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Self-propelled particle in an external potential: Existence of an effective temperature

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We study a stationary state of a single self-propelled, athermal particle in linear and quadratic external potentials. The self-propulsion is modeled as a fluctuating internal driving force evolving according to the Ornstein-Uhlenbeck process, independently of the state of the particle. Without an external potential, in the long time limit, the self-propelled particle moving in a viscous medium performs diffusive motion, which allows one to identify an effective temperature. We show that in the presence of a linear external potential the stationary state distribution has an exponential form with the sedimentation length determined by the effective temperature of the free self-propelled particle. In the presence of a quadratic external potential the stationary state distribution has a Gaussian form. However, in general, this distribution is not determined by the effective temperature of the free self-propelled particle.

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I. INTRODUCTION

Recently, there has been a lot of interest in the static and dynamic properties of particles that are self-propelled and, thus, can move on their own accord [1–3]. These particles are said to move *actively* and to form *active matter*.

There are two motivations for the interest in active matter systems. First, these systems model static and dynamic properties of specific biological and physical systems in which self-propelled motion occurs. For example, a system of particles with the so-called run-and-tumble motion serves as a model for *Escherichia coli* bacteria [3]. Similarly, a system of so-called active Brownian particles [2] is a model system for Janus colloidal particles [4]. The second motivation for the interest in active matter systems is the fundamental fascination with nonequilibrium physical systems and, in particular, with systems without detailed balance.

The present contribution is inspired by recent studies that showed that, at least in some cases, active matter systems can exhibit phenomena that are commonly found in standard (thermal, nonactive) systems. For example, Palacci *et al.* [5] found that a dilute active colloidal suspension under gravity exhibits qualitatively the same exponential density distribution as a standard dilute thermal colloidal system. Notably, the parameter that replaces the thermal system's temperature coincides with the effective temperature that was inferred from an independent measurement of the long-time diffusive motion of an active colloidal particle. More interestingly, behavior similar to that common in thermal systems was found in systems consisting of interacting active particles. For example, Bialké *et al.* [6] used computer simulations to show that a system of active Brownian particles can crystallize at sufficiently high densities. Next, Das *et al.* [7] used both a computer simulation and an integral equation theory to show that activity promotes phase separation in an active binary mixture. Finally, it was found that active systems can exhibit glassy dynamics. Berthier and Kurchan [8] analyzed a simple model active system inspired by the so-called spherical p-spin model and showed that it can exhibit kinetic arrest. This pioneering study was followed by two computer simulation investigations of systems of active Brownian particles [9,10] which showed that, generically, active systems exhibit glassy dynamics, but the onset of glassy behavior is pushed towards higher densities compared with systems of nonactive particles. In turn, the latter simulations inspired a very recent mode-coupling-like description of glassy dynamics in active systems [11].

Results of some of the investigations mentioned above [5,7] suggest an emergence of effective thermal behavior and, more importantly, effective temperature [12]. It should be noted, however, that other studies [6,13] question the usefulness of the notion of effective temperature. In particular, Fily and Marchetti [13] argue that this notion holds only in the dilute limit.

Our goal is to test the validity of effective temperature in a simple model. To keep the model exactly solvable we replace a system of interacting particles by a single active particle in an external field. Specifically, we compare the behavior of a single particle without any external potential (for which an effective temperature can be easily defined) with the behavior of the same particle in two different external potentials.

There is a number of different models of self-propelled motion [2]. Their common feature is that an active particle moves under an influence of an internal self-propulsion which evolves in some specified way, independently of the state of the particle. Here we will consider the continuous time, one-dimensional version of the model introduced by Berthier [10,14]. In the original model of Ref. [10] Monte Carlo dynamics with correlated trial moves was used (in standard Monte Carlo dynamics subsequent trial moves are uncorrelated [15]). In our model, the particle is subjected to an internal self-propulsion force and, possibly, a conservative force originating from an external potential. The self-propulsion force has a vanishing average, a finite mean-square and a finite persistence (i.e., relaxation) time. We choose a rather simple evolution for the self-propulsion: we assume that the self-propulsion force evolves according to the Ornstein-Uhlenbeck stochastic process, independently of the state of the particle and, in particular, of any external force acting on it. Our choice of the self-propulsion force evolution leads to relatively simple equations of motion for the probability distribution of the active particle, with stationary state distributions that can be derived analytically.

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We will be mostly concerned with the stationary state distributions under the influence of an external potential. In particular, we will show that even though the self-propulsion force evolves on its own, nontrivial correlations between the position of the particle and the self-propulsion force can develop.

We should mention that motion of a single active particle has already been considered a number of times in the literature, e.g., in Refs. [16–19]. Some of these studies were concerned with a very interesting time dependence of the mean-square displacement [16] and a surprising form of a stationary state velocity distribution [17] for specific models of active motion of a single particle without any external force. Other studies focused on the time dependence of the motion under the influence of external potentials [18]. Finally, Enculescu and Stark [19] investigated the stationary state distribution in a constant external force, which is one of the cases considered in the present contribution. We will compare their results to ours in Sec. III.

We start by briefly discussing the motion of a free self-propelled particle. We note that by using the theoretical apparatus developed to analyze Brownian motion we can easily derive the long-time-scale description of the active motion and define the free particle effective temperature. Next, we analyze the self-propelled particle under the influence of a constant force. We show that the stationary state probability distribution has the usual exponential form and that it can be expressed in terms of the free particle's effective temperature. Finally, we analyze the self-propelled particle under the influence of a harmonic force. We show that in this case the stationary state probability distribution has the familiar Gaussian form. However, in general, it is not determined by the effective temperature obtained from the motion of the free self-propelled particle. We also examine an effective temperature defined through a fluctuation-dissipation relation. We end the paper with a brief discussion of the results, which should be applicable in a broader context.

II. FREE SELF-PROPELLED PARTICLE

The free active particle moves in a viscous medium under the influence of an internal self-propulsion. Although the motion of real swimming bacteria or self-propelled Janus particles is force free, we follow previous studies [18] and describe the self-propulsion as an effective internal driving force. We assume that viscous dissipation dominates and consequently the motion is overdamped. The medium (solvent) is characterized by the single-particle friction coefficient ξ_0. We assume that the active particle is big enough so that any random force originating from the solvent's fluctuations is negligible. Thus, the particle is non-Brownian and, since it moves in a viscous medium, its velocity is proportional to the force acting on it.

The self-propulsion force evolves according to the Ornstein-Uhlenbeck stochastic process. Specifically, the average value of the force relaxes to zero on the time scale characterized by the inverse rate γ^{-1} and instantaneous force changes by random, uncorrelated increments due to an internal noise. As a consequence, the self-propulsion force acquires a finite, nonzero mean square. We note that the statistics of the force is qualitatively similar to that in the one-dimensional version of the standard model of active Brownian particles (see, e.g., Ref. [18]). In the latter model the spatial motion of the particle is one-dimensional but the direction of the self-propulsion moves via rotational diffusion resulting in the zero average and a finite, nonzero mean square of the self-propulsion force along the direction of the spatial motion. The motivation for our specific model is that its equations of motion are linear which allows to find stationary state distributions analytically.

The time evolution of our system is described by the following equations of motion:

$$\partial_t x(t) = \xi_0^{-1} f(t), \tag{1}$$

$$\partial_t f(t) = -\gamma f(t) + \eta(t). \tag{2}$$

Equation (1) describes overdamped motion of the particle, and Eq. (2) describes the evolution of the self-propulsion force. In Eq. (2) $\eta(t)$ is a white Gaussian noise with the autocorrelation function given by

$$\langle \eta(t)\eta(t')\rangle_{\text{noise}} = 2D_f \delta(t-t'), \tag{3}$$

where $\langle \dots \rangle_{\text{noise}}$ denotes averaging of a Gaussian white noise η.

Equivalently, the motion of the self-propelled particle can be described by a joint probability distribution for the particle's position and the self-propulsion force, $P(x,f;t)$. The equation of motion for this distribution reads:

$$\partial_t P(x,f;t) = -\frac{f}{\xi_0}\frac{\partial P(x,f;t)}{\partial x} + \frac{\partial}{\partial f}\left[\gamma f P(x,f;t) + D_f \frac{\partial}{\partial f} P(x,f;t)\right]. \tag{4}$$

It can be easily showed that in the stationary state

$$P^{ss}(x,f) \propto \exp\left(-\frac{\gamma f^2}{2D_f}\right) \tag{5}$$

and $\langle f^2\rangle = D_f/\gamma$. Here and in the following $\langle \dots \rangle$ denotes averaging over the stationary distribution of the position and self-propulsion force.

We note that Eq. (4) is formally equivalent to the so-called Fokker-Planck equation that describes the motion of a Brownian particle on a time scale on which its velocity relaxation can be observed [20]. Indeed, replacing f/ξ_0 by the particle's velocity v changes Eq. (4) into the Fokker-Planck equation. Consequently, we can use the well-known theoretical analyzes of Brownian motion [20,21] for the present case of self-propelled motion. We see immediately that the long-time motion of the self-propelled particle is diffusive and the diffusion constant is equal to

$$D = \langle f^2\rangle/\left(\xi_0^2\gamma\right) = D_f/(\xi_0\gamma)^2. \tag{6}$$

Since the particle is moving in a viscous medium, using the standard Einstein relation between the temperature and friction, and diffusion constant allows us to define an effective temperature,

$$T_{eff} = D\xi_0 = D_f/(\xi_0\gamma^2) \tag{7}$$

(we use a system of units in which the Boltzmann constant k_B is equal to 1).

We should note that this long-time diffusive motion of the self-propelled particle, with diffusion constant given by Eq. (6), is established on the time scale much longer than γ^{-1}. We might expect that the long-time motion may become different [or at least that the diffusion constant becomes different from (6)] if there is another comparable or shorter time scale in the problem. We should note in this context that interesting self-propulsion-related phenomena are observed for slowly relaxing self-propulsion forces, i.e., precisely when γ^{-1} is *not* the shortest time scale in the problem.

III. SELF-PROPELLED PARTICLE UNDER THE INFLUENCE OF A CONSTANT FORCE: SEDIMENTATION

If there is an external, conservative, time-independent force acting on the particle, the equation of motion for the position of the self-propelled particle has the following form:

$$\partial_t x(t) = \xi_0^{-1}\{f(t) + F^{\text{ext}}[x(t)]\}, \tag{8}$$

where $F^{\text{ext}}(x) = -\partial_x V^{\text{ext}}(x)$ is the external, conservative, time-independent force acting on the particle. Equation (8) needs to be augmented by the equation of motion for the self-propulsion force [Eq. (2)]. We emphasize that the evolution of the self-propulsion force is unchanged.

As in the case of a free self-propelled particle, we can describe the time dependence of the state of the particle through the joint probability distribution of the position and the self-propulsion force, which satisfies the following evolution equation:

$$\partial_t P(x,f;t) = -\frac{1}{\xi_0}\frac{\partial}{\partial x}\{[f + F^{\text{ext}}(x)]P(x,f;t)\} + \frac{\partial}{\partial f}\left[\gamma f P(x,f;t) + D_f \frac{\partial}{\partial f}P(x,f;t)\right]. \tag{9}$$

We should emphasize that since the self-propulsion force evolves independently of the external force, equation of motion (9) is qualitatively *different* from the Fokker-Planck equation for the joint probability distribution of the position and velocity of a Brownian particle moving under the influence of an external force. Thus, we cannot use the theoretical apparatus developed in Refs. [20,21].

In the remainder of this section we briefly analyze the stationary state of a self-propelled particle under the influence of a constant external force, which models sedimentation in a dilute active colloidal suspension [5]. In the next section we investigate a self-propelled particle in a harmonic potential.

We note that the stationary state of a self-propelled particle under the influence of a constant external force was also considered by Enculescu and Stark [19]. They considered the standard model of active Brownian motion [2], which models the experimental system of Palacci *et al.* In this model the amplitude of the self-propulsion force is constant and its direction changes via rotational diffusion of the active particle. Enculescu and Stark showed that in this case the stationary state probability distribution can be found perturbatively. In contrast, we will show that for our model a closed form of the stationary state distribution can be derived.

For a single self-propelled particle under the influence of a constant gravitational force, $F^{\text{ext}}(x) = -mg$, the equation of motion has the following form:

$$\partial_t P(x,f;t) = -\frac{1}{\xi_0}\frac{\partial}{\partial x}[(f - mg)P(x,f;t)] + \frac{\partial}{\partial f}\left[\gamma f P(x,f;t) + D_f \frac{\partial}{\partial f}P(x,f;t)\right], \tag{10}$$

where g is the gravitational acceleration and m is the mass of the particle.

Note that Eq. (10) is only valid above a lower wall, which we assume to be located at $x = 0$. For a hard wall, this equation has to be accompanied by a boundary term that ensures that the current through the lower wall vanishes. For a wall modeled by a continuous potential (e.g., a repulsive power law potential), a term proportional to the gradient of the wall potential needs to be added to Eq. (10).

For a hard wall, the consequence of the boundary term is that at the wall, for each value of the self-propulsion force, the current through the wall vanishes. In contrast, as we show below, for the solution of homogeneous Eq. (10) only the current integrated over all self-propulsions vanishes. The complete solution of the sedimentation problem is a sum of the solution of the homogeneous Eq. (10) and a term due to the boundary condition. We expect that relative magnitude of the latter term will decrease with increasing distance from the wall and that far from the wall the stationary state probability distribution will approach the solution of the homogeneous Eq. (10). This has indeed been found by Enculescu and Stark [19] through a numerical analysis of the sedimentation problem for the standard model of active Brownian particles. For the remainder of this section we will use the term stationary state distribution for the time-independent solution of the homogeneous Eq. (10).

We note that the so-called drift coefficients [20] in Eq. (10) are linear in x and f. This fact suggests looking for a stationary distribution having a Gaussian form. It can be shown that the following distribution is a stationary solution of Eq. (10):

$$P^{ss}(x,f) \propto \exp(-ax - bf^2 - cf), \tag{11}$$

where $a = mg\xi_0\gamma^2/D_f$, $b = \gamma/(2D_f)$ and $c = -a/(\xi_0\gamma)$.

According to the stationary state distribution (11) there is a nonzero local stationary state self-propulsion:

$$\langle f\rangle_{lss} = \int df\, f\, P^{ss}(f|x) = -\frac{c}{2b} = mg, \tag{12}$$

where $\langle\ldots\rangle_{lss}$ denotes the local stationary state average or, more precisely, the stationary state average over self-propulsion under the condition that the particle is at position x. In other words, $P^{ss}(f|x)$ in Eq. (12) is the conditional stationary state distribution of the self-propulsion force,

$$P^{ss}(f|x) = P^{ss}(x,f)/P^{ss}(x), \tag{13}$$

where $P^{ss}(x)$ is the stationary state distribution of the particle's positions, $P^{ss}(x) = \int df\, P^{ss}(x,f)$.

Nonzero average self-propulsion follows from the condition that the current in the stationary state, integrated over all self-propulsions, should vanish. Let us define the current density integrated over all self-propulsions through the continuity equation for the probability distribution of positions,

$$\partial_t P(x;t) = -\partial_x j(x;t), \tag{14}$$

where $P(x;t) = \int df P(x,f;t)$. Thus the current density is given by

$$\xi_0^{-1}\left[\int df f P(x,f;t) - mgP(x;t)\right], \tag{15}$$

and therefore in the stationary state we need to have $\langle f \rangle_{lss} = mg$.

It follows from Eq. (11) that the stationary state distribution of positions is exponential,

$$P^{ss}(x) = \int df P^{ss}(x,f) \propto \exp(-x/\delta_{eff}), \tag{16}$$

where the so-called sedimentation length $\delta_{eff} = 1/a = D_f/(mg\xi_0\gamma^2)$. We note that the sedimentation length of a dilute system of nonactive Brownian particles at temperature T is given by $\delta = T/(mg)$. We can thus conclude that the sedimentation length of a dilute system of self-propelled particles has the same form as that of nonactive Brownian particles if instead of the equilibrium temperature one uses an effective temperature T_{eff} of a free self-propelled particle, $T_{eff} = D_f/(\xi_0\gamma^2)$. This agrees with the experimental result of Palacci *et al.* [5]

We note that the consistency between the free particle effective temperature and the sedimentation length is not obvious. In fact, Enculescu and Stark [19] showed that already for the standard model of active Brownian motion the free particle effective temperature is equal to the parameter determining the sedimentation length only in the lowest nontrivial order in the strength of the self-propulsion. They predicted that the difference between the free particle effective temperature and the parameter determining the sedimentation length should become apparent for strengths of the self-propulsion somewhat larger than those used in experiments of Palacci *et al.* In addition, Tailleur and Cates [22] showed that for the run-and-tumble model of active particles the stationary state distribution in a linear potential has the exponential form, but the free particle effective temperature does not determine the sedimentation length.

IV. SELF-PROPELLED PARTICLE IN A HARMONIC POTENTIAL

We show in this section that the effective temperature defined through the long-time diffusive motion of a free self-propelled particle does not always determine the stationary state probability distribution of the particle's position in an external harmonic potential. To analyze this finding a little further, we investigate the particle's position autocorrelation function and the linear response to an external perturbation, and use these analyzes to examine a fluctuation-dissipation relation-based effective temperature.

A. Stationary state probability distribution

For a single self-propelled particle in a harmonic potential, $V^{ext}(x) = \frac{1}{2}kx^2$, the equation of motion for the joint probability distribution of the position and self-propulsion force has the following form:

$$\partial_t P(x,f;t) = -\frac{1}{\xi_0}\frac{\partial}{\partial x}[(f-kx)P(x,f;t)] + \frac{\partial}{\partial f}\left[\gamma f P(x,f;t) + D_f\frac{\partial}{\partial f}P(x,f;t)\right], \tag{17}$$

where k is the force constant that determines the strength of the potential.

Again, we note that the so-called drift coefficients [20] in Eq. (17) are linear in x and f and therefore a stationary distribution has a Gaussian form,

$$P^{ss}(x,f) \propto \exp(-ax^2 - bf^2 - cfx), \tag{18}$$

where $a = k\xi_0(\gamma + k/\xi_0)^2/(2D_f)$, $b = (\gamma + k/\xi_0)/(2D_f)$ and $c = -k(\gamma + k/\xi_0)/D_f$.

It follows from Eq. (18) that the stationary distribution of the particle's positions is also Gaussian,

$$P^{ss}(x) = \int df P^{ss}(x,f) = \left[\frac{a - c^2/(4b)}{\pi}\right]^{1/2} \exp\{-[a - c^2/(4b)]x^2\}, \tag{19}$$

where $a - c^2/(4b) = (k/2)(\gamma + k/\xi_0)\gamma\xi_0/D_f$. If we were to define an effective temperature through the relation $P^{ss}(x) \propto \exp[-V^{ext}(x)/T_{eff}]$, we would get

$$T_{eff} = D_f/[\gamma\xi_0(\gamma + k/\xi_0)]. \tag{20}$$

We note that this effective temperature is different from that defined through the long-time diffusive motion of the free self-propelled particle, Eq. (7). We note, furthermore, that in the present problem there are two different time scales. First, there is the time scale on which the self-propulsion force forgets its initial value. As for the free particle, this time scale is proportional to γ^{-1}. Second, there is the characteristic time scale for the relaxation of a particle moving in a viscous medium under the influence of a harmonic potential. This time scale is proportional to ξ_0/k. If the former time scale is much shorter than the latter, $\gamma^{-1} \ll \xi_0/k$, the effective temperature (20) coincides with the effective temperature of the free self-propelled particle (7). In the opposite case, $\gamma^{-1} \gg \xi_0/k$, which is the interesting strong self-propulsion limit, the effective temperature (20) approaches $D_f/(k\gamma)$ and can be significantly lower than that of the free self-propelled particle (7).

In contrast to the case of a constant external force, the self-propulsion distribution of a particle moving under the influence of a harmonic force agrees with that of the free self-propelled particle,

$$P^{ss}(f) = \int dx P^{ss}(x,f) = \left[\frac{b - c^2/(4a)}{\pi}\right]^{1/2} \exp\{-[b - c^2/(4a)]f^2\}, \tag{21}$$

where $b - c^2/(4a) = \gamma/(2D_f)$.

However, there is still nonzero local stationary state self-propulsion,

$$\langle f\rangle_{lss} = \int df f P^{ss}(f|x) = -\frac{cx}{2b} = kx. \quad (22)$$

This result is not unexpected since the joint stationary state distribution (18) does not factorize into distributions of positions and self-propulsions. Physically, this happens because particles with larger (albeit temporary) self-propulsions are able to venture farther into the high potential energy regions.

Finally, we note that, as in the case of a constant external force, in the stationary state the current density, integrated over all self-propulsions, vanishes,

$$\xi_0^{-1}\left[\int df f P^{ss}(x,f) - kxP^{ss}(x)\right] = 0. \quad (23)$$

B. Particle's position autocorrelation function

We use standard methods [20] to derive coupled equations of motion for the time-dependent autocorrelation function of the position of the self-propelled particle, $\langle x(t)x(0)\rangle$, and the correlation function between the self-propulsion force at time t and the position at the initial time, $\langle f(t)x(0)\rangle$,

$$\partial_t\langle x(t)x(0)\rangle = \xi_0^{-1}\langle f(t)x(0)\rangle - k\xi_0^{-1}\langle x(t)x(0)\rangle, \quad (24)$$

$$\partial_t\langle f(t)x(0)\rangle = -\gamma\langle f(t)x(0)\rangle. \quad (25)$$

Usually, equations of motion for these two functions would involve other, more complicated time-dependent correlation functions. The equations above are closed due to the simplicity of the external potential. Initial conditions for Eqs. (24)–(25) are

$$\langle x(0)x(0)\rangle \equiv \langle x^2\rangle = D_f/[k\gamma\xi_0(\gamma + k/\xi_0)], \quad (26)$$

$$\langle f(0)x(0)\rangle \equiv \langle fx\rangle = k\langle x^2\rangle, \quad (27)$$

where second equalities in Eqs. (26)–(27) follow from the stationary state distribution (18).

Equations of motion (24)–(25) can be easily solved. The second equation, Eq. (25), is independent of the first. Its solution reads

$$\langle f(t)x(0)\rangle = e^{-\frac{k}{\xi_0}t}k\langle x^2\rangle. \quad (28)$$

Substituting Eq. (28) into Eq. (24) and integrating we get the following expression for the particle's position autocorrelation function:

$$\langle x(t)x(0)\rangle = \left(\frac{\gamma}{\gamma - k/\xi_0}e^{-\frac{k}{\xi_0}t} + \frac{k/\xi_0}{k/\xi_0-\gamma}e^{-\gamma t}\right)\langle x^2\rangle. \quad (29)$$

We note two qualitatively different behaviors in two limiting cases identified in the previous subsection. If the self-propulsion force relaxation time is the shortest relevant time scale, $\gamma^{-1} \ll \xi_0/k$, we get

$$\langle x(t)x(0)\rangle \approx e^{-\frac{k}{\xi_0}t}\frac{D_f}{k\gamma^2\xi_0}. \quad (30)$$

In this case the self-propelled particle's position autocorrelation function has the same form as the autocorrelation function of a nonactive Brownian particle in equilibrium in an external harmonic potential.

In the opposite limit, $\gamma^{-1} \gg \xi_0/k$, we get

$$\langle x(t)x(0)\rangle \approx e^{-\gamma t}\frac{D_f}{k^2\gamma}. \quad (31)$$

We note that in this limit the time dependence of the particle's position autocorrelation function is slaved to the evolution of the self-propulsion force. Interestingly, the autocorrelation function is independent of the friction coefficient ξ_0.

C. Linear response to an external force

To calculate a linear response function we consider the self-propelled particle in the harmonic potential and under an influence of a weak time-dependent force. In this case, the evolution equation for the joint probability distribution of the position and the self-propulsion force has the following form:

$$\begin{aligned}\partial_t P(x,f;t) = &-\frac{1}{\xi_0}\frac{\partial}{\partial x}[(f-kx)P(x,f;t)] \\ &+\frac{\partial}{\partial f}\left[\gamma f P(x,f;t) + D_f\frac{\partial}{\partial f}P(x,f;t)\right] \\ &-\frac{1}{\xi_0}\frac{\partial}{\partial x}[f^{\text{ext}}(t)P(x,f;t)]. \end{aligned} \quad (32)$$

Here $f^{\text{ext}}(t)$ is a weak, time-dependent force which, following the analysis of the linear response in equilibrium [23], we take to be position independent.

To examine the linear response we linearize Eq. (32) with respect to the external force. To this end we substitute into Eq. (32) the distribution of the form

$$P(x,f;t) = P^{ss}(x,f) + \delta P(x,f;t). \quad (33)$$

Here $\delta P(x,f;t)$ is the difference between the probability distribution in the presence of the force and its stationary state form. Next, we assume that $\delta P(x,f;t)$ is of the same order as the weak external force $f^{\text{ext}}(t)$ and we keep only terms of the lowest (linear) order in the weak external force. Using the fact that $P^{ss}(x,f)$ is the stationary state distribution without the force we get the following equation for $\delta P(x,f;t)$:

$$\begin{aligned}\partial_t \delta P(x,f;t) = &-\frac{1}{\xi_0}\frac{\partial}{\partial x}[(f-kx)\delta P(x,f;t)] \\ &+\frac{\partial}{\partial f}\left[\gamma f \delta P(x,f;t) + D_f\frac{\partial}{\partial f}\delta P(x,f;t)\right] \\ &-\frac{1}{\xi_0}\frac{\partial}{\partial x}[f^{\text{ext}}(t)P^{ss}(x,f)]. \end{aligned} \quad (34)$$

We assume that the force is turned on at $t = 0$ and thus the initial condition for $\delta P(x,f;t)$ is $\delta P(x,f;t=0) = 0$.

Our goal is to calculate the time-dependent change of the particle's position, $\delta\langle x(t)\rangle = \int dx df x \delta P(x,f;t)$. To this end we use Eq. (34) to derive coupled equations of motion for

$\delta\langle x(t)\rangle$ and $\delta\langle f(t)\rangle = \int dx\, df f \delta P(x,f;t)$,

$$\partial_t \delta\langle x(t)\rangle = \frac{1}{\xi_0}\delta\langle f(t)\rangle - \frac{k}{\xi_0}\delta\langle x(t)\rangle + \frac{1}{\xi_0} f^{\rm ext}(t), \quad (35)$$

$$\partial_t \delta\langle f(t)\rangle = -\gamma\delta\langle f(t)\rangle. \quad (36)$$

The initial conditions for these equations are $\delta\langle x(t=0)\rangle = 0 = \delta\langle f(t=0)\rangle$.

Equations (35) can be easily solved. We get $\delta\langle f(t)\rangle \equiv 0$ and

$$\delta\langle x(t)\rangle = \frac{1}{\xi_0}\int_0^t dt' e^{-\frac{k}{\xi_0}(t-t')} f^{\rm ext}(t'), \quad (37)$$

and thus the response function is given by

$$R(t) = \frac{1}{\xi_0} e^{-\frac{k}{\xi_0}t}. \quad (38)$$

D. Fluctuation-dissipation relation

The form of the joint stationary state distribution (18) suggests an effective temperature can be defined for both rapidly evolving self-propulsion force (in which case T_{eff} is the same as the one defined through diffusive motion of the free particle) and for the more interesting slowly evolving self-propulsion force (strong self-propulsion limit). Physically, in the former case the existence of an effective temperature is expected but in the latter case it seems to be related to the special form of the interaction potential. Here, to investigate this a little further, we examine a different way to introduce an effective temperature, one that uses a fluctuation-dissipation relation (FDR).

Following a recent review [24] we define a frequency-dependent fluctuation-dissipation relation-based effective temperature,

$$T_{eff}^{FDR}(\omega) = \frac{\omega {\rm Re} C(\omega)}{\chi''(\omega)}, \quad (39)$$

where ${\rm Re}C(\omega)$ is the real part of the one-sided Fourier transform of the particle's position autocorrelation function, ${\rm Re}C(\omega) = {\rm Re}\int_0^\infty e^{i\omega t}\langle x(t)x(0)\rangle$, and $\chi''(\omega)$ is the imaginary part of the one-sided Fourier transform of the response function, $\chi''(\omega) = {\rm Im}\int_0^\infty e^{i\omega t} R(t)$.

Using explicit forms of the autocorrelation function and the response function we get

$$T_{eff}^{FDR}(\omega) = \frac{D_f}{\xi_0(\omega^2+\gamma^2)}. \quad (40)$$

In principle, the fluctuation-dissipation relation-based effective temperature is frequency-dependent and, thus, the fluctuation-dissipation relation is violated. A more appropriate interpretation of Eq. (40) is that, in the limit of small-frequency perturbations, $\omega \ll \gamma$, the fluctuation-dissipation relation is recovered and $T_{eff}^{FDR}(\omega)$ coincides with the effective temperature obtained from the long-time diffusive motion of the free self-propelled particle. We note that a similar agreement of effective temperatures measured in different ways has been found by Loi *et al.* [12]. We shall emphasize, however, that in the strong self-propulsion limit, $\gamma^{-1} \gg \xi_0/k$, the free self-propelled particle-based effective temperature does not determine the stationary state distribution.

V. DISCUSSION

We have found that even for our simple model of active motion the most natural effective temperature defined through the long-time diffusive motion of a free self-propelled particle does not always determine the stationary state distribution in an external field, even in the dilute (single particle) limit. This finding nicely complements earlier work of Enculescu and Stark who showed that Palacci *et al.*'s result for the standard model of active Brownian particles, that the free particle effective temperature and a temperature-like parameter determining the sedimentation length are identical, is only valid for not too strong self-propulsion.

For the single active particle in a harmonic potential, we have calculated a frequency-dependent fluctuation-dissipation relation-based effective temperature. It has a well-defined low-frequency limit, which, for the simple model considered in the present contribution, coincides with the effective temperature of the free self-propelled particle. This explicit, exact calculation complements earlier studies of the fluctuation-dissipation relation-based effective temperature of active matter [12].

Although the fluctuation-dissipation relation-based effective temperature can be defined for active systems, it remains to be seen what properties of active systems it determines. In particular, it does not necessarily determine the stationary state of the self-propelled particle in an external potential. In this regard the answer to the question of the usefulness of the effective temperature is similar to that for thermal systems where, in spite of the large body of work [24], the situation is still far from well understood.

Finally, we shall emphasize a point that we expect to be quite general. Even though the self-propulsion force evolves independently of the state of the self-propelled particle (and independently of the interaction of this particle with an external force or with other particles), nontrivial correlations between self-propulsion force and the particle's position can develop. We also expect that in the case of many interacting self-propelled particles, correlations between self-propulsions and distances between particles can develop. These correlations imply the appearance of a nontrivial anisotropic pair distribution function which is thought to be responsible for the instability of a single phase uniform state in some systems of self-propelled particles [25]. It is possible that similar correlations are also present in jammed states of active matter [26].

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