Strong local passivity in finite quantum systems

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Passive states of quantum systems are states from which no system energy can be extracted by any cyclic (unitary) process. Gibbs states of all temperatures are passive. Strong local (SL) passive states are defined to allow any general quantum operation, but the operation is required to be local, being applied only to a specific subsystem. Any mixture of eigenstates in a system-dependent neighborhood of a nondegenerate entangled ground state is found to be SL passive. In particular, Gibbs states are SL passive with respect to a subsystem only at or below a critical system-dependent temperature. SL passivity is associated in many-body systems with the presence of ground state entanglement in a way suggestive of collective quantum phenomena such as quantum phase transitions, superconductivity, and the quantum Hall effect. The presence of SL passivity is detailed for some simple spin systems where it is found that SL passivity is neither confined to systems of only a few particles nor limited to the near vicinity of the ground state.

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I. INTRODUCTION

The maximum energy that can be extracted from a physical system—the work that it can do—by an applied process is a fundamental thermodynamic problem of continuing interest [1,2]. This problem is typically posed [3–5] for a quantum system with Hamiltonian $\mathbf{H}$ in terms of cyclic (unitary) processes in which the system, initially isolated, is coupled at time $t = 0$ to external sources of work with combined potential $\mathbf{V}(t)$ and later decoupled from them at time $t = \tau$, creating for $t \in [0,\tau]$ a time-dependent system Hamiltonian $\mathbf{H}(t) = \mathbf{H} + \mathbf{V}(t)$, with associated free energy $F(t) = -kT \ln \mathrm{Tr}\{\exp[-\mathbf{H}(t)/kT]\}$. System states for which no cyclic process can extract a positive amount of energy from the system are called passive [3,5]. More specifically, since in this context the change in free energy is zero, a system in a passive state can do no positive work. An important result for finite quantum systems is that Gibbs states are passive. Indeed, this is the *no perpetuum mobile* version of the second law of thermodynamics for equilibrium as formulated by Thomson [6,7].

We introduce in this paper a variant of passivity we call strong local (SL) passivity, which identifies a new collective quantum phenomenon exhibited by multipartite systems. We find for finite quantum systems with a nondegenerate, entangled ground state that states in a neighborhood of the ground state are SL passive. In particular, though all Gibbs states (of any temperature) are passive, only Gibbs states at or below a critical, system-dependent temperature are SL passive. This means for many-body systems that, for any state close to the ground state, the ground state entanglement and nondegeneracy sufficiently constrain the system's subsystems to collectively inhibit energy release from any subsystem. Ground state entanglement is a recognized root cause of other collective quantum phenomena, including quantum phase transitions [8], superconductivity [9], and the quantum Hall effect [10].

A system state is defined to be SL passive if no general (Kraus, operator-sum) quantum operation $\mathcal{G}$ applied locally to a subsystem can extract positive energy from the system. We are restricted, in other words, to operations of the form $\mathcal{G} \otimes \mathcal{I}$ where $\mathcal{I}$ is the identity operation for the rest of the system. The system dynamics driven by $\mathbf{H}$ can have, generally, a nonlocal component. So that the effect of $\mathcal{G}$ applied locally is not confounded with the time evolution accompanying any nonlocal component of $\mathbf{H}$, we include in SL passivity's definition the idealization that $\mathcal{G}$ proceeds much faster than the system's natural unitary evolution $\exp(-i\mathbf{H}\tau/\hbar)$ due to $\mathbf{H}$. In fact, fast local operations are of main interest in applications; in, for example, circuit-based quantum information processing, gates must operate faster than the background evolution of the physical substrate. For sufficiently fast $\mathcal{G}$ and the system in a state ρ, the energy extracted by $\mathcal{G}$ is effectively

$$\Delta E(\rho) = \mathrm{Tr}[\mathbf{H}\rho] - \mathrm{Tr}\left[\mathbf{H}(\mathcal{G} \otimes \mathcal{I})(\rho)\right]. \tag{1}$$

The local energy $\Omega_\circ$ of a subsystem is defined to be the maximum of $\Delta E(\rho)$ for any $\mathcal{G}$ [11]. Note that $\Omega_\circ \geqslant 0$ and that ρ is SL passive if and only if $\Omega_\circ = 0$. Local energy for SL passive states is analogous to ergotropy introduced for state passivity [12].

Our definition of SL passivity reflects two modifications of the usual notion of state passivity. First, any general quantum operation $\mathcal{G}$ expressible in terms of Kraus operators [13] is allowed, relaxing the restriction to unitary operations. Second, only a subsystem is accessible, making $\mathcal{G}$ local to only that subsystem. It is easy to check that neither modification alone yields interesting physics. Suppose we define SL passivity to allow any general quantum operation $\mathcal{G}$ but do not narrow the operation's scope to a subsystem. For any finite quantum system with a ground state $|E_0\rangle$ and eigenstates $|E_k\rangle$ of higher energy, a $\mathcal{G}$ can be constructed from Kraus operators $\mathbf{K}_k = |E_0\rangle\langle E_k|$ so that $\mathcal{G}(\rho) = |E_0\rangle\langle E_0|$ for any system state ρ. By this definition, only a ground state can be SL passive. Or, suppose we narrow the scope of the operation to a subsystem but still require a unitary operation; that is, we allow only operations of the form $\mathcal{U} \otimes \mathcal{I}$ where $\mathcal{U}$ is a local unitary operation on the subsystem. Here again nothing results; $\mathcal{U} \otimes \mathcal{I}$

is unitary, so for systems with identifiable subsystems all passive states, including Gibbs states, would be SL passive. Only together do the two modifications have an unexpected and interesting result.

Conditions for SL passivity are presented as a theorem in the following section. Then in Sec. III the presence of SL passivity is detailed in some simple quantum spin systems. Section IV addresses the possible extent of SL passivity in a system in terms of type of state, number of system particles, and size of the ground state's SL passivity neighborhood. We close in the last section with some summary remarks.

II. MAIN RESULT

The setting of our main result for SL passivity is a finite quantum system $\mathcal{S}$ [described by a complex Hilbert space $\mathcal{H}$ with $d = \dim(\mathcal{H}) < \infty$] with a subsystem (or component) $\mathcal{C}$ whereby $\mathcal{H} = \mathcal{H}_c \otimes \mathcal{H}_{\bar{c}}$, where $\mathcal{H}_c$ and $\mathcal{H}_r$ are the Hilbert spaces associated with $\mathcal{C}$ and the rest of $\mathcal{S}$, respectively. The Hamiltonian of $\mathcal{S}$ is

$$\mathbf{H} = \sum_{k=0}^{d-1} E_k |E_k\rangle\langle E_k|, \tag{2}$$

with eigenstates $|E_k\rangle$ and associated eigenenergies $E_0 \leqslant E_1 \leqslant \ldots \leqslant E_{d-1}$. The Schmidt decomposition of the ground state $|E_0\rangle$ is [13]

$$|E_0\rangle = \sum_s \sqrt{q_s}|c_s\rangle|r_s\rangle, \tag{3}$$

where $\sum_s q_s = 1$ and $|c_s\rangle$ and $|r_s\rangle$ are, respectively, orthonormal states of $\mathcal{H}_c$ and $\mathcal{H}_r$. The ground state $|E_0\rangle$ is fully entangled if all the q_s in (3) are positive [13,14].

We will be concerned mostly with system states ρ of $\mathcal{S}$ that commute with $\mathbf{H}$, in other words, eigenmixtures

$$\rho = \sum_{k=0}^{d-1} p_k |E_k\rangle\langle E_k| \tag{4}$$

that are statistical mixtures of the eigenstates $|E_k\rangle$ with population probabilities p_k such that $\sum_k p_k = 1$. Eigenmixtures (4) disallow coherences among the system eigenstates but still include the important case of Gibbs states for which

$$p_k = \frac{1}{\mathcal{Z}} \exp\left(-\frac{E_k}{kT}\right), \tag{5}$$

where k is Boltzmann's constant, T is Gibbs temperature, and $\mathcal{Z} = \mathrm{Tr}[\exp(-\mathbf{H}/kT)]$ is the partition function. Eigenmixtures play a distinctive role in connection with system passivity; for example [4], a state of a finite quantum system is passive if and only if it is an eigenmixture with $p_k \geqslant p_{k'}$ for $E_k < E_{k'}$. We will see that with respect to SL passivity eigenmixtures play a similarly prominent role. We now state our main result.

Theorem. Let $\mathcal{S}$ be a finite quantum system with Hamiltonian (2) and a specified subsystem $\mathcal{C}$ such that $\mathcal{C}$ is fully entangled with the rest of $\mathcal{S}$ in the ground state $|E_0\rangle$. Suppose further that $|E_0\rangle$ is nondegenerate; that is, $E_0 < E_1$. Then a threshold ground state population probability $p_* < 1$ exists such that any eigenmixture ρ in (4) with $p_0 \geqslant p_*$ is SL passive.

Proof. Let $\mathcal{G}$ be a general quantum operation [13] on subsystem $\mathcal{C}$. For states σ on $\mathcal{H}_c$,

$$\mathcal{G}(\sigma) = \sum_\mu \mathbf{K}_\mu \,\sigma\, \mathbf{K}_\mu^\dagger, \tag{6}$$

with Kraus operators $\mathbf{K}_\mu$ on $\mathcal{H}_c$ such that

$$\sum_\mu \mathbf{K}_\mu^\dagger \mathbf{K}_\mu = \mathbf{I}. \tag{7}$$

With $\mathcal{S}$ initially in the eigenstate $|E_k\rangle$, the system energy loss due to $\mathcal{G}$ is

$$\Delta E_k = E_k - \mathrm{Tr}\left[\mathbf{H}(\mathcal{G}\otimes\mathcal{I})(|E_k\rangle\langle E_k|)\right].$$

A calculation involving Eqs. (2) and (7) and the completeness identity $\sum_k |E_k\rangle\langle E_k| = \mathbf{I}$ then yields

$$\Delta E_k = \sum_{k'\neq k}(E_k - E_{k'})\sum_\mu |\langle E_{k'}|\mathbf{K}_\mu|E_k\rangle|^2. \tag{8}$$

Because $|E_0\rangle$ is nondegenerate, $\Delta E_0 \leqslant 0$ in (8), and

$$\Delta E_0 = 0 \Leftrightarrow \langle E_m|\mathbf{K}_\mu|E_0\rangle = 0 \;\forall m \neq 0 \tag{9}$$

$$\Leftrightarrow \mathbf{K}_\mu|E_0\rangle = \lambda_\mu|E_0\rangle$$
$$\Leftrightarrow \sum_s \sqrt{q_s}(\mathbf{K}_\mu|s\rangle_c - \lambda_\mu|s\rangle_c)|s\rangle_r = 0 \tag{10}$$

$$\Leftrightarrow \mathbf{K}_\mu = \lambda_\mu \mathbf{I}_c, \tag{11}$$

where (9) holds because $|E_0\rangle$ is nondegenerate, (10) follows from (3), and (11) is due to $q_s \neq 0$ for all s. Now consider $\mathbf{K}_\mu$ in a neighborhood of the trivial operator $\lambda_\mu \mathbf{I}_c$. The Kraus operators $\mathbf{K}_\mu$ are trace class (hence compact) acting on the finite-dimensional Hilbert space $\mathcal{H}_c$, so

$$\mathbf{K}_\mu = \lambda_\mu \mathbf{I} + \sum_\gamma \theta_\gamma \mathbf{J}_{\mu\gamma} + \frac{1}{2}\sum_{\gamma,\gamma'}\theta_\gamma\theta_{\gamma'}\mathbf{J}_{\mu\gamma\gamma'} \tag{12}$$

for small $\theta_\gamma > 0$ to order $O(\theta_\gamma\theta_{\gamma'}\theta_{\gamma''})$. Put (12) into (8), with $\chi_{\mu\gamma} = \langle E_{k'}|\mathbf{J}_{\mu\gamma}|E_k\rangle$. Then, using $\langle E_{k'}|\lambda_\mu \mathbf{I}|E_k\rangle = 0$ for $k' \neq k$, we have

$$\Delta E_k = \sum_{\gamma,\gamma'}\theta_\gamma\theta_{\gamma'}\sum_{k'\neq k}(E_k - E_{k'})\sum_\mu \chi_{\mu\gamma}^\dagger \chi_{\mu\gamma'} \tag{13}$$

to order $O(\theta_\gamma\theta_{\gamma'}\theta_{\gamma''})$ for each $k \geqslant 1$. The remarkable feature of (13), and the key to the proof, is that no term linear in θ_γ appears for $k \geqslant 1$. (Linear terms do appear when we attempt to adjust the proof for states that are not eigenmixtures.) The absence of linear terms in (13) means that $\Delta E_k/\Delta E_0$ does not diverge for $\theta_\gamma \to 0$. So, for any $k \geqslant 1$ and any nontrivial $\mathcal{G}$, there exists $p_* < 1$ such that

$$\frac{1-p_0}{p_0}\left|\frac{\Delta E_k}{\Delta E_0}\right| \leqslant 1$$

for all $p_0 \geqslant p_*$. Since $\Delta E_0 < 0$ for all nontrivial $\mathcal{G}$,

$$\frac{p_k}{1-p_0}p_0\Delta E_0 + p_k|\Delta E_k| \leqslant 0,$$

from which, by summing, follows

$$p_0 \Delta E_0 + \sum_{k=1}^{d-1} p_k |\Delta E_k| \leqslant 0 \tag{14}$$

for all $p_0 \geqslant p_*$. If the state ρ of $\mathcal{S}$ is an eigenmixture as in (4), the system energy $\Delta E(\rho)$ in (1) extracted by the local operation $\mathcal{G}$ is $\Delta E(\rho) = \sum_{k=0}^{d-1} p_k \Delta E_k$, and we conclude from (14) that, for any eigenmixture ρ, $\Delta E(\rho) \leqslant 0$ for all $p_0 \geqslant p_*$. ∎

Corollary. The Gibbs states of a finite quantum system with a nondegenerate, fully entangled ground state are SL passive with respect to a subsystem for all temperatures $T \leqslant T_*$ for some critical temperature $T_* > 0$.

Our theorem is stated for a system with one identified subsystem. For a many-particle system governed by, say, a particle-symmetric Hamiltonian, SL passivity with respect to one particle implies SL passivity for all, and the theorem then says that the system's particles in an eigenmixture sufficiently near $|E_0\rangle$ are constrained by $|E_0\rangle$'s entanglement and nondegeneracy to collectively disallow energy release from *any* particle.

III. TWO-PARTICLE SYSTEMS

Our theorem and its corollary can be seen at work in a variety of multiparticle quantum systems. We detail this in this section in examples of two-particle systems.

Let $\mathcal{S}_2$ be a pair of coupled spin-$\frac{1}{2}$ particles with Hamiltonian

$$\mathbf{H} = \kappa \boldsymbol{\sigma}_1^x \boldsymbol{\sigma}_2^x + \boldsymbol{\sigma}_1^z + \boldsymbol{\sigma}_2^z, \tag{15}$$

where the Pauli operator terms $\boldsymbol{\sigma}_1^z$ and $\boldsymbol{\sigma}_2^z$ reflect the presence of an external magnetic field transverse to the coupling and $\kappa > 0$ is the coupling's relative strength. The pair $\mathcal{S}_2$ has a fully entangled ground state and eigenenergies

$$E_0 = -m, \ E_1 = -\kappa, \ E_2 = \kappa, \ E_3 = m, \tag{16}$$

where $m = \sqrt{\kappa^2 + 4}$.

Consider a general quantum operation $\mathcal{G}$ of the form of (6) applied to a particle of $\mathcal{S}_2$. For a spin-$\frac{1}{2}$ particle, $\mathcal{G}$ requires at most four Kraus operators [13]:

$$\mathbf{K}_\mu = \begin{pmatrix} s_\mu & t_\mu \\ u_\mu & v_\mu \end{pmatrix}, \tag{17}$$

with complex-valued elements $s_\mu, t_\mu, u_\mu, v_\mu$. In terms of these elements, condition (7) for the $\mathbf{K}_\mu$ becomes

$$\begin{aligned} \mathbf{s}^\dagger \mathbf{s} + \mathbf{u}^\dagger \mathbf{u} &= 1, \\ \mathbf{t}^\dagger \mathbf{t} + \mathbf{v}^\dagger \mathbf{v} &= 1, \\ \mathbf{s}^\dagger \mathbf{t} + \mathbf{u}^\dagger \mathbf{v} &= 0, \end{aligned} \tag{18}$$

where $\mathbf{s} = (s_1\ s_2\ s_3\ s_4)^\top$, etc.

The local energy $\Omega_\circ$ in (21) depends solely on η and ξ in (20), which depend in turn on the eigenmixture ρ through only the two probability differences δ_0 and δ_1. Because of this limited dependence on ρ, an eigenmixture can be SL passive without being passive. For $\kappa = 2$ in $\mathcal{S}_2$, for example, the eigenmixture $(p_0, p_1, p_2, p_3) = (.96, 0, .04, 0)$ is SL passive ($\Omega_\circ = 0$) but not passive ($p_2 > p_1$). In general, passivity is neither necessary nor sufficient for SL passivity.

For the pair $\mathcal{S}_2$ in an eigenmixture state ρ, the energy extracted by applying $\mathcal{G}$ locally to a particle in $\mathcal{S}_2$ is, after elementary calculation,

$$\begin{aligned} \Delta E(\rho) = &(1-\eta)\mathbf{u}^\dagger \mathbf{u} - (1+\eta)\mathbf{t}^\dagger \mathbf{t} \\ &+ \xi \frac{\mathbf{s}^\dagger \mathbf{v} + \mathbf{v}^\dagger \mathbf{s} + \mathbf{u}^\dagger \mathbf{t} + \mathbf{t}^\dagger \mathbf{u}}{2} - \xi, \end{aligned} \tag{19}$$

where

$$\eta = \frac{2}{m}\delta_0, \qquad \xi = \frac{\kappa^2}{m}\delta_0 + \kappa \delta_1, \tag{20}$$

with $\delta_0 = p_0 - p_3$ and $\delta_1 = p_1 - p_2$. The maximum of (19) subject to (18) is the local energy $\Omega_\circ$ of a particle in $\mathcal{S}_2$. We twice apply the dominance argument in [11], once for $\xi \geqslant 0$ and then again for $\xi < 0$. We find that the local energy in a particle of $\mathcal{S}_2$ is $\Omega_\circ = \Omega_\circ(\eta, \xi)$ where

$$\Omega_\circ(\eta,\xi) = \begin{cases} \sqrt{\frac{1-\eta^2+\xi^2}{1-\eta^2}} - \xi - \eta, & |\eta\xi| < 1 - \eta^2 \\ |\xi| + |\eta| - \xi - \eta, & \text{otherwise} \end{cases}. \tag{21}$$

Shown in the top display of Fig. 1 is a contour plot of $\Omega_\circ$ for $\kappa = 2$. The plot's diamond-shaped domain given by $|\delta_0| + |\delta_1| \leqslant 1$ is all possible combinations of δ_0 and δ_1. Of

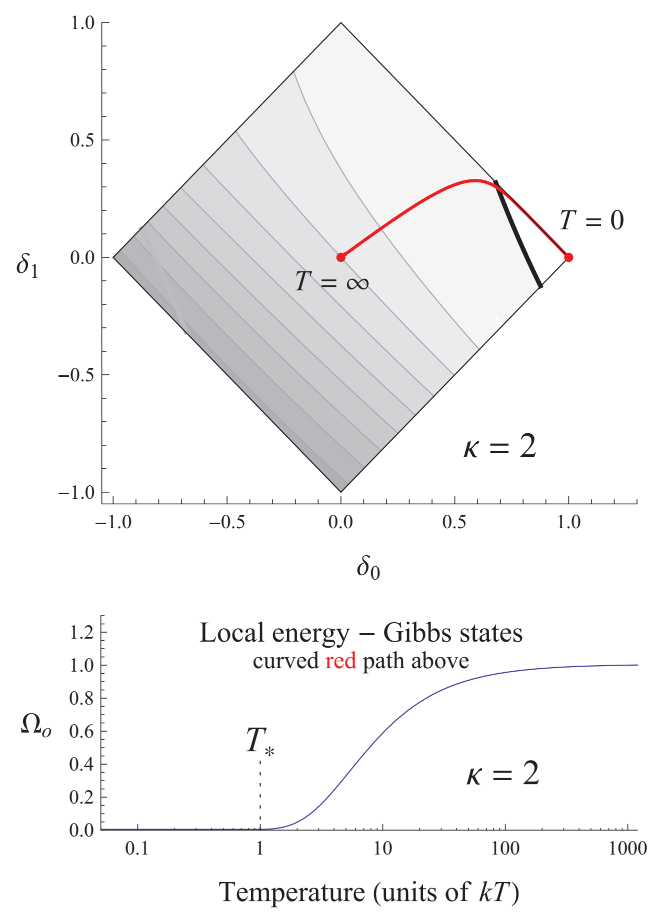

FIG. 1. (Color online) Local energy $\Omega_\circ$ for $\kappa = 2$ in $\mathcal{S}$ with system state ρ parametrized by the probability differences δ_0, δ_1. Darker shading indicates greater local energy. $\Omega_\circ = 0$ for any eigenmixture ρ in a neighborhood of the ground state $|E_0\rangle$. Gibbs states are SL passive below the critical temperature T_*.

special interest in the δ_0,δ_1 parameter space is the three-sided region at right that includes $\delta_0 = 1$. This region consists of the ground state $|E_0\rangle$ and all eigenmixtures ρ that are small departures from it. These states are all SL passive; they all have $\eta,\xi \geqslant 0$ with zero local energy, $\Omega_O = |\xi| + |\eta| - \xi - \eta = 0$. The extent of the SL passive neighborhood around $|E_0\rangle$ can be quantified by considering the three-sided $\Omega_\circ = 0$ region in Fig. 1's top display. A sufficient condition for $\Omega_\circ = 0$ is that the system eigenmixture have $\delta_0 \geqslant \delta_*$ where

$$\delta_* = \frac{\kappa + \sqrt{(3m^2 + 2\kappa m - 8)}}{2(m^2 + \kappa m - 2)}$$

is the δ_0 coordinate of the bottom corner of the $\Omega_O = 0$ region (see the top display in Fig. 1). A sufficient condition for $\delta_0 \geqslant \delta_*$ is, in turn,

$$p_0 \geqslant p_* = \frac{1 + \delta_*}{2}. \tag{22}$$

Any eigenmixture ρ of the form of (4) with ground state probability $p_0 \geqslant p_*$ has zero local energy. The threshold probability p_* in (22) is a decreasing function of κ with, for example, $p_* = .9383$ for $\kappa = 2$. We conclude that the neighborhood of $|E_0\rangle$ of zero local energy and SL passivity can be substantial. We pursue this further in the following section.

The Gibbs states (4) with population probabilities (5) of the particle pair $\mathcal{S}_2$ have partition function

$$\mathcal{Z} = \sum_{j=0}^{3} \exp\left(-\frac{E_j}{kT}\right) = 2\left(\cosh\frac{\kappa}{kT} + \cosh\frac{m}{kT}\right)$$

and, in particular,

$$\delta_0 = \frac{2}{\mathcal{Z}}\sinh\frac{m}{kT}, \quad \delta_1 = \frac{2}{\mathcal{Z}}\sinh\frac{\kappa}{kT}. \tag{23}$$

The bottom display in Fig. 1 uses (23) to show local energy varying by temperature for $\kappa = 2$ through the Gibbs states (the red path in the top display), from the $T = 0$ ground state at $(\delta_0,\delta_1) = (1,0)$ to the $T = \infty$ completely mixed state at $(\delta_0,\delta_1) = (0,0)$. The Gibbs states exit the $\Omega_O = 0$ region at a nonzero temperature T_* for any $\kappa > 0$, where the critical temperature T_* is determined by the condition $|\eta\xi| < 1 - \eta^2$ in (21) with (23) used in (20).

The particle pair $\mathcal{S}_2$ with Hamiltonian (15) is a special case of the class of two-particle systems $\mathcal{S}_{2,\gamma}$ with Hamiltonian

$$\mathbf{H} = \kappa\left(\frac{1+\gamma}{2}\boldsymbol{\sigma}_1^x\boldsymbol{\sigma}_2^x + \frac{1-\gamma}{2}\boldsymbol{\sigma}_1^y\boldsymbol{\sigma}_2^y\right) + \boldsymbol{\sigma}_1^z + \boldsymbol{\sigma}_2^z, \tag{24}$$

where $\gamma \in [0,1]$ is the coupling anisotropy. The coupling is isotropic when $\gamma = 0$ and fully anisotropic when $\gamma = 1$ as in (15). (Re)define $m = \sqrt{\gamma^2\kappa^2 + 4}$ for (24). Then the eigenenergies of (24) are those in (16). The class $\mathcal{S}_{2,\gamma}$ of anisotropic systems is interesting because it allows us to see firsthand the roles of ground state nondegeneracy and entanglement in our theorem: for $\gamma = 0$ and $\kappa < 2$ the ground state $|E_0\rangle$ is nondegenerate but separable, and for $\gamma \in (0,1)$ it is degenerate if and only if

$$\kappa = \frac{2}{\sqrt{1-\gamma^2}}. \tag{25}$$

To determine the local energy in a particle of $\mathcal{S}_{2,\gamma}$, we again consider a general local quantum operation $\mathcal{G}$ applied to a particle of $\mathcal{S}_{2,\gamma}$, where $\mathcal{G}$ has Kraus operators (17) with constraints (18). The energy extracted from $\mathcal{S}_{2,\gamma}$ in an eigenmixture state ρ by the local operation $\mathcal{G}$ is, after some calculation,

$$\Delta E(\rho) = (1-\eta)\mathbf{u}^\dagger\mathbf{u} - (1+\eta)\mathbf{t}^\dagger\mathbf{t} + \xi\frac{\mathbf{s}^\dagger\mathbf{v} + \mathbf{v}^\dagger\mathbf{s}}{2} + \mu\frac{\mathbf{u}^\dagger\mathbf{t} + \mathbf{t}^\dagger\mathbf{u}}{2} - \xi, \tag{26}$$

where

$$\eta = \frac{2\delta_0}{m}, \quad \xi = \frac{\gamma^2\kappa^2\delta_0}{m} + \kappa\delta_1, \quad \mu = \frac{\gamma\kappa^2\delta_0}{m} + \gamma\kappa\delta_1, \tag{27}$$

with $\delta_0 = p_0 - p_3$ and $\delta_1 = p_1 - p_2$. To maximize (26) subject to (18), we again take essentially the approach in [11] and find that the maximum of $\Delta E(\rho)$ in (26) subject to constraints (18) is the unconstrained maximum of

$$\omega(\alpha,\beta) = (1-\eta)\sin^2\alpha - (1+\eta)\sin^2\beta + |\xi|\cos\alpha\cos\beta + |\mu|\sin\alpha\sin\beta - \xi. \tag{28}$$

The maximum of $\omega(\alpha,\beta)$ is the local energy of an $\mathcal{S}_{2,\gamma}$ particle. Using (23) for δ_0 and δ_1 in (27), we numerically maximize $\omega(\alpha,\beta)$ to find (see Fig. 2) the Gibbs states critical temperatures T_* as a function of the coupling strength κ for selected anisotropies γ. The points in Fig. 2's inset show the values of κ where, for different $\gamma \in (0,1)$, T_* falls to zero. The superposed curve in the inset is condition (25) for degeneracy. For $\gamma \in (0,1)$ we see that, consistent with our theorem, $T_* = 0$ wherever the system coupling parameters combine in (25) to make $|E_0\rangle$ degenerate. For $\gamma = 0$, $|E_0\rangle$ is nondegenerate and separable for $\kappa < 2$, degenerate for $\kappa = 2$, and nondegenerate and entangled for $\kappa > 2$, while Fig. 2 shows that $T_* > 0$ (the extant ground state neighborhood of SL passivity) only for

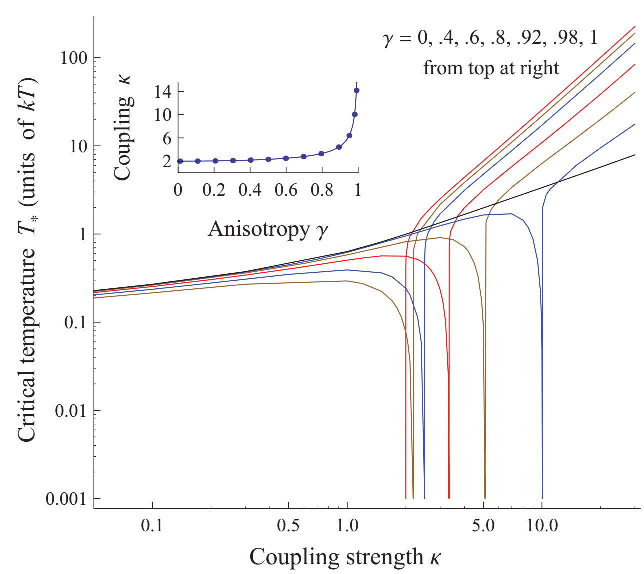

FIG. 2. (Color online) Critical temperatures T_* below which $\Omega_\circ = 0$ for selected coupling anisotropies γ. The points in the inset are coupling strengths κ where $T_* = 0$. The superposed curve in the inset is condition (25) for ground state degeneracy.

$\kappa > 2$. These various cases illustrate the point of our theorem: a nondegenerate, entangled ground state is sufficient to create a ground state neighborhood of SL passivity.

IV. SL PASSIVITY'S EXTENT

We explore in three directions the extent of SL passivity's presence in finite quantum systems. We first ask whether the threshold ground state probability p_* identified in our theorem applies also for system states that are not eigenmixtures. We show by an example that p_* does not necessarily apply to system states with some coherence among eigenstates; that is, given some coherence, a state with ground state population probability $p_0 > p_*$ can fail to be SL passive. This shows that eigenmixtures, which play a distinctive role in state passivity, are also special for SL passivity. We then go on to ask whether SL passivity is limited to only systems of small dimension. We show by the example of a Heisenberg chain of N spin-$\frac{1}{2}$ particles that SL passivity can be a nonvanishing feature of a many-particle system. Finally, we ask whether SL passivity is always confined only to eigenmixtures near the ground state. We saw in the previous section that for the two-particle system $\mathcal{S}_2$ any eigenmixture with large enough ground state population probability ($p_0 > .9383$ for $\kappa = 2$) is SL passive. In this section we offer an example of a two-particle system in which the ground state's SL passivity neighborhood extends all the way to the completely mixed state ($p_0 = .25$) and in which, in particular, Gibbs states of *any* temperature are SL passive. Our point with this section's examples is that eigenmixtures are central to SL passivity and that, among eigenmixtures, SL passivity is neither limited to only quantum systems of a few particles nor necessarily confined to only the near vicinity of the ground state.

Suppose the state ρ of the particle pair $\mathcal{S}_2$ of the previous section is an eigenmixture. The local energy of a particle is then $\Omega_\circ = \Omega_\circ(\eta,\xi)$ in (21), and a threshold ground state population probability $p_* < 1$ exists such that ρ with $p_0 \geqslant p_*$ is SL passive. Now introduce some coherence to the eigenmixture ρ and consider the system state

$$\rho' = \rho + r(|E_2\rangle\langle E_0| + |E_0\rangle\langle E_2|), \tag{29}$$

with real coherence r where $|r| \leqslant \sqrt{p_0 p_2}$. We find after some calculation that, for $\mathcal{S}_2$ in the state ρ', the energy extracted by a general quantum operation $\mathcal{G}$ applied locally to a particle is

$$\begin{aligned}\Delta E(\rho') = {} & \Omega_\circ(\eta,\xi) + \frac{r}{\sqrt{m(m+2)}}[\kappa(\mathbf{t}^\dagger\mathbf{s} + \mathbf{s}^\dagger\mathbf{t}) \\ & + \kappa^2(\mathbf{u}^\dagger\mathbf{s} + \mathbf{s}^\dagger\mathbf{u}) + (m+2)(\mathbf{u}^\dagger\mathbf{v} + \mathbf{v}^\dagger\mathbf{u}) \\ & - \kappa(m+2)(\mathbf{t}^\dagger\mathbf{v} + \mathbf{v}^\dagger\mathbf{t})].\end{aligned} \tag{30}$$

Suppose $\mathcal{G}$ has a single Kraus operator $\mathbf{K} = \exp(-i\phi\boldsymbol{\sigma}^y)$. The energy (30) extracted by this (unitary) $\mathcal{G}$ is

$$\Delta E(\rho') = \frac{2\sin^2\phi}{m\kappa}\left[rA\cot\phi - \eta - 2\xi\right], \tag{31}$$

where

$$A = \frac{2}{\kappa}\sqrt{\frac{m-2}{m}}[2 + (m+\kappa)(\kappa+1)].$$

For $\mathcal{S}_2$ with any degree of coupling κ and any ρ' in (29) with nonzero coherence r, we can pick the angle ϕ associated with $\mathcal{G}$ so that $\Delta E(\rho')$ is positive. This is an example in which the smallest amount of coherence added to a SL passive eigenmixture ρ renders the resulting state ρ' not SL passive, allowing energy to be extracted from the system. Eigenmixtures play a distinctive role in state passivity; this shows that they do also in SL passivity.

The particle pair $\mathcal{S}_2$ is a case of an N-particle closed Heisenberg spin chain $\mathcal{S}_N$ with Hamiltonian

$$\mathbf{H} = \kappa\prod_{i=1}^{N}\boldsymbol{\sigma}_i^x\boldsymbol{\sigma}_{i+1}^x + \sum_{i=1}^{N}\boldsymbol{\sigma}_i^z, \tag{32}$$

where $\boldsymbol{\sigma}_{N+1}^x \equiv \boldsymbol{\sigma}_1^x$. For each $N \geqslant 2$ the ground state $|E_0\rangle$ is nondegenerate and fully entangled. Suppose $\mathcal{S}_N$ is in state (4) with $d = 2^N$ and we apply a general quantum operation $\mathcal{G}$ to a particle. As in the two-particle case, $\mathcal{G}$ involves at most four Kraus operators (17) with the constraints (18). We seek the system energy $\Delta E(\rho)$ in (1) extracted by $\mathcal{G}$ from $\mathcal{S}_N$ and find, remarkably, that the extracted energy $\Delta E(\rho)$ has the same form as (19) for all $N \geqslant 2$, where η and ξ vary according to N. Therefore, for $N \geqslant 2$ the local energy in a particle is as in (21) with η and ξ depending on N. We suppose ρ is a Gibbs state for each N and then solve $|\xi\eta| = 1 - \eta^2$ in (21) to obtain the critical temperature T_* for zero local energy. Figure 3 shows the results of these calculations for spin chains $\mathcal{S}_N$ of up to six particles. (For $N \geqslant 6$ the curves for T_* are visually indistinguishable.) We see that T_* increases with N but that this increase quickly becomes vanishingly small. This could be expected for a closed chain's ring topology. A particle is most strongly affected by its two immediate neighbors, while any added particle joins the chain as a most distant particle. Increasing N only adds distant neighbors with vanishingly less effect, and Fig. 3 reflects this. Most importantly, Fig. 3 shows that SL passivity and zero local energy are not limited to only a few particles; theoretically, a neighborhood of SL passivity can exist with no diminution in systems of arbitrarily many particles.

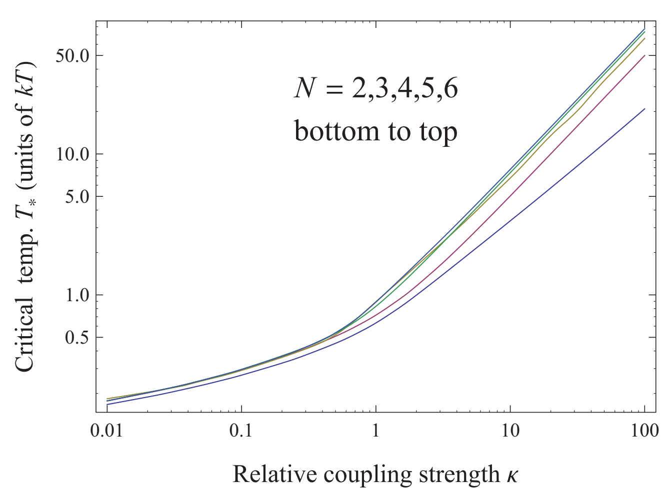

FIG. 3. (Color online) Gibbs states' critical temperatures T_* for SL passivity and zero local energy in N-particle spin chains. Curves for $N = 6$ and beyond are visually indistinguishable.

Thus far the spin systems in our examples have all exhibited a SL passivity neighborhood of bounded extent with, specifically, a finite critical Gibbs temperature $T_* < \infty$. Now consider the system $\mathcal{S}_X$ of two Heisenberg XXX-coupled spin particles with Hamiltonian

$$\mathbf{H}_X = \boldsymbol{\sigma}_1^x\boldsymbol{\sigma}_2^x + \boldsymbol{\sigma}_1^y\boldsymbol{\sigma}_2^y + \boldsymbol{\sigma}_1^z\boldsymbol{\sigma}_2^z, \tag{33}$$

eigenenergies $E_0 = -3$, $E_1 = E_2 = E_3 = 1$, and corresponding eigenstates

$$|E_0\rangle = \frac{|10\rangle - |01\rangle}{\sqrt{2}}, \quad |E_2\rangle = |00\rangle,$$

$$|E_1\rangle = \frac{|10\rangle + |01\rangle}{\sqrt{2}}, \quad |E_3\rangle = |11\rangle.$$

The ground state $|E_0\rangle$ of $\mathcal{S}_X$ is nondegenerate and entangled so we conclude that $|E_0\rangle$ has a neighborhood of SL passivity. To determine this neighborhood's extent, we derive for $\mathcal{S}_X$ the energy (1) extracted by a general quantum operation (6) on one of $\mathcal{S}_X$'s two particles, finding that

$$\Delta E(\rho) = -(p_0 + p_1 - 2p_2)\mathbf{u}^\dagger\mathbf{u} - (p_0 + p_1 - 2p_3)\mathbf{t}^\dagger\mathbf{t} - (p_0 - p_1)(2 - \mathbf{s}^\dagger\mathbf{v} - \mathbf{v}^\dagger\mathbf{s}) \tag{34}$$

for any eigenmixture ρ with population probabilities p_0, p_1, p_2, p_3. For Gibbs states $p_0 \geqslant p_1 \geqslant p_2 \geqslant p_3$. Also, $\mathbf{u}^\dagger\mathbf{u} \geqslant 0$, $\mathbf{t}^\dagger\mathbf{t} \geqslant 0$, and $\mathbf{s}^\dagger\mathbf{v} + \mathbf{v}^\dagger\mathbf{s} \leqslant \mathbf{s}^\dagger\mathbf{s} + \mathbf{v}^\dagger\mathbf{v} \leqslant 2$. We readily conclude then that $\Delta E(\rho) \leqslant 0$ in (34) and that the local energy in a particle of $\mathcal{S}_X$ is $\Omega_\circ = 0$ for any Gibbs state ρ; that is, $T_* = \infty$. Thus $\mathcal{S}_X$ is a quantum system whose Gibbs states of all temperatures are both passive and SL passive.

V. SUMMARY AND FINAL REMARKS

We summarize the work presented in this paper by emphasizing that SL passivity provides a framework for determining the energy that is locally accessible in multipartite quantum systems. This newly identified property of states in finite quantum systems is a variant of the standard notion of state passivity, where the nature of the operation on the multipartite system is both (1) relaxed to allow any general quantum operation and (2) restricted in its application to a subsystem. These countervailing modifications yield unexpected and interesting results. Passive states are known to be eigenmixtures that have no population probability inversion, and, in particular, all Gibbs states are passive. While eigenmixtures are similarly important to SL passivity, the conditions for and the extent of SL passivity within these states are more subtle. If the ground state is nondegenerate and entangled, then the system exhibits a neighborhood of SL passivity around the ground state. Using Gibbs state temperature to gauge this neighborhood's extent, we saw by example that the Gibbs state critical temperature for SL passivity can be $T_* = 0$ (when the ground state is separable or degenerate), positive and finite, or even $T_* = \infty$ (in which case all the Gibbs states are SL passive). The existence of systems with $T_* = \infty$ decisively establishes that SL passivity is not limited to only the near vicinity of the ground state; its extent can be considerable. Remarkably, SL passivity can extend, also, without diminution to high-dimensional systems of arbitrarily many particles. The Gibbs critical temperature of an N-particle Heisenberg ring, for example, quickly converges for increasing N to a positive limit value $T_* > 0$. For such a system in a state of SL passivity, the system particles act collectively to block energy release from any one particle.

Our theorem concerns energy extracted by a local operation when the system state is near the ground state. A complementary result can be stated for *adding* energy when the system state is near the *maximum* energy eigenstate $|E_{d-1}\rangle$. Let $\mathcal{S}$ be a finite quantum system with Hamiltonian (2) and a specified subsystem $\mathcal{C}$. Suppose that $|E_{d-1}\rangle$ is nondegenerate and that, in $|E_{d-1}\rangle$, $\mathcal{C}$ is fully entangled with the rest of $\mathcal{S}$. Then a threshold maximum energy state population probability $q_* < 1$ exists such that no energy can be added to the system by any local quantum operation on $\mathcal{C}$ when the system state is an eigenmixture as in (4) with $p_{d-1} \geqslant q_*$. The two threshold probabilities p_* and q_* associated with a subsystem $\mathcal{C}$ are not generally equal. The proof of this complementary result parallels that of our theorem.

Strong local passivity is only newly discovered, and it is premature to anticipate applications. We note, though, that in the anisotropically coupled particle pair $\mathcal{S}_{2,\gamma}$ the critical Gibbs temperature T_* is highly sensitive to the strength of the external magnetic field (reflected in the parameter κ) when condition (25) is close to satisfied. In fact, under conditions close to (25), Fig. 2 shows T_* varying over orders of magnitude in response to only a small change in κ. The critical temperature T_* is a proxy for the extent of the SL passivity neighborhood, and with a suitable initial system state varying T_* can switch on and off the SL passivity of a subsystem, locking up or allowing the release of energy. This suggests that a system such as $\mathcal{S}_{2,\gamma}$ might be a sensitive detector of small changes in the external magnetic field, or by actively modulating the external field $\mathcal{S}_{2,\gamma}$ might be used as a switch for energy release. These comments, while only speculative, suggest potential possibilities.

The notion of SL passivity raises a host of theoretical questions. In particular, SL passivity makes a new connection between the local versus global paradigm in quantum information science and the standard notion of passivity in thermodynamics, potentially advancing, for example, the theory of quantum Maxwell demons for subsystems.

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