

# Role of the interpretation of stochastic calculus in systems with cross-correlated Gaussian white noises

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We derive the Fokker-Planck equation for multivariable Langevin equations with cross-correlated Gaussian white noises for an arbitrary interpretation of the stochastic differential equation. We formulate the conditions when the solution of the Fokker-Planck equation does not depend on which stochastic calculus is adopted. Further, we derive an equivalent multivariable Ito stochastic differential equation for each possible interpretation of the multivariable Langevin equation. To demonstrate the usefulness and significance of these general results, we consider the motion of Brownian particles. We study in detail the stability conditions for harmonic oscillators with two white noises, one of which is additive, random forcing, and the other, which accounts for fluctuations of either the damping or the spring coefficient, is multiplicative. We analyze the role of cross correlation in terms of the different noise interpretations and confirm the theoretical predictions via numerical simulations. We stress the interest of our results for numerical simulations of stochastic differential equations with an arbitrary interpretation of the stochastic integrals.

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## I. INTRODUCTION

A great variety of natural phenomena and technological devices involve the interaction of a system with a random environment. In light of the central limit theorem, the latter is often considered to be Gaussian. If the surroundings vary on a much faster time scale than the intrinsic dynamics of the system, the fluctuations of the environment can be considered white. This Gaussian white noise idealization has provided a successful description of systems driven by real noises with small correlation times for many applications [1]. The role of environmental fluctuations on the dynamical evolution of a system can often be captured by assuming that characteristic parameters fluctuate around their mean value. In such cases the noise appears in the stochastic evolution (Langevin) equation in a multiplicative way. The effects of multiplicative noise on the dynamics of the system have been investigated, for instance, in models of population dynamics [2–6], tumor growth [7–9], epidemics [10–13], lasers [14–17], nematic liquid crystals [18], and oscillators [19–23].

A system can be driven by two or more external noises. If these noises have a common origin, they are cross correlated. Cross-correlated noises can also arise if the action of the random environment affects the internal fluctuations of the system. The effect of cross correlations between noises has been studied in a variety of applications, e.g., in single-mode lasers [24–26], in bistable systems [27–32], in stochastic resonance [33–39], in anomalous diffusion [40], in noise-induced transport [41,42], in tumor growth [9,43–48], in vegetation models [49], in metapopulation models [50], in barrier crossing and transitions from metastable states [51–59], and in nonequilibrium phase transitions in spatially extended systems [60].

We are interested in studying the effect of cross-correlated multiplicative and additive white Gaussian noises. As usual, we encounter the technical problem of having to adopt an appropriate interpretation of stochastic differential equations

(SDEs) with multiplicative Gaussian white noise. The Ito and Stratonovich interpretations are most commonly employed, but other interpretations are possible [1,61–78]. The predicted dynamical behavior of a system at the macroscopic level can depend on the stochastic interpretation. The latter is part of the model and must be chosen on physical grounds. The interpretation of the stochastic integrals is also crucially important for numerical simulations of Langevin equations. Care must be taken to ensure that the integration algorithm is consistent with the adopted interpretation [79,80]. The results we derive below provide clear guidance how to deal with the various interpretations of stochastic calculus in analytical and numerical approaches to modeling dynamic systems with Gaussian white noises. In particular, we delineate the class of multivariable Langevin equations where the Fokker-Planck equation does not depend on the choice of stochastic calculus, even if the white noise is multiplicative. For this class, the physics of the system is the same, no matter what stochastic calculus is adopted. Any interpretation of the stochastic integrals can be employed in the analytical and numerical treatment of such systems, a very advantageous fact from a modeling view point.

We provide the foundation to investigate the dynamics of systems with cross-correlated Gaussian white noises for an arbitrary interpretation of the stochastic integrals, including the Ito [81], Stratonovich [82], and Klimontovich [83] interpretations. To illustrate the value of our general results, we investigate the dynamics of the classical harmonic oscillator with a fluctuating damping coefficient or with a fluctuating potential, i.e., a random frequency, driven by an additive random force. The investigation of oscillators subjected to various random influences is relevant and timely since oscillators are used to describe a large variety of applications [20–23,84–92] and have recently been considered for harvesting energy from ambient fluctuations to power small self-contained sensors and actuators [93–101].

The paper is organized as follows. In Sec. II, we present general results for  $n$ -variable Langevin equations with  $p$  cross-correlated Gaussian white noises. We formulate the difference scheme that corresponds to these equations and obtain the Fokker-Planck equation, which explicitly depends on the parameters describing how the white noises are interpreted. We obtain the condition when the Fokker-Planck equation does not depend on the interpretation of the stochastic integrals, even for multiplicative noise. Finally, we derive a system of Ito equations that is statistically equivalent to the system of Langevin equations for a given interpretation of the noise terms. We apply these results in Sec. III to the motion of Brownian particles in a potential subjected to two cross-correlated, additive, and multiplicative noises, deriving the class of systems with interpretation-independent Fokker-Planck equations and formulating the moment equations for the Brownian particle. In Sec. IV, we focus on the behavior of the randomly driven harmonic oscillator with either a fluctuating damping coefficient or a fluctuating spring coefficient and clarify the role of cross correlation. Our main results are discussed in Sec. V and some technical details of our calculations are provided in Appendices A and B.

## II. FOKKER-PLANCK EQUATION FOR DIFFERENT INTERPRETATIONS OF THE STOCHASTIC CALCULUS

### A. Difference scheme for Langevin equations

Consider a general system of  $n$  Langevin equations, which can be interpreted as an  $n$ -variable Langevin equation, for the state variables  $u_i(t)$ , driven by  $p$  cross-correlated Gaussian white noises  $\xi_j(t)$ :

$$\frac{du_i(t)}{dt} = f_i(\mathbf{u}(t)) + \sum_{j=1}^p g_{ij}(\mathbf{u}(t))\xi_j(t). \quad (2.1)$$

Here,  $\mathbf{u}(t) = [u_1(t), u_2(t), \dots, u_n(t)]$  with  $i = 1, 2, \dots, n$  is the state vector,  $f_i(\mathbf{u}(t))$  are the drift terms, and  $g_{ij}(\mathbf{u}(t))$  are the elements of the  $n \times p$  matrix  $(g_{ij})$ . If the elements of this matrix depend on  $\mathbf{u}(t)$ , the noises  $\xi_j(t)$  appear in (2.1) in a multiplicative way. In that case, we need to specify how these stochastic equations should be interpreted. To this end, we introduce the following implicit difference scheme:

$$\begin{aligned} \delta u_i &= u_i(t + \tau) - u_i(t) \\ &= f_i(\mathbf{u}(t))\tau + \sum_{j=1}^p g_{ij}[\mathbf{u}(t + \lambda_j \tau)]\delta W_j, \end{aligned} \quad (2.2)$$

corresponding to (2.1). Here,  $\tau$  is a small time, the parameters  $\lambda_j$  lie in the interval  $[0, 1]$ , and  $\delta W_j = W_j(t + \tau) - W_j(t)$  are the increments of Wiener processes  $W_j(t)$ . The latter are characterized by

$$\langle \delta W_i \rangle = 0, \quad \langle \delta W_i \delta W_j \rangle = 2C_{ij}\tau, \quad (2.3)$$

where  $\langle \dots \rangle$  denotes averaging over the realizations of these processes, and  $C_{ij} = C_{ji}$  are the elements of the  $p \times p$  correlation matrix  $(C_{ij})$ . We expand  $g_{ij}[\mathbf{u}(t + \lambda_j \tau)]$  up to first

order in  $\tau$ :

$$\begin{aligned} g_{ij}[\mathbf{u}(t + \lambda_j \tau)] &= g_{ij}\left(\mathbf{u}(t) + \frac{d\mathbf{u}(t)}{dt}\lambda_j \tau\right) \\ &= g_{ij}(\mathbf{u}(t)) + \sum_{l=1}^n \frac{\partial g_{ij}(\mathbf{u}(t))}{\partial u_l(t)} \frac{du_l(t)}{dt} \lambda_j \tau. \end{aligned} \quad (2.4)$$

Taking into account that  $du_l(t)/dt = \lim_{\tau \rightarrow 0} \delta u_l/\tau$ , we can rewrite (2.2) with linear accuracy in  $\tau$  as

$$\begin{aligned} \delta u_i &= f_i(\mathbf{u}(t))\tau + \sum_{j=1}^p g_{ij}(\mathbf{u}(t))\delta W_j \\ &+ \sum_{l=1}^n \sum_{k,j=1}^p \lambda_j \frac{\partial g_{ij}(\mathbf{u}(t))}{\partial u_l(t)} g_{lk}(\mathbf{u}(t))\delta W_k \delta W_j. \end{aligned} \quad (2.5)$$

This difference scheme, in which the noise terms are interpreted by means of the parameters  $\lambda_j$ , completely specifies the meaning of the Langevin equations (2.1) and constitutes our *first main result*. Note that the cases with  $\lambda_j = 0, \frac{1}{2},$  and  $1$  correspond to the Ito [81], Stratonovich [82], and Klimontovich [83] interpretations of these equations, respectively. The parameters  $\lambda_j$  are part of the model and must be chosen on physical grounds.

### B. Derivation of Fokker-Planck equations from Langevin equations

To derive the Fokker-Planck equation associated with the Langevin equations (2.1), which are interpreted in the sense of (2.5), we use the approach of [102] generalized to the multivariable case. Let  $P(\mathbf{u}, t)$  be the probability density that the random variable  $\mathbf{u}(t)$  takes the value  $\mathbf{u}$ . Expansion of its increment,  $\delta P(\mathbf{u}, t) = P(\mathbf{u}, t + \tau) - P(\mathbf{u}, t)$  up to first order in  $\tau$  yields  $\delta P(\mathbf{u}, t) = \tau \partial P(\mathbf{u}, t)/\partial t$ . On the other hand, using the definition  $P(\mathbf{u}, t) = \langle \delta[\mathbf{u} - \mathbf{u}(t)] \rangle$ , where  $\delta(\mathbf{u})$  is the Dirac  $\delta$  function, and the relation  $\mathbf{u}(t + \tau) = \mathbf{u}(t) + \delta\mathbf{u}$ , we obtain another representation of the increment  $\delta P(\mathbf{u}, t) = \langle \delta[\mathbf{u} - \mathbf{u}(t) - \delta\mathbf{u}] \rangle - \langle \delta[\mathbf{u} - \mathbf{u}(t)] \rangle$ . We perform a Taylor expansion of  $\delta[\mathbf{u} - \mathbf{u}(t) - \delta\mathbf{u}]$  up to second order in  $\delta\mathbf{u}$  and find

$$\begin{aligned} \tau \frac{\partial}{\partial t} P(\mathbf{u}, t) &= - \left\langle \sum_{i=1}^n \delta u_i \frac{\partial}{\partial u_i} \delta[\mathbf{u} - \mathbf{u}(t)] \right\rangle \\ &+ \frac{1}{2} \left\langle \sum_{i,l=1}^n \delta u_i \delta u_l \frac{\partial^2}{\partial u_i \partial u_l} \delta[\mathbf{u} - \mathbf{u}(t)] \right\rangle. \end{aligned} \quad (2.6)$$

To carry out the averaging procedure, we first substitute (2.5) into (2.6). Then, taking into account that the increments  $\delta W_j$  are of order  $\tau^{1/2}$  [see (2.3)], we keep on the right-hand side of (2.6) only terms up to first order in  $\tau$ . As a result, using the well-known properties of the  $\delta$  function, the two-stage

procedure of averaging [102], and (2.3), we obtain

$$\begin{aligned} & \left\langle \sum_{i=1}^n \delta u_i \frac{\partial}{\partial u_i} \delta[\mathbf{u} - \mathbf{u}(t)] \right\rangle \\ &= \tau \sum_{i=1}^n \frac{\partial}{\partial u_i} [f_i(\mathbf{u}) P(\mathbf{u}, t)] \\ &+ 2\tau \sum_{i,l=1}^n \sum_{j=1}^p \lambda_j \frac{\partial}{\partial u_i} \left( \frac{\partial g_{ij}(\mathbf{u})}{\partial u_l} S_{jl}(\mathbf{u}) P(\mathbf{u}, t) \right) \end{aligned} \quad (2.7)$$

and

$$\begin{aligned} & \left\langle \sum_{i,l=1}^n \delta u_i \delta u_l \frac{\partial^2}{\partial u_i \partial u_l} \delta[\mathbf{u} - \mathbf{u}(t)] \right\rangle \\ &= 2\tau \sum_{i,l=1}^n \sum_{j=1}^p \frac{\partial^2}{\partial u_i \partial u_l} g_{ij}(\mathbf{u}) S_{jl}(\mathbf{u}) P(\mathbf{u}, t), \end{aligned} \quad (2.8)$$

where

$$S_{jl}(\mathbf{u}) = \sum_{k=1}^p C_{jk} g_{lk}(\mathbf{u}) \quad (2.9)$$

are the elements of the  $p \times n$  matrix  $(S_{jl})$ .

Substituting (2.7) and (2.8) into (2.6) we obtain the Fokker-Planck equation

$$\begin{aligned} \frac{\partial}{\partial t} P(\mathbf{u}, t) &= - \sum_{i=1}^n \frac{\partial}{\partial u_i} f_i(\mathbf{u}) P(\mathbf{u}, t) \\ &- 2 \sum_{i,l=1}^n \sum_{j=1}^p \lambda_j \frac{\partial}{\partial u_i} \left( \frac{\partial g_{ij}(\mathbf{u})}{\partial u_l} S_{jl}(\mathbf{u}) P(\mathbf{u}, t) \right) \\ &+ \sum_{i,l=1}^n \sum_{j=1}^p \frac{\partial^2}{\partial u_i \partial u_l} g_{ij}(\mathbf{u}) S_{jl}(\mathbf{u}) P(\mathbf{u}, t), \end{aligned} \quad (2.10)$$

corresponding to the system of Langevin equations (2.1). This formula constitutes our *second main result*, viz., the Fokker-Planck equation for a general system of Langevin equations with multiplicative and cross-correlated noises with an arbitrary interpretation of the stochastic calculus.

As usual, the solution of this equation must be normalized,

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} du_1 \dots du_n P(\mathbf{u}, t) = 1, \quad (2.11)$$

and satisfy the initial condition

$$P(\mathbf{u}, 0) = \delta[\mathbf{u} - \mathbf{u}(0)], \quad (2.12)$$

where the initial state vector  $\mathbf{u}(0)$  is assumed to be nonrandom.

### C. Langevin equations with interpretation-independent Fokker-Planck equations

According to (2.10), the probability density  $P(\mathbf{u}, t)$  depends in general on the parameters  $\lambda_j$ , which specify how the white noises in the Langevin equations (2.1) are interpreted. If these noises represent an approximation of “real” noises, the parameters  $\lambda_j$  need to be determined using additional

information, physical or otherwise, about the system. If the Langevin equations (2.1) are considered in a general theoretical context without reference to a specific system, any value of  $\lambda_j$  from the interval  $[0, 1]$  is allowed. In this context, it is important to answer the question if there are systems, driven by multiplicative white noises, whose statistical properties do not depend on the interpretation of these noises. In other words, we are interested in finding the conditions when the parameters  $\lambda_j$  do not affect the probability density  $P(\mathbf{u}, t)$ . It follows from (2.10) that this occurs if

$$\sum_{i,l=1}^n \frac{\partial}{\partial u_i} \left[ \frac{\partial g_{ij}(\mathbf{u})}{\partial u_l} S_{jl}(\mathbf{u}) P(\mathbf{u}, t) \right] = 0 \quad (2.13)$$

for all  $j$ . Using (2.9), it can be easily verified that the condition (2.13) holds for any correlation matrix  $(C_{ij})$ , if the elements of the matrix  $(g_{ij})$  satisfy the condition

$$\sum_{l=1}^n \frac{\partial g_{ij}(\mathbf{u})}{\partial u_l} g_{lk}(\mathbf{u}) = 0 \quad (2.14)$$

for all admissible values of  $i, j$ , and  $k$ . This condition constitutes our *third main result*. It disproves the conventional view in the field that the solution of Langevin equations with multiplicative noise always depends on the choice of noise interpretation. If (2.14) is satisfied, then the Fokker-Planck equation for a general system of Langevin equations with multiplicative and cross-correlated noises is independent of the interpretation of stochastic calculus. The concern about which interpretation of the stochastic integrals is the physically correct choice does not arise for such systems. This result also has practical consequences; it implies that any acceptable integration algorithm may be used in the numerical simulations of such Langevin equations.

Let us first consider the one-dimensional case  $n = 1$ , in which (2.14) reduces to

$$\frac{dg_{1j}(u_1)}{du_1} g_{1k}(u_1) = 0 \quad (2.15)$$

( $j, k = 1, \dots, p$ ). This condition is fulfilled in the trivial case of additive noises, where  $dg_{1j}(u_1)/du_1 = 0$ , but it does not hold for multiplicative noises. This implies that the probability density  $P(u_1, t)$  of one-variable systems with multiplicative white noises always depends on their interpretation. Remarkably, there exist multivariable systems,  $n \geq 2$ , where this dependence is absent. Such a situation occurs, for example, when the matrix elements  $g_{ij}(\mathbf{u})$  depend only on the variables  $u_1, \dots, u_s$  with  $1 \leq s \leq n - 1$ , i.e.,  $g_{ij}(\mathbf{u}) = g_{ij}(u_1, \dots, u_s)$ , and all elements in the first  $s$  rows of the matrix  $(g_{ij})$  are equal to zero, i.e.,  $g_{ij}(u_1, \dots, u_s) = 0$  if  $1 \leq i \leq s$ . In this case, the condition (2.14) holds for all  $i, j, k$ , and despite the fact that the white noises are multiplicative, the probability density  $P(\mathbf{u}, t)$  does not depend on  $\lambda_j$ . A special but important case of such systems will be considered in Sec. III.

### D. Equivalent Ito equations

If the set of Langevin equations

$$\frac{du_i(t)}{dt} = f_i(\mathbf{u}(t)) + \tilde{f}_i(\mathbf{u}(t)) + \sum_{j=1}^p g_{ij}(\mathbf{u}(t)) \xi_j(t), \quad (2.16)$$

with the additional drift terms

$$\tilde{f}_i(\mathbf{u}(t)) = 2 \sum_{l=1}^n \sum_{j=1}^p \lambda_j \frac{\partial g_{ij}(\mathbf{u}(t))}{\partial u_l(t)} S_{jl}(\mathbf{u}(t)), \quad (2.17)$$

is interpreted in the Ito sense, defined by the difference scheme

$$\delta u_i = [f_i(\mathbf{u}(t)) + \tilde{f}_i(\mathbf{u}(t))]\tau + \sum_{j=1}^p g_{ij}(\mathbf{u}(t))\delta W_j, \quad (2.18)$$

then it gives rise to exactly the same Fokker-Planck equation as (2.10). In other words, we can always describe a system by a set of Ito SDEs and take into account the different interpretations required by the physics of the problem via additional drift terms that depend on the parameters  $\lambda_j$ . With this notation, the Fokker-Planck equation (2.10) can be rewritten as

$$\begin{aligned} \frac{\partial}{\partial t} P(\mathbf{u}, t) = & - \sum_{i=1}^n \frac{\partial}{\partial u_i} \{ [f_i(\mathbf{u}) + \tilde{f}_i(\mathbf{u})] P(\mathbf{u}, t) \\ & + \sum_{i,l=1}^n \sum_{j=1}^p \frac{\partial^2}{\partial u_i \partial u_l} [g_{ij}(\mathbf{u}) S_{jl}(\mathbf{u}) P(\mathbf{u}, t)] \}. \end{aligned} \quad (2.19)$$

We emphasize that the solution of (2.1), interpreted in a general sense according to the difference scheme (2.5), and the solution of (2.16), interpreted in the Ito sense according to the difference scheme (2.18), have the same statistical properties. This constitutes our *fourth main result*: We have shown that the set of Langevin equations (2.1) for each choice of the stochastic integrals has an equivalent set of SDEs in the Ito sense. This result is of practical importance for numerical simulations of Langevin equations with any choice of the noise interpretation. Ito SDEs are typically more suitable for implementing numerical integration algorithms. We illustrate this point in Sec. III, where we study the role of parameters  $\lambda_j$  by solving numerically the equivalent Ito equations (2.16). Note also that if  $\lambda_j = \frac{1}{2}$ , then (2.17) represents the Wong-Zakai term [103], which connects the stochastic equations in the Ito and Stratonovich interpretations.

### III. BROWNIAN PARTICLES IN ONE DIMENSION

To demonstrate the utility of our four main results, we study the dynamics of a Brownian particle moving in a potential  $V(x)$ , subjected to an additive random force  $\xi_2(t)$  and a multiplicative noise  $\xi_1(t)$ . Both noises are assumed to be Gaussian, white, and cross correlated. The Langevin equation reads as

$$m\ddot{x} + b\dot{x} + V'(x) + h(x, \dot{x})\xi_1(t) = \xi_2(t), \quad (3.1)$$

where  $m$  is the mass of the particle,  $V(x)$  is the potential, and the prime denotes the derivative with respect to  $x$ . The damping coefficient is  $b$ , and  $h(x, \dot{x})$  is a deterministic function of the particle position  $x = x(t)$  and velocity  $\dot{x} = dx(t)/dt$ . The noises  $\xi_i(t)$ , with  $i = 1, 2$ , are Gaussian white noises characterized by zero means  $\langle \xi_i(t) \rangle = 0$  and correlation functions

$$\langle \xi_i(t) \xi_j(t') \rangle = 2\delta(t - t') \begin{cases} D_i, & i = j \\ rD_{12}, & i \neq j. \end{cases} \quad (3.2)$$

Here,  $D_1$  is the intensity of multiplicative noise  $\xi_1(t)$ ,  $D_2$  the intensity of additive noise  $\xi_2(t)$ ,  $D_{12} = \sqrt{D_1 D_2}$ , and  $r, |r| \leq 1$ , a parameter characterizing the cross correlation of the noises.

Rewriting the Langevin equation (3.1) as a system of two first-order equations,

$$\frac{dx}{dt} = v, \quad (3.3a)$$

$$\frac{dv}{dt} = -\frac{1}{m}[bv + V'(x) + h\xi_1(t) - \xi_2(t)], \quad (3.3b)$$

where  $h = h(x, v)$ , and comparing (3.3) with (2.1), we obtain  $n = p = 2$ ,  $u_1 = x$ ,  $u_2 = v$ ,  $f_1(\mathbf{u}) = v$ ,  $f_2(\mathbf{u}) = -(bv + V'(x))/m$ , and

$$(g_{ij}) = \begin{pmatrix} 0 & 0 \\ -h/m & 1/m \end{pmatrix}. \quad (3.4)$$

Comparing (3.2) with (2.3) and applying the relation  $\int_t^{t+\tau} dt' \xi_i(t') = \delta W_i$ , we find the correlation matrix

$$(C_{ij}) = \begin{pmatrix} D_1 & rD_{12} \\ rD_{12} & D_2 \end{pmatrix}. \quad (3.5)$$

Using (3.4) and the definition (2.9), we obtain the matrix

$$(S_{ij}) = \begin{pmatrix} 0 & -D_1 h/m + rD_{12}/m \\ 0 & D_2/m - rD_{12}h/m \end{pmatrix}. \quad (3.6)$$

The definition (2.17) implies that one of the additional drift terms vanishes,  $\tilde{f}_1(\mathbf{u}) = 0$ , and the other is given by

$$\tilde{f}_2(\mathbf{u}) = \frac{2\lambda_1}{m^2} (D_1 h - rD_{12}) \frac{\partial h}{\partial v}. \quad (3.7)$$

Collecting the above results, we obtain from (2.10) the Fokker-Planck equation for the probability density  $P = P(x, v, t)$ :

$$\begin{aligned} \frac{\partial}{\partial t} P + v \frac{\partial}{\partial x} P - \frac{1}{m} \frac{\partial}{\partial v} [bv + V'(x)] P \\ + \frac{2\lambda_1}{m^2} \frac{\partial}{\partial v} (D_1 h - rD_{12}) \frac{\partial h}{\partial v} P \\ - \frac{1}{m^2} \frac{\partial^2}{\partial v^2} (D_1 h^2 + D_2 - 2rD_{12}h) P = 0. \end{aligned} \quad (3.8)$$

#### A. Interpretation-independent case: $h = h(x)$

According to (3.4), the matrix  $(g_{ij})$  satisfies the condition (2.14), i.e., the statistical properties of the oscillator do not depend on  $\lambda_1$ , if and only if  $h = h(x)$ . In this case,  $\tilde{f}_1(\mathbf{u}) = \tilde{f}_2(\mathbf{u}) = 0$ , and the Fokker-Planck equation (3.8) reduces to

$$\begin{aligned} \frac{\partial}{\partial t} P + v \frac{\partial}{\partial x} P - \frac{1}{m} \frac{\partial}{\partial v} [bv + V'(x)] P \\ - \frac{1}{m^2} [D_1 h^2(x) + D_2 - 2rD_{12}h(x)] \frac{\partial^2}{\partial v^2} P = 0. \end{aligned} \quad (3.9)$$

Since  $(\sqrt{D_1}h - r\sqrt{D_2})^2 \geq 0$ , the condition  $D_1 h^2(x) + D_2 - 2rD_{12}h(x) \geq 0$  always holds.

This constitutes an illustration of our second main result. *If the multiplicative noise term in (3.1) depends only on the position of the Brownian particle and not on its velocity, i.e.,  $h = h(x)$ , then all interpretations of the stochastic integrals*

in (3.1), in particular the Ito and Stratonovich interpretations, are equivalent. Consequently, the problem of interpretation of the stochastic integrals does not arise and any satisfactory integration method may be used in the numerical simulations of such Langevin equations.

### B. Moment equations

To derive the equations for the moments of the particle position  $x$  and velocity  $v$ , we apply the difference scheme (2.5) to (3.3) and obtain

$$\delta x = v\tau, \quad (3.10a)$$

$$\begin{aligned} \delta v = & -\frac{1}{m}[bv + V'(x)]\tau - \frac{1}{m}(h\delta W_1 - \delta W_2) \\ & + \frac{\lambda_1}{m^2}h\frac{\partial h}{\partial v}(\delta W_1)^2 - \frac{\lambda_1}{m^2}\frac{\partial h}{\partial v}\delta W_1\delta W_2. \end{aligned} \quad (3.10b)$$

Taking averages, dividing by  $\tau$ , performing the limit  $\tau \rightarrow 0$ , and using (2.3), we find the obvious result  $\langle v \rangle = d\langle x \rangle/dt$  and the following equation for the mean particle position:

$$\begin{aligned} m\frac{d^2\langle x \rangle}{dt^2} + b\frac{d\langle x \rangle}{dt} + \langle V'(x) \rangle - \frac{2\lambda_1}{m}D_1\left\langle h\frac{\partial h}{\partial v} \right\rangle \\ + \frac{2\lambda_1}{m}rD_{12}\left\langle \frac{\partial h}{\partial v} \right\rangle = 0. \end{aligned} \quad (3.11)$$

It shows, in particular, that the interpretation of the multiplicative noise  $\xi_1(t)$  does not affect  $\langle x \rangle$ , if  $h = h(x)$ . This is in accordance with our previous finding that *all* statistical characteristics of the solution of equations (3.3) with  $h = h(x)$  do not depend on the parameter  $\lambda_1$ . In contrast, if the function  $h$  depends on  $v$ , then the evolution of  $\langle x \rangle$  is influenced by  $\lambda_1$ . Note that if  $h$  nonlinearly depends on  $v$ , then (3.11) can not be directly integrated because it is coupled to the equations for higher-order moments.

To derive the equations for the second moments, we first use (3.10) to write the increments  $\delta(x^2) \simeq 2xv\tau$ ,  $\delta(xv) \simeq v^2\tau + x\delta v$ , and  $\delta(v^2) \simeq 2v\delta v + (\delta v)^2$  up to first order in  $\tau$ . Proceeding in the same way as in the previous case, we find

$$\frac{d\langle x^2 \rangle}{dt} = 2\langle xv \rangle, \quad (3.12a)$$

$$\begin{aligned} \frac{d\langle v^2 \rangle}{dt} = & -\frac{2b}{m}\langle v^2 \rangle - \frac{2}{m}\langle vV'(x) \rangle + \frac{2}{m^2}D_1\langle h^2 \rangle + \frac{2}{m^2}D_2 \\ & - \frac{4}{m^2}rD_{12}\langle h \rangle + \frac{4\lambda_1}{m^2}D_1\left\langle v h \frac{\partial h}{\partial v} \right\rangle \\ & - \frac{4\lambda_1}{m^2}rD_{12}\left\langle v \frac{\partial h}{\partial v} \right\rangle, \end{aligned} \quad (3.12b)$$

$$\begin{aligned} \frac{d\langle xv \rangle}{dt} = & -\frac{b}{m}\langle xv \rangle - \frac{1}{m}\langle xV'(x) \rangle + \langle v^2 \rangle \\ & + \frac{2\lambda_1}{m^2}D_1\left\langle x h \frac{\partial h}{\partial v} \right\rangle - \frac{2\lambda_1}{m^2}rD_{12}\left\langle x \frac{\partial h}{\partial v} \right\rangle. \end{aligned} \quad (3.12c)$$

Equation (3.1) has a wide spectrum of applicability. For example, it can be used to describe the stochastic motion of a linear or nonlinear oscillator of mass  $m$  with random

friction and random external force. If  $h = 0$  and the potential is periodic, then (3.1) corresponds to the equation for a Brownian motor [42]. In this case, the Fokker-Planck equation (3.8) does not depend on  $\lambda_1$  and is independent of the stochastic interpretation.

## IV. BROWNIAN HARMONIC OSCILLATOR

In the following, we will be more specific and focus on two important cases. We assume that  $V(x) = kx^2/2$ , i.e., the Brownian particle moves in a harmonic potential, and that either  $h = bv$  or  $kx$ . The first one corresponds to the harmonic oscillator with additive noise and a fluctuating damping coefficient, and the second one to the harmonic oscillator with additive noise and a fluctuating spring coefficient, which gives rise to a random frequency. The statistical properties of the harmonic oscillator with a fluctuating parameter have been studied extensively (see, e.g., [23] and references therein). However, the effects arising from the cross correlation between additive noise and multiplicative noise and from the interpretation of the multiplicative noise have not been addressed previously. Such effects will be relevant for the problem of energy harvesting from ambient fluctuations. Typically, studies in that area deal with oscillators driven by additive noise (see, e.g., [104–106]). The problem of additional random effects from fluctuations in the parameters of the oscillator appears not to have been addressed.

### A. Fluctuating damping coefficient

The model of a harmonic oscillator with a fluctuating damping coefficient, first introduced in the context of water wave generation by turbulent air flow [87], is widely used in different areas [23,88–92]. A more general version of this model that accounts for the influence of two cross-correlated white noises is described by (3.3) with  $h = bv$ . According to (3.8), the Fokker-Planck equation reads as

$$\begin{aligned} \frac{\partial}{\partial t}P + v\frac{\partial}{\partial x}P - \frac{1}{m}\frac{\partial}{\partial v}(bv + kx)P \\ + \frac{2\lambda_1 b}{m^2}\frac{\partial}{\partial v}(D_1bv - rD_{12})P \\ - \frac{1}{m^2}\frac{\partial^2}{\partial v^2}(D_1b^2v^2 + D_2 - 2rD_{12}bv)P = 0. \end{aligned} \quad (4.1)$$

If the white noises are not correlated with each other, i.e.,  $r = 0$ , then the solution of this equation satisfies the symmetry condition  $P(x, v, t) = P(-x, -v, t)$ . In the case of cross-correlated noises, where  $r \neq 0$ , this symmetry is broken,  $P(x, v, t) \neq P(-x, -v, t)$ .

Since the Fokker-Planck equation (4.1) depends on the parameter  $\lambda_1$ , all statistical characteristics of the oscillator also depend on  $\lambda_1$ . Let us first analyze this dependence for the mean particle position  $\langle x \rangle$ , which in accordance with (3.11) is governed by the equation

$$m\frac{d^2\langle x \rangle}{dt^2} + b\left(1 - 2\lambda_1\frac{b}{m}D_1\right)\frac{d\langle x \rangle}{dt} + k\langle x \rangle = -2\lambda_1\frac{b}{m}rD_{12}. \quad (4.2)$$

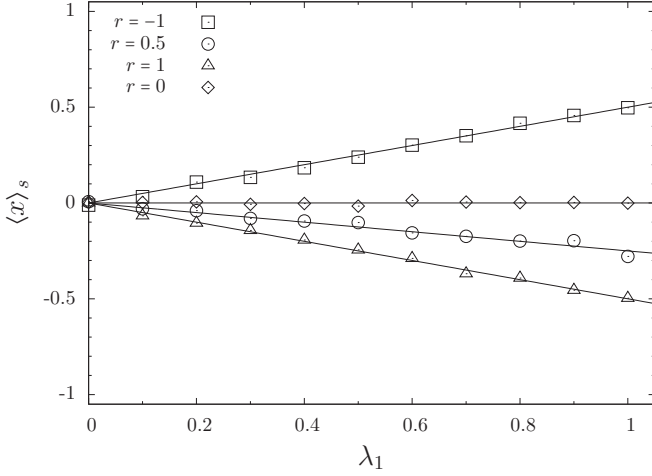


FIG. 1. Mean particle position in the steady state as a function of the parameter  $\lambda_1$  for different values of the parameter  $r$ . The solid lines represent the theoretical result (4.4), and the symbols show the simulation data. Both theoretical and numerical results are obtained for  $D_1 = D_2 = 0.25$ ,  $m = b = k = x_0 = 1$ , and  $v_0 = 0$ .

If the intensity of the multiplicative noise satisfies the condition  $D_1 < D_1^{(1)}$ , where

$$D_1^{(1)} = \frac{m}{2\lambda_1 b} = \frac{1}{4\lambda_1 \omega_d} \quad (4.3)$$

is the critical intensity of the multiplicative noise for the first moments and  $\omega_d = b/2m$  is the characteristic damping (angular) frequency, then the effective damping coefficient  $b(1 - 4\lambda_1 \omega_d D_1)$  is positive and (4.2) has the steady-state solution

$$\langle x \rangle_s = \langle x(\infty) \rangle = -\frac{r D_{12}}{k D_1^{(1)}}. \quad (4.4)$$

Equation (4.4) shows that *cross correlations between the white noises,  $r \neq 0$ , induce a shift of the mean particle position away from the origin, if the interpretation of the multiplicative noise differs from the Ito one,  $\lambda_1 > 0$ .* (Note that  $1/D_1^{(1)} \propto \lambda_1$ .) To confirm this interesting result, we have solved numerically the system of equations (3.3) with  $h = bv$  and calculated the mean position for large times. Figure 1 shows that our simulation results are in excellent agreement with the theoretical predictions.

If  $D_1 = D_1^{(1)}$ , the effective damping coefficient vanishes and, as follows from (4.2), the mean particle position  $\langle x \rangle$  oscillates periodically with the angular frequency  $\omega_0 = \sqrt{k/m}$  around the mean value (4.4). For  $D_1 > D_1^{(1)}$ , the function  $\langle x \rangle$  also oscillates, but its amplitude tends to infinity as  $t \rightarrow \infty$ . All these regimes follow from the exact solution of (4.2):

$$\begin{aligned} \langle x \rangle = & \exp(-\omega_1 t) \left( x_0 \cos(\Omega t) + \frac{1}{\Omega} (v_0 + \omega_1 x_0) \sin(\Omega t) \right) \\ & - \frac{r D_{12}}{k D_1^{(1)}} \left[ 1 - \exp(-\omega_1 t) \left( \cos(\Omega t) + \frac{\omega_1}{\Omega} \sin(\Omega t) \right) \right], \end{aligned} \quad (4.5)$$

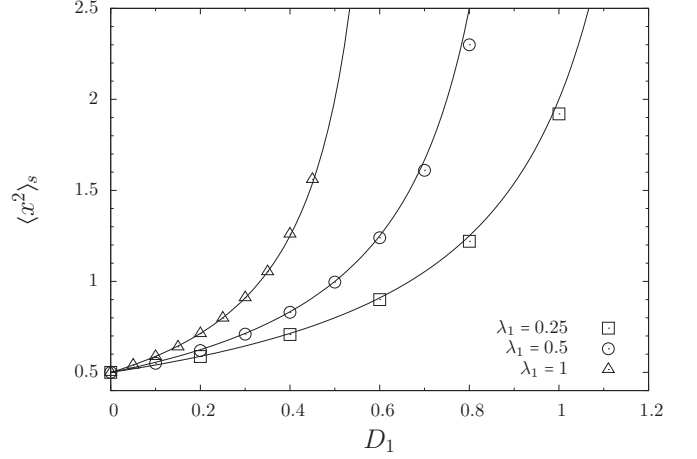


FIG. 2. Second moment of the particle position in the steady state as a function of the noise intensity  $D_1$  for different values of the parameter  $\lambda_1$ . The solid lines are obtained from (4.9), and the symbols indicate the simulation results. The parameters are  $D_2 = 0.25$ ,  $b = 0.5$ ,  $m = k = x_0 = 1$ , and  $r = 0$ ,  $v_0 = 0$ . According to (4.7),  $D_1^{(2)} = \frac{4}{3}$ , 1, and  $\frac{2}{3}$  for  $\lambda_1 = 0.25$ , 0.5, and 1, respectively.

where  $x_0 = x(0)$  and  $v_0 = v(0)$  are the deterministic initial conditions for the particle position and velocity, respectively,  $\Omega = \sqrt{\omega_0^2 - \omega_1^2}$ , and

$$\omega_1 = \omega_d(1 - 4\lambda_1 \omega_d D_1). \quad (4.6)$$

To find the second moments in the steady state, we substitute the function  $h = bv$  into the system of equations (3.12) and assume that the moments do not depend on time. Solving the corresponding system of algebraic equations for  $D_1 < D_1^{(2)}$ , where

$$D_1^{(2)} = \frac{2\lambda_1}{1 + 2\lambda_1} D_1^{(1)} \quad (4.7)$$

is the critical intensity of multiplicative noise for the second moments, we obtain  $\langle xv \rangle_s = 0$ ,

$$\langle v^2 \rangle_s = \frac{D_2}{b^2(1 + 2\lambda_1)(D_1^{(2)} - D_1)}, \quad (4.8)$$

and

$$\langle x^2 \rangle_s = \frac{1}{\omega_0^2} \langle v^2 \rangle_s + \langle x \rangle_s^2. \quad (4.9)$$

For  $\lambda_1 = \frac{1}{2}$  and  $r = 0$ , these expressions coincide with those given in [23] (see also [86]).

According to (4.8) and (4.9), the interpretation of the multiplicative noise strongly affects the second moments, mainly because of the dependence of  $D_1^{(2)}$  on  $\lambda_1$ . The cross correlation of noises is responsible for the second term  $\langle x \rangle_s^2$  in (4.9). (Recall that  $\langle x \rangle_s = 0$  if  $r = 0$ .) An important feature of these moments is that  $\langle x^2 \rangle_s \rightarrow \infty$  and  $\langle v^2 \rangle_s \rightarrow \infty$  as the noise intensity  $D_1$  tends to  $D_1^{(2)}$  from below. As Fig. 2 illustrates, the simulation data agree well with the theoretical dependence of  $\langle x^2 \rangle_s$  on  $D_1$ , calculated using (4.7)–(4.9) and (4.4). If  $D_1$  exceeds the critical value  $D_1^{(2)}$ , the oscillator approaches for  $t \rightarrow \infty$  a state where the moments  $\langle x^2 \rangle_s$  and  $\langle v^2 \rangle_s$  do not exist.

The higher moments can be calculated in the same way, but the calculations become more and more cumbersome as the order of the moments increases. Therefore, we present in Appendix A the results for the third and fourth moments only. The moments of arbitrary order  $n$ ,  $n \geq 1$ , exist if  $D_1 < D_1^{(n)}$ , where

$$D_1^{(n)} = \frac{2\lambda_1}{n-1+2\lambda_1} D_1^{(1)} \quad (4.10)$$

is the critical intensity of the multiplicative noise for the  $n$ th moments. If  $D_1$  tends to  $D_1^{(n)}$  from below, the  $n$ th moments diverge as  $(D_1^{(n)} - D_1)^{-1}$ , except the moment  $\langle x^{n-1} v \rangle_s$  which equals zero for all  $n \geq 1$ . Since  $D_1^{(n)} < D_1^{(l)}$  for  $n > l$ , the existence of the  $n$ th moments implies that the  $l$ th moments exist too. At the same time, for any noise intensity satisfying the condition  $D_1 < D_1^{(1)}$ , there always exists a critical order  $n_{cr}$ , such that all moments with  $n < n_{cr}$  do exist, while the moments with  $n \geq n_{cr}$  do not exist. Specifically, if  $D_1^{(n+1)} \leq D_1 < D_1^{(n)}$ , then  $n_{cr} = n + 1$ .

### B. Fluctuating spring coefficient

The harmonic oscillator with a fluctuating spring coefficient and additive white noise is described by (3.3) in which  $h = kx$ . According to (3.9), in this case the Fokker-Planck equation reads as

$$\begin{aligned} \frac{\partial}{\partial t} P + v \frac{\partial}{\partial x} P - \frac{1}{m} \frac{\partial}{\partial v} (bv + kx) P \\ - \frac{1}{m^2} (D_1 k^2 x^2 + D_2 - 2r D_{12} kx) \frac{\partial^2}{\partial v^2} P = 0. \end{aligned} \quad (4.11)$$

If  $r = 0$ , this equation reduces to that discussed in [107], and its solution satisfies the symmetry condition  $P(x, v, t) = P(-x, -v, t)$ . As in the previous case, the cross correlation of the white noises,  $r \neq 0$ , breaks this symmetry.

Since  $h = h(x)$ , the Fokker-Planck equation (4.11) does not depend on the parameter  $\lambda_1$ . Consequently, all statistical properties of the oscillator that follow from the solution of this equation are the same for any interpretation of the multiplicative white noise. In particular, according to (3.11) the mean particle position

$$\langle x \rangle = \exp(-\omega_d t) \left( x_0 \cos(\omega t) + \frac{1}{\omega} (v_0 + \omega_d x_0) \sin(\omega t) \right), \quad (4.12)$$

$\omega = \sqrt{\omega_0^2 - \omega_d^2}$ , does not depend on the statistical characteristics of the noises, and  $\langle x \rangle \rightarrow 0$  as  $t \rightarrow \infty$ . This conclusion is confirmed in Fig. 3, where we show the analytical and simulation results for the time dependence of  $\langle x \rangle$ , obtained from (4.12) and from the system of equations (3.3), respectively.

Substituting  $h = kx$  into (3.12), we easily find that in the steady state  $\langle xv \rangle_s = 0$ , and

$$\langle v^2 \rangle_s = \frac{D_2}{km(D_1^{(2)} - D_1)}, \quad \langle x^2 \rangle_s = \frac{1}{\omega_0^2} \langle v^2 \rangle_s, \quad (4.13)$$

where

$$D_1^{(2)} = \frac{b}{k} = 2 \frac{\omega_d}{\omega_0^2} \quad (4.14)$$

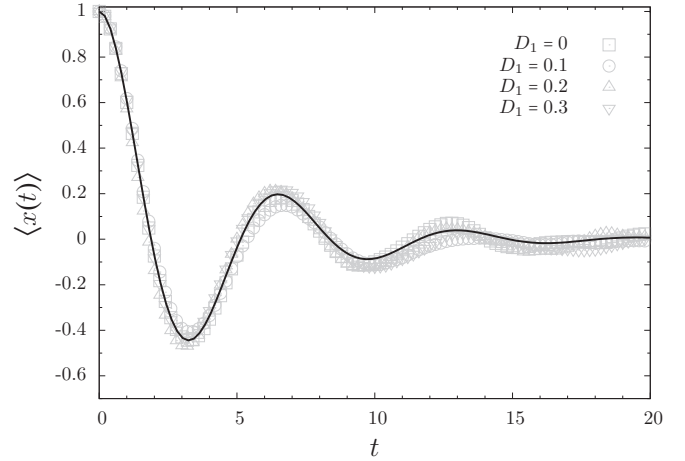


FIG. 3. Mean particle position as a function of time. The solid line represents the theoretical result (4.12), and the symbols show the results obtained from the numerical solution of (3.3) with  $h = kx$  and the parameters  $D_2 = 0.25$ ,  $b = 0.5$ ,  $m = k = x_0 = 1$ , and  $r = v_0 = 0$ .

is the critical noise intensity for the second moments. In contrast to the first moments, which exist for any noise intensity  $D_1$ , the second moments exist only if  $D_1 < D_1^{(2)}$ . If  $D_1$  approaches  $D_1^{(2)}$  from below, then  $\langle v^2 \rangle_s \rightarrow \infty$  and  $\langle x^2 \rangle_s \rightarrow \infty$ , and the second moments do not exist if  $D_1 > D_1^{(2)}$ . In Fig. 4, we show that the theoretical dependence of  $\langle x^2 \rangle_s$  on  $D_1$  agrees with numerical results. The expressions for the third and fourth moments and the corresponding critical values of the noise intensity  $D_1$  are listed in Appendix B.

### V. CONCLUSIONS

We have studied the effects of the interpretation of the stochastic calculus on the cross-correlated multiplicative Gaussian white noises governing a multivariable Langevin

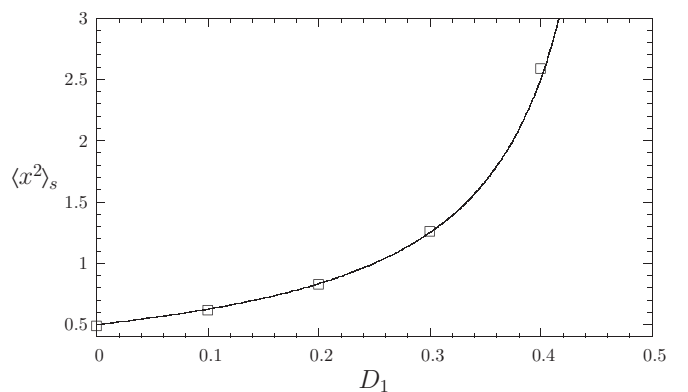


FIG. 4. Second moment of the particle position in the steady state as a function of the noise intensity  $D_1$ . The solid line represents the theoretical result (4.13), and the symbols indicate the simulation results obtained by numerical solution of (3.3) with  $h = kx$ . In accordance with our theoretical prediction, the latter do not depend on the parameter  $\lambda_1$ . The simulation parameters are chosen to be the same as in Fig. 3.

equation. We have allowed for the possibility that the physics of the problem dictates that each white noise must be interpreted in a different way. Our main results are the following:

(1) In Sec. II A, we have constructed a difference scheme to specify rigorously the meaning of Langevin equations with cross-correlated additive and multiplicative Gaussian white noises for any choice of the noise interpretation.

(2) In Sec. II B, we have derived the Fokker-Planck equation for the probability density of the system described by Langevin equations, using the two-stage procedure of averaging. Our result is completely general and valid for any choice of the noise interpretation.

(3) In Sec. II C, we have derived the condition when the Fokker-Planck equation does not depend on the interpretation of the multiplicative noises. Our result shows that the solutions of multivariable Langevin equations are not always affected by the interpretation of the multiplicative noises, in contrast to the one-variable case. This is a helpful result from a practical point of view since it implies that any satisfactory algorithm may be used to integrate numerically any set of Langevin equations that fulfills condition (2.14).

(4) In Sec. II D, we have shown that a set of Langevin equations, where each noise may be interpreted in a different way, has an equivalent set of Ito stochastic differential equations, in the sense that both sets have the same Fokker-Planck equation and statistical properties. This is a second helpful result for scientists and engineers who need to employ stochastic differential equations to model dynamical systems. The Ito stochastic integral has many desirable mathematical properties, e.g., the martingale property, not shared by the other possible choices, and consequently the Ito calculus is the most well developed and complete stochastic calculus, both from an analytical as well as a numerical view point [80,108]. To quote Gardiner [63, p. 92]: “The Ito integral is mathematically and technically the most satisfactory, but unfortunately, it is not always the most natural choice physically.” Our result shows that it is always possible to work with an equivalent multivariable Ito SDE, if the choice of calculus matters, i.e., condition (2.14) is not satisfied, and if the physics dictates an interpretation of the stochastic integrals that differs from the Ito one.

To demonstrate the value of our general results, we have studied Brownian particles moving in a one-dimensional potential under the effect of two cross-correlated Gaussian white noises, one of which is additive and one of which is multiplicative. We found that the statistics does not depend on the interpretation of the stochastic integral, if the multiplicative noise term depends only on the position of the Brownian particle and not its velocity.

For the case of a harmonic potential, we have determined the conditions for the stability of the moments and have in particular investigated the dependence of the stability conditions on the stochastic interpretation. In the case where the multiplicative noise describes the fluctuations of the damping coefficient, the statistical characteristics depend on the interpretation of the multiplicative noise. We have obtained the interesting result that the cross correlation between additive and multiplicative noises breaks the symmetry of the probability density with respect to the simultaneous change

of sign of the particle position and velocity. For the steady state of the oscillator, we have calculated the moments of the particle position and velocity up to the fourth order. We have also shown that for each intensity of the multiplicative noise there is a critical order such that all moments of higher order do not exist. A particular noteworthy result is the fact that cross correlation between the Gaussian white noises induces a shift in the stationary mean particle position away from the origin for all interpretations of the multiplicative noise, except for the Ito interpretation.

If the multiplicative noise describes the fluctuations of the spring coefficient, i.e., of the potential, then the statistical characteristics of the oscillator do not depend on the noise interpretation. As in the previous case, we have calculated the stationary moments up to the fourth order and have determined the critical intensities of the multiplicative noise (see Appendices A and B). As the intensity of the multiplicative noise approaches these critical intensities from below, the corresponding moments diverge. The main theoretical predictions for the harmonic oscillator with two cross-correlated Gaussian white noises have been confirmed by numerical simulations.

## ACKNOWLEDGMENTS

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## APPENDIX A: HIGHER MOMENTS: FLUCTUATING DAMPING COEFFICIENT

If  $D_1 < D_1^{(3)} = \lambda_1 D_1^{(1)} / (1 + \lambda_1)$ , then  $\langle x^2 v \rangle_s = 0$ , and the other third moments in the steady state can be obtained as follows:

$$\langle x v^2 \rangle_s = \frac{D_2}{b^2(1 + 2\lambda_1)(D_1^{(2)} - D_1)} \langle x \rangle_s - \frac{3\lambda_1 m^2 r D_{12}}{2b^3(1 + \lambda_1)(1 + 2\lambda_1)(D_1^{(2)} - D_1)(D_1^{(3)} - D_1)} \times \langle v^2 \rangle_s, \quad (\text{A1a})$$

$$\langle v^3 \rangle_s = -\frac{3\lambda_1 r D_{12}}{b(1 + \lambda_1)(D_1^{(3)} - D_1)} \langle v^2 \rangle_s, \quad (\text{A1b})$$

$$\langle x^3 \rangle_s = \frac{2m D_2}{b^2 k (D_1^{(2)} - D_1)} \langle x \rangle_s - \frac{2\lambda_1 b r D_{12}}{k m} \langle x^2 \rangle_s - \frac{3\lambda_1 m^3 r D_{12}}{b^3 k (1 + \lambda_1)(D_1^{(2)} - D_1)(D_1^{(3)} - D_1)} \langle v^2 \rangle_s. \quad (\text{A1c})$$

Here,  $\langle x \rangle_s$ ,  $\langle v^2 \rangle_s$ , and  $\langle x^2 \rangle_s$  are given by (4.4), (4.8), and (4.9), respectively.

The fourth moments can also be easily calculated, but the result is lengthy. Here, we note only that  $\langle x^3 v \rangle_s = 0$ , and the



other moments satisfy the system of equations

$$km\langle x^4 \rangle_s - 3m^2\langle x^2v^2 \rangle_s = -2\lambda_1 br D_{12}\langle x^3 \rangle_s, \quad (\text{A2a})$$

$$b^2(1 + 2\lambda_1)(D_1^{(2)} - D_1)\langle x^2v^2 \rangle_s - m^2\langle xv^3 \rangle_s = D_2\langle x^2 \rangle_s, \quad (\text{A2b})$$

$$m^2\langle v^4 \rangle_s - 6b^2(1 + \lambda_1)(D_1^{(3)} - D_1)\langle xv^3 \rangle_s - 3km\langle x^2v^2 \rangle_s = 6b(2 + \lambda_1)r D_{12}\langle xv^2 \rangle_s, \quad (\text{A2c})$$

$$b^2(3 + 2\lambda_1)(D_1^{(4)} - D_1)\langle v^4 \rangle_s + km\langle xv^3 \rangle_s = 3D_2\langle v^2 \rangle_s - 8\lambda_1 br D_{12}\langle v^3 \rangle_s, \quad (\text{A2d})$$

where  $D_1 < D_1^{(4)} = 2\lambda_1 D_1^{(1)}/(3 + 2\lambda_1)$ . All the critical noise intensities are determined by (4.10).

## APPENDIX B: HIGHER MOMENTS: FLUCTUATING SPRING COEFFICIENT

In this case, the third moments in the stationary state are given by

$$\langle x^2v \rangle_s = 0, \quad (\text{B1a})$$

$$\langle xv^2 \rangle_s = \frac{k}{2m}\langle x^3 \rangle_s, \quad (\text{B1b})$$

$$\langle v^3 \rangle_s = -\frac{k^2}{2bm}\langle x^3 \rangle_s, \quad (\text{B1c})$$

$$\langle x^3 \rangle_s = -\frac{2r D_{12}}{k(D_1^{(3)} - D_1)}\langle x^2 \rangle_s. \quad (\text{B1d})$$

These moments exist only if the condition  $D_1 < \min\{D_1^{(2)}, D_1^{(3)}\}$  holds, where  $D_1^{(2)}$  is given by (4.14). Here,

$$D_1^{(3)} = \frac{1}{4}(2 + \xi)D_1^{(2)} \quad (\text{B2})$$

is the critical noise intensity for the third moments, and

$$\xi = \frac{km}{b^2} = \left(\frac{\omega_0}{2\omega_d}\right)^2. \quad (\text{B3})$$

The fourth moments are expressed through the moments of the same or smaller order as follows:

$$\langle x^4 \rangle_s = \frac{b}{4k^3} D_2\langle x^2 \rangle_s \frac{6 + 11\xi + 4\xi^2 - (4k/b)(3 + 4\xi - 12r^2 - 10r^2\xi)D_1}{(3 + 2\xi)(D_1^{(4)} - D_1)(D_1^{(3)} - D_1)}, \quad (\text{B4a})$$

$$\langle xv^3 \rangle_s = \frac{m}{3b}\langle v^4 \rangle_s - \frac{k^2}{3bm}\langle x^4 \rangle_s, \quad (\text{B4b})$$

$$\langle x^2v^2 \rangle_s = \frac{k}{3m}\langle x^4 \rangle_s, \quad (\text{B4c})$$

$$\left(1 + \frac{1}{3}\xi\right)\langle v^4 \rangle_s = \frac{k^3}{3b^2m}\left(1 + \frac{3b}{m}D_1\right)\langle x^4 \rangle_s - \frac{3k^2}{bm^2}r D_{12}\langle x^3 \rangle_s + \frac{3k}{bm^2}D_2\langle x^2 \rangle_s, \quad (\text{B4d})$$

and  $\langle x^3v \rangle_s = 0$ . These moments exist if  $D_1 < D_1^{(4)}$ , where

$$D_1^{(4)} = \frac{3 + 4\xi}{3(3 + 2\xi)}D_1^{(2)} \quad (\text{B5})$$

is the critical noise intensity for the fourth moments. Since  $D_1^{(4)} < D_1^{(3)}$  and  $D_1^{(4)} < D_1^{(2)}$ , the existence of the fourth moments implies the existence of the third and second moments.

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