaled cumulant generating function. Carrying out the integration by the saddle point method, the large deviation principle is obtained

$$P(M, J, V) \sim e^{-V I_{\mathcal{M}}(m, J)}, \quad (6)$$

with the rate function,

$$I_{\mathcal{M}}(m, J) = z^* m + \lambda_{\mathcal{M}}(z^*, J), \quad (7)$$

and where $z^*(m, J)$ is the solution, supposedly unique, of the saddle point equation,

$$\frac{\partial}{\partial z} \lambda_{\mathcal{M}}(z, J) = -m. \quad (8)$$

We stress that the large deviation principle originates from extensivity with respect to volume. Extensivity with respect to time [14] will not be considered. Therefore, regarding the latter quantity as the “free energy” of the new ensemble,

$$P(\varphi, z, J, V) = \frac{1}{K_{\mathcal{M}}(z, J, V)} P(\varphi, J, V) e^{z\mathcal{M}(\varphi)}, \quad (9)$$

obtained by imposing the exponential bias on the prior, the rate function remains identified with the “thermodynamic potential” associated with yet another ensemble,

$$P(\varphi, M, J, V) = \frac{1}{P(M, J, V)} P(\varphi, J, V) \delta[M - \mathcal{M}(\varphi)], \quad (10)$$

in which the bias is implemented rigidly through the phase space restriction $M = \mathcal{M}(\varphi)$. To make contact with familiar ground, if the prior was the uniform ensemble $P(\varphi, V) = 1/|\Omega(V)|$ and \mathcal{M} was the energy of the system, then $P(\varphi, z, V)$ and $P(\varphi, M, V)$, respectively, would be the usual canonical ensemble at the inverse temperature $\beta = -z$ and the micro-canonical ensemble with energy $E = M$.

We stress that the above chain of relations holds in general, without limitations to equilibrium. Therefore, the quantity $I_{\mathcal{M}}(m, J)$ plays two distinct roles [8, 11, 16]: On one hand it acts as the rate function regulating the occurrence of rare events in the prior ensemble, and on the other hand it is the thermodynamic potential accounting for the average properties in the constrained ensemble $P(\varphi, M, J, V)$. In particular, if the extra correlations due to the bias are responsible for singularities in the free energy, amenable to a phase transition, the same singularities arise in the unbiased fluctuations. Consequently, the same phenomenon, in principle, could be observed following different experimental protocols, either by leaving the system unbiased and monitoring fluctuations or, alternatively, by arranging the appropriate bias aimed to typically render the effect of interest.

III. THE GAUSSIAN MODEL

In order to produce a concrete and simple realization of the above ideas, let us consider the Gaussian model, which describes a system of volume V , with a scalar order parameter field $\varphi(\vec{x})$ and the bilinear energy functional,

$$\mathcal{H}[\varphi] = \frac{1}{2} \int_V d\vec{x} [(\nabla\varphi)^2 + r\varphi^2(\vec{x})], \quad (11)$$

where r is a non-negative mass. The system is prepared in equilibrium at the temperature T_I . At the time $t = 0$ is instantaneously quenched to the lower temperature T_F . The dynamics, without conservation of the order parameter, are governed by the overdamped Langevin equation [12, 19],

$$\dot{\varphi}(\vec{x}, t) = [\nabla^2 - r]\varphi(\vec{x}, t) + \eta(\vec{x}, t), \quad (12)$$

where $\eta(\vec{x}, t)$ is the white Gaussian noise generated by the cold reservoir with zero average and correlator,

$$\langle \eta(\vec{x}, t) \eta(\vec{x}', t') \rangle = 2T_F \delta(\vec{x} - \vec{x}') \delta(t - t'). \quad (13)$$

Due to linearity, the problem can be diagonalized by Fourier transformation. Keeping in mind that the Fourier components $\varphi_{\vec{k}} = \int_V d\vec{x} \varphi(\vec{x}) e^{i\vec{k}\cdot\vec{x}}$ are complex, some care is needed in the identification of the independent variables. Let us denote by \mathcal{B} the set of all wave vectors with magnitude smaller than an ultraviolet cutoff Λ , caused by the existence of a microscopic length scale in the problem, such as an underlying lattice spacing. Since the reality of $\varphi(\vec{x})$ requires $\varphi_{-\vec{k}} = \varphi_{\vec{k}}^*$, the independent variables are φ_0 and the set of pairs $\{u_{\vec{k}} = \text{Re } \varphi_{\vec{k}}, v_{\vec{k}} = \text{Im } \varphi_{\vec{k}}\}$ with $\vec{k} \in \mathcal{B}_+$, where \mathcal{B}_+ is half of \mathcal{B} . More precisely, if \mathcal{B}_- is the set obtained by reversing all the

wave vectors in \mathcal{B}_+ , then \mathcal{B}_+ is such that $\mathcal{B}_+ \cap \mathcal{B}_- = \emptyset$ and $\mathcal{B}_+ \cup \mathcal{B}_- = \mathcal{B} - \{0\}$. However, rather than working with \mathcal{B}_+ , it is more convenient to let \vec{k} vary over the whole of \mathcal{B} by taking as independent real variables,

$$x_{\vec{k}} = \begin{cases} \varphi_0 & \text{for } \vec{k} = 0, \\ u_{\vec{k}} & \text{for } \vec{k} \in \mathcal{B}_+, \\ v_{\vec{k}}, & \text{for } \vec{k} \in \mathcal{B}_-. \end{cases} \quad (14)$$

With this convention, from Eq. (12) we get the equations of motion for a set of independent Brownian oscillators,

$$\dot{x}_{\vec{k}}(t) = -\omega_k x_{\vec{k}}(t) + \zeta_{\vec{k}}(t), \quad (15)$$

with the dispersion relation $\omega_k = (k^2 + r)$. The noise correlator is given by

$$\langle \zeta_{\vec{k}}(t) \zeta_{\vec{k}'}(t') \rangle = 2T_{F,k} V \delta_{\vec{k}, -\vec{k}'} \delta(t - t'), \quad (16)$$

where

$$T_{F,k} = \frac{T_F}{2\theta_k}, \quad (17)$$

and θ_k is the Heaviside step function with $\theta_0 = 1/2$. The energy functional (11) then takes the separable form

$$\mathcal{H}(\mathbf{x}) = \sum_{\vec{k}} \mathcal{H}_{\vec{k}}(x_{\vec{k}}), \quad (18)$$

with

$$\mathcal{H}_{\vec{k}}(x_{\vec{k}}) = \frac{1}{V} \theta_k \omega_k x_{\vec{k}}^2, \quad (19)$$

and where \mathbf{x} stands for the whole set $\{x_{\vec{k}}\}$.

Due to mode independence, the state of the system is factorized at all times $P(\mathbf{x}, t) = \prod_{\vec{k}} P_{\vec{k}}(x_{\vec{k}}, t)$ with the single-mode contributions given by

$$P_{\vec{k}}(x_{\vec{k}}, t) = Z_{\vec{k}}^{-1}(t) e^{-\beta_k(t) \mathcal{H}_{\vec{k}}(x_{\vec{k}})}, \quad (20)$$

$$Z_{\vec{k}}(t) = \sqrt{\frac{\pi V}{\beta_k(t) \theta_k \omega_k}}, \quad (21)$$

where $\beta_k^{-1}(t)$ is the effective temperature of the modes with wave vector magnitude k , defined from the average energy per degree of freedom [20],

$$\beta_k^{-1}(t) = 2 \langle \mathcal{H}_{\vec{k}}(t) \rangle = \frac{2}{V} \theta_k \omega_k \langle x_{\vec{k}}^2(t) \rangle, \quad (22)$$

which yields

$$\beta_k^{-1}(t) = (T_I - T_F) e^{-2\omega_k t} + T_F. \quad (23)$$

In this paper we will take $k_B = 1$ for the Boltzmann constant. As illustrated in the top panel of Fig. 1, initially the spectrum of effective temperatures is flat with $\beta_k(t = 0) = \beta_I$, which is the statement of energy equipartition. Then, as the system relaxes, the temperatures of the different modes acquire a k dependence, signaling the breaking of equipartition and departure from equilibrium. Eventually, convergence to the same final value β_F takes place as the system equilibrates and equipartition is restored. The probability distribution $P(\mathbf{x}, t)$ will be taken as the prior in the following.

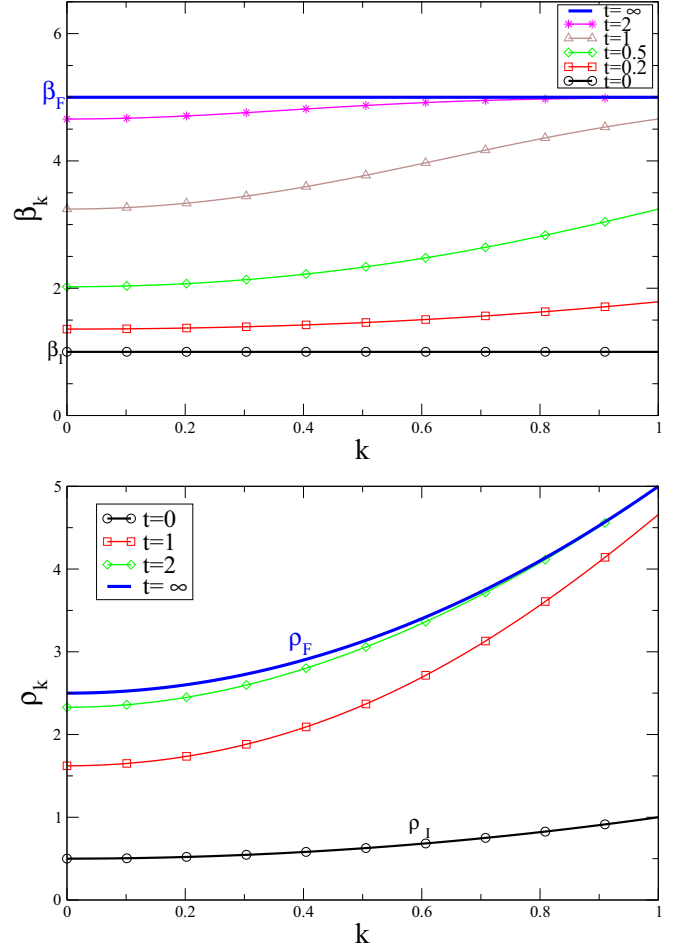


FIG. 1. (Color online) Spectra of inverse effective temperatures (top) and of ρ_k for the order parameter sample variance (bottom) with $r = 1, T_I = 1$, and $T_F = 0.2$.

IV. FLUCTUATIONS OF A MACROVARIABLE

Let us now consider a quadratic and separable macrovariable $\mathcal{M}(\mathbf{x}) = \sum_{\vec{k}} \mathcal{M}_{\vec{k}}(x_{\vec{k}})$ with $\mathcal{M}_{\vec{k}}(x_{\vec{k}}) = \frac{1}{V} \theta_k \mu_k x_{\vec{k}}^2$, whose coefficients μ_k are to be specified. According to the scheme of Sec. II, all the information on the fluctuations of $\mathcal{M}(\mathbf{x})$ at the generic time t is contained in the rate function (7) with $J = t$. The computation of this quantity requires the preliminary computation of the moment generating function. From the factorization of the prior and the separability of \mathcal{M} follows:

$$K_{\mathcal{M}}(z, t) = \prod_{\vec{k}} K_{\mathcal{M}, \vec{k}}(z, t), \quad (24)$$

with the single-mode factors given by

$$\begin{aligned} K_{\mathcal{M}, \vec{k}}(z, t) &= \int_{-\infty}^{\infty} dx_{\vec{k}} P_{\vec{k}}(x_{\vec{k}}, t) e^{z \mathcal{M}_{\vec{k}}(x_{\vec{k}})} \\ &= \frac{1}{\sqrt{1 - \rho_k^{-1}(t) z}}, \end{aligned} \quad (25)$$

where

$$\rho_k = \beta_k \omega_k / \mu_k = \frac{1}{2} \langle \mathcal{M}_{\vec{k}} \rangle^{-1}. \quad (26)$$

Inserting this result into Eq. (5), the saddle point equation (8) can be written as

$$m = \tilde{F}_{\mathcal{M}}(z, t, V), \quad (27)$$

where the function on the right hand side is given by

$$\tilde{F}_{\mathcal{M}}(z, t, V) = \frac{1}{V} \sum_{\bar{k}} \langle \mathcal{M}_{\bar{k}} \rangle_z, \quad (28)$$

and

$$\langle \mathcal{M}_{\bar{k}} \rangle_z = \frac{1}{2[\rho_{\bar{k}}(t) - z]} \quad (29)$$

is the average per mode in the biased ensemble (9). Recalling the definition (26) of ρ_k , the above equation can be rewritten as

$$\langle \mathcal{M}_{\bar{k}} \rangle_z = \frac{1}{\langle \mathcal{M}_{\bar{k}}(t) \rangle^{-1} - 2z}, \quad (30)$$

in which the biased and the prior averages enter in the same formal relationship as the dressed and the bare average in a Dyson-Schwinger-type equation [21,22] with $2z$ playing the role of the tadpole self-energy. Now, since truncating the self-energy skeleton expansion to the tadpole contribution amounts to a self-consistent mean-field approximation as in the large- N limit of an $O(N)$ model [22,23], we have that biased expectations can be viewed as arising from the mean-field approximation on an underlying interacting theory, whose free limit is given by the prior expectations. This turns out to be essential for the distinction between condensation as a typical phenomenon or as a rare fluctuation.

Transforming the sum in Eq. (28) into an integral, the saddle point equation (27) can be rewritten as

$$m = F_{\mathcal{M}}(z, t), \quad (31)$$

with

$$F_{\mathcal{M}}(z, t) = \frac{\Upsilon_d}{2} \int_0^\Lambda \frac{dk}{(2\pi)^d} \frac{k^{d-1}}{\rho_k(t) - z}, \quad (32)$$

where d is the space dimensionality, $\Upsilon_d = 2\pi^{d/2}/\Gamma(d/2)$ is the d -dimensional solid angle, and Γ is the Euler Γ function. The formal solution is given by

$$z^*(m, t) = F_{\mathcal{M}}^{-1}(m, t), \quad (33)$$

where $F_{\mathcal{M}}^{-1}$ is the inverse, with respect to z , of the function defined by Eq. (32). The existence of this solution depends on the domain of definition of $F_{\mathcal{M}}^{-1}$. If we assume \mathcal{M} to be positive, $F_{\mathcal{M}}^{-1}$ is defined for $z \leq \rho_{k_M}$, where k_M is the wave vector which minimizes ρ_k , and

$$F_{\mathcal{M}}(z, t) \leq m_C(t), \quad (34)$$

with

$$m_C(t) = F_{\mathcal{M}}(z = \rho_{k_M}, t). \quad (35)$$

The issue is whether this upper bound is finite or infinite. In this paper, for simplicity, we will limit the discussion to cases with $k_M = 0$. Then, if $[\rho_k(t) - \rho_0(t)]$ vanishes with k like k^α , for $d \leq \alpha$ the singularity is not integrable, $m_C(t)$ diverges, and the solution (33) exists for any $m \geq 0$. This is shown in the top panel of Fig. 2. Instead, if $d > \alpha$, the singularity is integrable, $m_C(t)$ is finite, and the solution (33) exists only

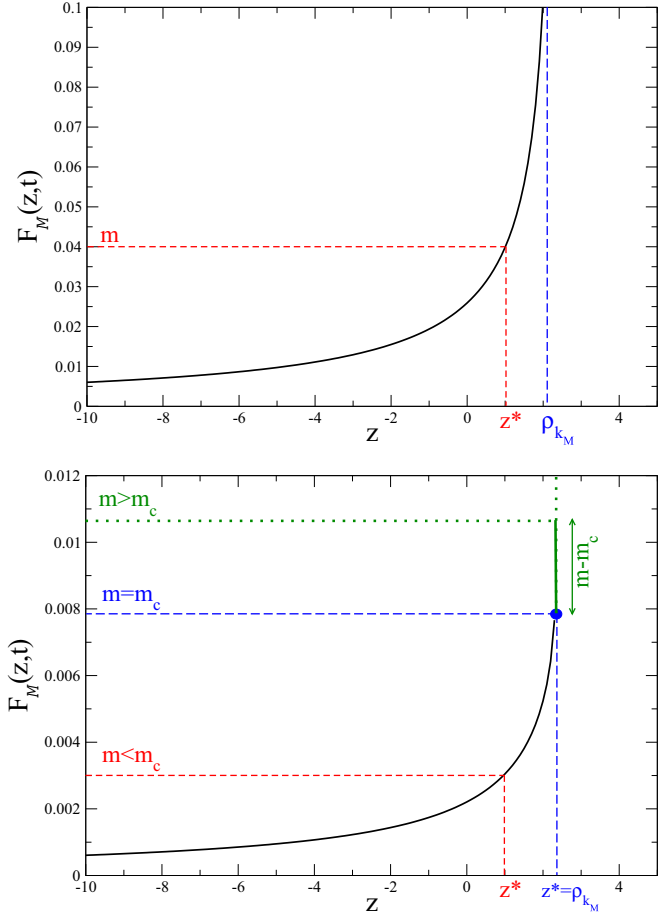


FIG. 2. (Color online) Typical behavior of $F_{\mathcal{M}}(z, t)$, obtained with $\mu_k = 2, t = 2, r = 1, T_I = 1$, and $T_F = 0.2$. Top panel: graphical solution of Eq. (31) with $d = 1$. Bottom panel: graphical solution of Eq. (36) with $d = 3$ and for $m < m_C$, $m = m_C$, and $m > m_C$.

for $m \leq m_C(t)$ (bottom panel of Fig. 2). In order to find the solution for $m > m_C(t)$ one must proceed as in the standard treatment of the BEC [1], separating the $k = 0$ term from the sum and rewriting Eq. (31) as

$$m = \frac{1}{V} \langle \mathcal{M}_0 \rangle_{z^*} + F_{\mathcal{M}}(z^*, t). \quad (36)$$

Then, $m_C(t)$ defines a critical line on the (t, m) plane separating the normal phase (below) from the condensed phase (above). Below, the first term on the right hand side of Eq. (36) is $O(1/V)$ and negligible, whereas above (see Fig. 2) takes the finite value $[m - m_C(t)]$ due to the “sticking” [1,2] of z^* to the m -independent value $z^* = \rho_0(t)$. Summarizing,

$$z^*(m, t) = \begin{cases} F_{\mathcal{M}}^{-1}(m, t) & \text{for } m \leq m_C(t), \\ \rho_0(t) & \text{for } m > m_C(t), \end{cases} \quad (37)$$

as is illustrated in the bottom panel of Fig. 2. What we have derived, so far, is condensation on average in the framework of the biased ensemble. That is, the transition from microscopic to macroscopic of the expectation $\langle \mathcal{M}_0 \rangle_{z^*}$, analogous to the BEC for the zero momentum occupation number. We emphasize, for future reference, that the occurrence of the transition requires: (i) that the intensive parameter ρ conjugate to \mathcal{M} does depend

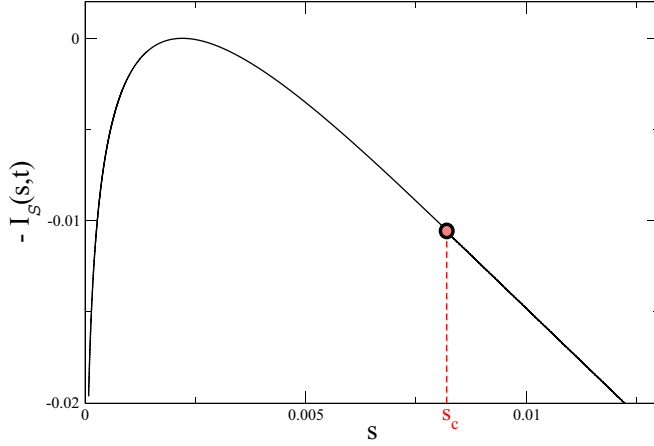


FIG. 3. (Color online) Rate function $I_S(s,t)$ for the sample variance discussed in Sec. V, s_c denotes the critical threshold. Parameters $\mu_k = 2$, $t = 2$, $r = 1$, $T_I = 1$, $T_F = 0.2$, and $d = 3$.

on k , i.e., that there exists a spectrum of parameters ρ_k and (ii) that the spectrum vanishes with k as k^α with $\alpha < d$.

In order to see the dual image of this transition in the fluctuations occurring in the prior ensemble [24], we must take a look at the rate function. Taking into account the definition (7) and the above result for $z^*(m,t)$, we have

$$I_M(m,t) = \begin{cases} z^*(m,t)m + \lambda_M[z^*(m,t),t] & \text{for } m \leq m_C(t), \\ \rho_0(t)(m - m_C) + I_M(m_C,t) & \text{for } m > m_C(t), \end{cases} \quad (38)$$

whose typical behavior is displayed in Fig. 3, obtained for the sample variance discussed below in Sec. V. Thus, the probability of a fluctuation with $m > m_C(t)$ is given by

$$P(M,t) \sim e^{-V\rho_0(t)(m-m_C)} e^{-V I_M(m_C,t)}. \quad (39)$$

On the other hand, the fluctuation probability can also be written as

$$P(M,t) = \int \prod_{\vec{k}} dM_{\vec{k}} P(\{M_{\vec{k}},t\}) \delta\left(M - \sum_{\vec{k}} M_{\vec{k}}\right), \quad (40)$$

where $\{M_{\vec{k}}\}$ is a configuration of the values taken by the single-mode observables $\mathcal{M}_{\vec{k}}$. The statement is simply that, once M has been fixed, the allowed microscopic events $\{M_{\vec{k}}\}$ are those on the hypersurface defined by the constraint $M = \sum_{\vec{k}} M_{\vec{k}}$ and that the probability $P(M,t)$ is obtained by summing over the shell. The probability of one such configuration is given by

$$P(\{M_{\vec{k}},t\}) = \prod_{\vec{k}} P_{\vec{k}}(M_{\vec{k}},t), \quad (41)$$

where $P_{\vec{k}}(M_{\vec{k}},t)$, appearing on the right hand side, using Eqs. (2), (24), and (25) is given by

$$P_{\vec{k}}(M_{\vec{k}},t) = \frac{e^{-\rho_k M_{\vec{k}}}}{\sqrt{\pi \rho_k^{-1} M_{\vec{k}}}} \theta(\rho_k^{-1} M_{\vec{k}}), \quad (42)$$

and θ is, again, the Heaviside step function. Now, inserting this result into Eq. (40) and comparing with Eq. (39), we obtain

$$P(M,t) = \int dM_0 P_0(M_0,t) \delta[M_0 - (M - M_C)] \\ \times \int \prod_{\vec{k} \neq 0} dM_{\vec{k}} P_{\vec{k}}(M_{\vec{k}},t) \delta\left(M_C - \sum_{\vec{k} \neq 0} M_{\vec{k}}\right),$$

which means that, for $m > m_C(t)$, the probability of the configurations $\{M_{\vec{k}}\}$ is concentrated on the subset of the shell singled out by the additional condition $M_0 = M - M_C$. This is condensation of fluctuations, in the sense that a fluctuation above the threshold M_C can occur only if the macroscopic fraction $M - M_C$ of it is contributed by the zero mode. As anticipated in Sec. I, the remarkable feature of this transition is that it takes place in a noninteracting system like the Gaussian model in which no transition on average can take place in and out of equilibrium. The explanation is in Eq. (30), which shows how the bias generates the interaction sustaining the transition, and the bias is generated once the size of the fluctuation has been fixed.

As an illustration, in the next sections we will analyze two specific cases. In the first one condensation occurs both in equilibrium and out of equilibrium, whereas in the second one it occurs exclusively as an out-of-equilibrium phenomenon.

V. ORDER PARAMETER SAMPLE VARIANCE

Let us consider the sample variance,

$$S[\varphi] = \int_V d\vec{x} \varphi^2(\vec{x}) = \frac{1}{V} \sum_{\vec{k}} \theta_k x_{\vec{k}}^2, \quad (43)$$

as the fluctuating macrovariable. This corresponds to $\mu_k = 2$, which is independent of k and yields $\rho_k(t) = \beta_k \omega_k / 2$. From the small k behavior $[\rho_k(t) - \rho_0(t)] \sim k^2$ follows $\alpha = 2$ for all times, including the initial and the final equilibrium states (bottom panel of Fig. 1). Therefore, denoting by s the density S/V , the critical value $s_c(t)$ is finite for $d > 2$ at all times. The critical line for $d = 3$ is displayed in the top panel of Fig. 4. In order to understand this phase diagram, one should keep in mind that fixing the value of s amounts to implementing a *spherical* constraint à la Berlin and Kac [2]. Let us first consider equilibrium in the time region $t \leq 0$ preceding the quench. Here, the critical line is horizontal and corresponds to the critical threshold $s_c(T_I)$ of the spherical model at the temperature T_I [25]. Then, according to the dual point of view expounded above, the two alternative readings of the equilibrium transition are that condensation can be observed either as the usual transition of the spherical model or as a rare event in the Gaussian model where the sample variance is free to fluctuate.

Consider, next, the relaxation regime after the quench for $t > 0$. As is evident from Fig. 4, there are two time regimes separated by the minimum of the critical line, about the characteristic time $\tau \sim r^{-1}$, which is the relaxation time of the slowest mode. In the first regime ($0 < t < \tau$) the system is strongly off-equilibrium, and the threshold $s_c(t)$ drops abruptly. In the second regime ($t > \tau$) the system gradually equilibrates to the final temperature, and $s_c(t)$ saturates slowly

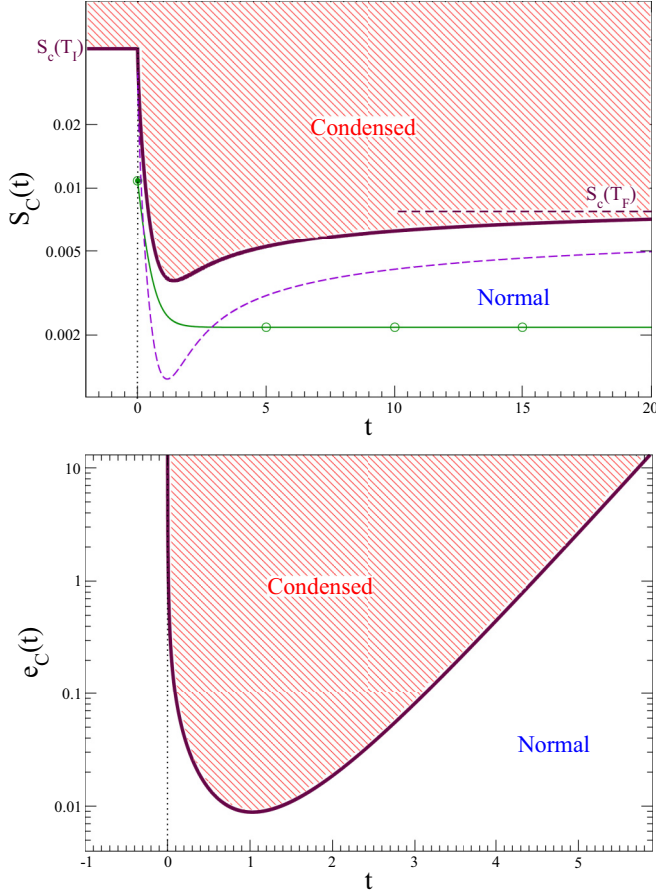


FIG. 4. (Color online) Top panel: Phase diagram of order parameter sample variance. The upper horizontal dashed line corresponds to $s_C(T_F)$. The green line is the plot of $\langle s(t) \rangle$. The lower dashed line is the difference $[s_C(t) - \langle s(t) \rangle]$. Bottom panel: Energy phase diagram. In both cases: $r = 1$, $T_I = 1$, $T_F = 0.2$, and $d = 3$.

toward the final equilibrium value $s_C(T_F) < s_C(T_I)$. A few observations are in order: (i) The plot of the unbiased average $\langle s(t) \rangle$ lies below the critical line, showing that condensation of fluctuations is always a rare event. However, the plot of $[s_C(t) - \langle s(t) \rangle]$ shows that the *rarity* of the condensation event varies with time and that the most favorable time window for condensation is around τ where the difference is minimized. Hence, condensation of the fluctuations is enhanced by the off-equilibrium dynamics. (ii) The nonmonotonicity of the critical line is a remarkable dynamical feature, leading to a reentrance phenomenon. Namely, when the transition is driven by t and s is kept fixed to a value in between $s_C(T_F)$ and $s_C(T_I)$, a fluctuation of this size at first is normal and then condenses, whereas for s in between the minimum of the critical line and $s_C(T_F)$, the fluctuation undergoes a second and reverse transition becoming normal again at late times. (iii) The dynamical condensation analyzed here is not related to the properties of the dynamical spherical model [26], which requires the spherical constraint to be imposed pathwise, namely, at all times after the quench. Here, instead, the evolution is unconstrained, and the spherical constraint is imposed only at the observation time t . Therefore, while in equilibrium the two experimental protocols, fluctuations

monitoring vs bias implementation, are in principle both realizable, the latter one requiring an instantaneous bias is hardly realizable off-equilibrium.

VI. ENERGY

As a second example, let us consider the energy (18) as the fluctuating macrovariable. This is representative of a different class of observables, whose fluctuations behave normally in equilibrium and undergo a condensation transition as an out-of-equilibrium phenomenon. This is due to $\mu_k = \omega_k$ from which follows $\rho_k = \beta_k$. Therefore, the k dependence of ρ_k disappears in equilibrium (Fig. 1) shifting the critical threshold to infinity. More in detail, denoting by e the energy density, the critical line is given by

$$e_C(t) = \int_0^\Lambda \frac{dk}{4\pi^2} \frac{k^2}{\beta_k(t) - \beta_0(t)}. \quad (44)$$

The corresponding phase diagram in the bottom panel of Fig. 4 is qualitatively different from the one in the top panel for the absence of the phase transition in equilibrium. This is due to the fact that, in equilibrium, the denominator $(\beta_k - \beta_0)$ under the integral vanishes identically for all k . This implies $\alpha \rightarrow \infty$ and the divergence of both $e_C(T_I)$ and of $e_C(T_F)$ for any space dimensionality d . However, as soon as the system is put off-equilibrium, equipartition is broken, and the spectrum of inverse effective temperatures develops a minimum at $k = 0$ (Fig. 1). Then, the integral becomes convergent for $d > 2$. Consequently, $e_C(t)$ drops down from infinity to a minimum around τ and, then, rises again toward infinity as the system reaches the final equilibrium state. The nonmonotonic shape of the critical line implies, also in this case, reentrance of the t -driven transition for all fluctuations above the minimum of the critical line.

VII. CONCLUSIONS

To summarize, we have analyzed the behavior of fluctuations of macrovariables in the Gaussian model, both in equilibrium and in off-equilibrium relaxation following a sudden temperature quench. For a certain class of bilinear variables there is condensation in the behavior of large deviations in the sense that the $k = 0$ mode contributes a macroscopic amount to the fluctuations. The transition in the fluctuations is dual to an ordinary transition, sustained by an effective mean-field interaction in the constrained or biased system. Different from previous work on condensation, we have considered equilibrium followed by relaxation through a nonstationary process in which the time evolution plays an essential role. Also, essential is the k -space structure of macrovariables and the dispersion relation in the prior model, which is a feature not present in models with identically distributed variables [11]. In this respect, the case of energy fluctuations is particularly interesting as an instance in which the k dependence of the conjugate intensive parameters ρ is dynamically generated and, with it, the occurrence of condensation. Finally, duality is a general property, not limited to the case of a noninteracting prior. Future work will be devoted to the investigation of fluctuation singularities in the case of interacting systems.

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