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Anomalous diffusion induced by enhancement of memory

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We introduced simple microscopic non-Markovian walk models which describe the underlying mechanism of anomalous diffusions. In the models, we considered the competitions between randomness and memory effects of previous history by introducing the probability parameters. The memory effects were considered in two aspects: one is the perfect memory of whole history and the other is the latest memory enhanced with time. In the perfect memory model superdiffusion was induced with the relation of the Hurst exponent H to the controlling parameter p as $H = p$ for $p > 1/2$, while in the latest memory enhancement models, anomalous diffusions involving both superdiffusion and subdiffusion were induced with the relations $H = (1+\alpha)/2$ and $H = (1-\alpha)/2$ for $0 \leqslant \alpha \leqslant 1$, where α is the parameter controlling the degree of the latest memory enhancement. Also we found that, although the latest memory was only considered, the memory improved with time results in the long-range correlations between steps and the correlations increase as time goes on. Thus we suggest the memory enhancement as a key origin describing anomalous diffusions.

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I. INTRODUCTION

Random walks [1] have played a key role in statistical physics for over a century. They were proposed to stochastically formulate transport phenomena and macroscopic diffusion observables were calculated in long-time and short-distance limits of them [2]. It is well known that the key quantity characterizing the random walks or diffusion phenomena, the mean-squared displacement (MSD) $\langle x^2(t)\rangle$, grows linearly with time. However, Hurst found the persistence of hydrologic time series indicating that the MSD behaves in a nonlinear way [3–5] and, recently, such phenomena have been observed in many different systems such as chaotic [6], biophysical [7–11], economic systems [12,13], etc. The nonlinear behavior is recognized as anomalous diffusions compared with the linear behavior that is regarded as normal diffusion, and is characterized in terms of the MSD,

$$\langle x^2(t)\rangle \sim t^{2H}. \tag{1}$$

Here $\langle \cdots \rangle$ means average over independent realizations, i.e., ensemble average, in general, in nonequilibrium. H is called the anomalous diffusion or the Hurst exponent which classifies superdiffusion ($H > 1/2$), in which the past and future random variables are positively correlated and thus persistence is exhibited, and subdiffusion ($0 < H < 1/2$), which behaves in the opposite way, showing antipersistence.

The Hurst exponent, however, does not provide any information on the underlying physical mechanism of anomalous diffusion, and so a variety of models to describe the mechanism have been proposed [14–18] but they do not give any a universal mechanism but rather suggest very distinct origins, separately. The representative models among them are the fractional Brownian motion (fBM) [14], the Lévy flights [16,19,20,22], and the continuous-time random walks (CTRW) [15,21,22]. In the fBM, long-ranged temporal correlations between steps is given so that MSD scales like Eq. (1) within the range of $0 < H < 1$, and thus fBM describes both subdiffusion and superdiffusion; however, its correlation is mathematically constructed and it shows stationary behaviors unlike the nonstationary nature shown in real experiments and systems. Meanwhile, the other two models mimic further specific systems and describe only one region of anomalous diffusions, respectively. In Lévy flights, step-length distribution follows the power-law asymptotic behavior, so that the average distance per a step is infinite, which invokes superdiffusions. In the CTRW model a time interval between two consecutive steps is a continuous random variable which is drawn according to the waiting time distribution (WTD). For the WTD possessing the finite average of waiting time the MSD is linearly dependent on time, that is, the normal diffusive behavior is shown, while for the cases where the WTD behaves asymptotically as power laws and thus possesses an infinite average of waiting time, subdiffusive behaviors are induced. Also CTRW and Lévy walks have been generalized to reflect more physical realities by considering coupled space-time memory or various correlations between steps [23–28].

In recent years, a microscopic non-Markovian model with perfect memory of previous history was proposed, in which a walker jumps persistently or antipersistently according to prior steps with a probability parameter [29]. Below the critical value of the control parameter, the model shows normal diffusive behaviors while, above it, superdiffusive behaviors. Due to its simpleness, the microscopic memory effect, the key origin of anomalous diffusion, was easily applied to other models, among which Cressoni *et al.* suggested that the loss of recent memory rather than the distant past can induce persistence, which is related to the repetitive behaviors and psychological symptoms of Alzheimer's disease [30]. In [31], it was shown that, by adding a possibility that a walker does not move at all in the model of [29], diffusive, superdiffusive, and subdiffusive behaviors can exhibit in different parameter regimes. It has the advantage to describe the anomalous diffusion within a single model just by changing the parameters; however, in this case, the subdiffusive property may be caused by the staying behavior rather than the memory effect and thus superdiffusion and subdiffusion are not induced by a single origin.

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Thus, although anomalous diffusions have been described by various origins separately, more general origins which can describe the nonstationary mechanisms in both superdiffusions and subdiffusions are still questionable. To answer this, we focus on two features: microscopic memory effect varying with time and the competition between Markovian and non-Markovian processes which are realized by simple stochastic models. In the first model, non-Markovian processes induced by the full memory of the entire history and Markovian processes constructed by the original random walk compete by a probability parameter. In the second model, non-Markovian processes are induced by the latest memory rather than full memory and its realizations vary with time. From these models we find that, in the regime where non-Markovian nature prevails, superdiffusion is induced by the perfect memory, while the latest memory enhanced with time causes subdiffusions as well as superdiffusions.

II. MODEL WITH PERFECT MEMORY

First, we define a simple microscopic non-Markovian model in which a walker moves depending on full memory of its entire history with probability p and at random with probability $1-p$. The random walker starts at origin and randomly moves either one step to the right or left at time $t=1$, so the position of the walker becomes $x_1=\sigma_1$ with $\sigma_1 = 1$ or -1. Then the random variable σ_1 is preserved in the set $\{\sigma\}$ to memory the entire history of the walking process. At time t, the stochastic evolution equation becomes

$$x_{t+1} = x_t + \sigma_{t+1}, \tag{2}$$

with

$$\sigma_{t+1} = \begin{cases} \sigma_{t'}, & \text{with probability} \quad p, \\ +1 \text{ or } -1, & \text{with probability} \quad 1\text{-}p. \end{cases} \tag{3}$$

Here $t' \leqslant t$ and the random variable σ_{t+1} is chosen from the set $\{\sigma_t'\}$ with equal probability $1/t$. For the case of probability $1-p$, σ_{t+1} is chosen in 1 or -1 with equal probability 1/2 at random. It differentiates this model from that of [29] where $\sigma_{t+1} = -\sigma_{t'}$, which makes for competition between positive correlation of random variables and negative correlation rather than randomness in the process.

In order to compute the mean displacement $\langle x_t\rangle$, we first note that for a given previous history $\{\sigma_t\}$, the conditional probability that $\sigma_{t+1}=\sigma$ can be written as

$$P[\sigma_{t+1}=\sigma|\{\sigma_t\}] = \frac{1-p}{2} + p\frac{t_\sigma}{t} = \frac{1}{2t}\sum_{k=1}^{t}(p\sigma_k\sigma + 1), \tag{4}$$

where t_σ is the total number of steps having σ in the past. For $t \geqslant 1$ the conditional mean value of σ_{t+1} in a given realization is given by

$$\langle \sigma_{t+1}|\{\sigma_t\}\rangle = \sum_{\sigma=\pm1}\sigma P[\sigma_{t+1}=\sigma|\{\sigma_t\}] = \frac{p}{t}x_t, \tag{5}$$

where the displacement from the origin becomes $x_t = \sum_{k=1}^{t}\sigma_k$ if the walker starts at $x=0$. On averaging Eq. (5) over all realizations of the process, the conventional mean value of σ is given by

$$\langle\sigma_{t+1}\rangle = \frac{p}{t}\langle x_t\rangle \tag{6}$$

and by using the average of Eq. (2) the recursion relation is obtained as

$$\langle x_{t+1}\rangle = \left(1+\frac{p}{t}\right)\langle x_t\rangle. \tag{7}$$

The solution of Eq. (7) is given as

$$\langle x_t\rangle = \langle\sigma_1\rangle\frac{\Gamma(t+p)}{\Gamma(1+p)\Gamma(t)} \sim t^p \quad \text{for} \quad t \gg 1. \tag{8}$$

When $\langle\sigma_1\rangle \neq 0$ the mean displacement increases monotonically following the power law. It is the same behavior as that of [29] although the exponent is different, which indicates that the persistence due to the full memory makes the walker move away from the origin with time on average.

The recursion relation of the MSD also can be computed from Eq. (4) and Eq. (2) as follows:

$$\langle x_{t+1}^2\rangle = 1 + \left(1+\frac{2p}{t}\right)\langle x_t^2\rangle, \tag{9}$$

and its solution [29] is asymptotically obtained as

$$\langle x_t^2\rangle = \frac{t}{1-2p} \quad \text{for} \quad p < 1/2, \tag{10}$$

$$\langle x_t^2\rangle = t\ln t \quad \text{for} \quad p = 1/2, \tag{11}$$

$$\langle x_t^2\rangle = \frac{t^{2p}}{(2p-1)\Gamma(2p)} \quad \text{for} \quad p > 1/2. \tag{12}$$

For $p < 1/2$, the MSD depends linearly on time and the mean displacement follows the power law $\langle x_t\rangle \sim t^p$ with the exponent smaller than 1/2, so that the variance $\Delta(t) = \langle x_t^2\rangle - \langle x_t\rangle^2$ remains normally diffusive for large t. Specially when at $t=1$ a walker moves to the right or left with equal probability 1/2 so that $\langle\sigma_1\rangle = 0$, the variance increases asymptotically linearly with time having the diffusion coefficient $D = 1/2(1-2p)$. While for $p > 1/2$ the MSD follows the power law $\langle x_t^2\rangle \sim t^{2p}$, which is of the same order as the square of the mean, but with a different prefactor. Hence it results in the superdiffusion with the relation between the Hurst exponent and the parameter $H=p$. The marginal superdiffusive phenomena is shown for $p=1/2$. These results have been confirmed by computer simulations as shown in Fig. 1. The critical parameter $p_c = 1/2$ means that the superdiffusive phenomena occur when the persistence induced by full memory prevails in the process against the randomness. It can be compared to the results of [29] in which the critical value of the parameter ($p_c = 3/4$) is larger than that of this model, which indicates that the antipersistent rule invokes more randomness in the process than just random choices used in this model. The steps made by anticorrelation with previous steps do not continuously retain an antipersistent nature but rather bring about random nature changing the directions of steps. Therefore, it is difficult to embody subdiffusion phenomena from the perfect memory effect and thus we need to consider an another approach to describe anomalous diffusions comprising subdiffusions.

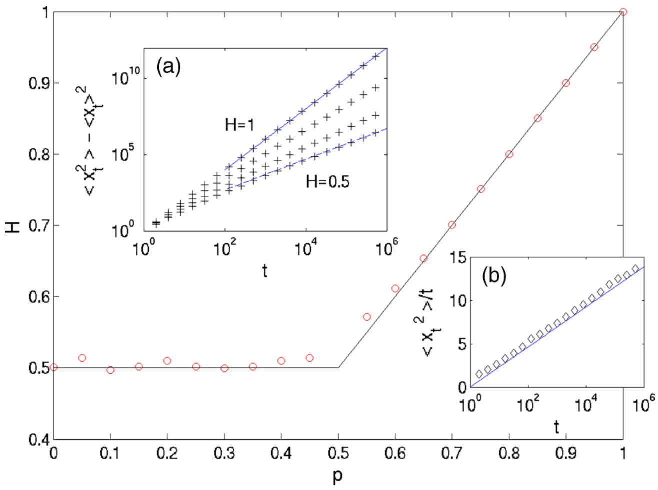

FIG. 1. (Color online) Inset (a) shows the plots of the variance $\langle x_t^2\rangle - \langle x_t\rangle^2$ for $p = 0.4, 0.6, 0.8$, and 1 from the bottom to the top. For $p = 1$, the data are in excellent agreement with the solid line indicating $H = 1$, while for $p = 0.4$, the data are in a good agreement with the dashed line $H = 0.5$. The data were measured with the initial condition where a walker moves to the right or left with equal probability 1/2 and 10^4 independent realizations. The main plot shows the Hurst exponent H versus the parameter p. The solid line is $H = 0.5$ for $p < 0.5$ and $H = p$ for $p > 0.5$. It confirms the analytic results of Eqs. (10) and (12), which show that the persistence vanished in the regimes $p < 0.5$ and there are the persistence with the relation $H = p$ for $p > 0.5$. The case of $p = 0.5$ is shown in the inset (b) which shows the marginal behavior, $\langle x_t^2\rangle/t$, increases logarithmically.

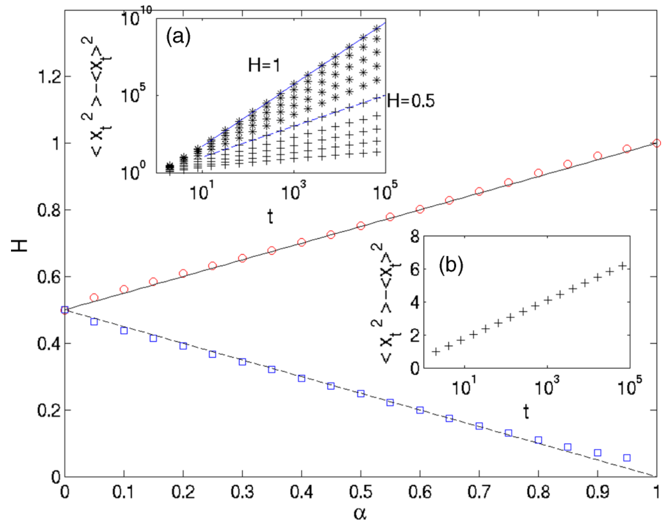

FIG. 2. (Color online) Inset (a) shows the plots of $\langle x_t^2\rangle - \langle x_t\rangle^2$ with $\alpha = 0.2, 0.4, 0.6, 0.8$, and 1 for the pLMEM (star symbols) and with $\alpha = 0, 0.2, 0.4, 0.6$, and 0.8 for the nLMEM (plus symbols). For the models, the mean displacement is nearly zero numerically so that the variance is equal to the MSD. The solid and dashed lines represent the case of $H = 1$ and $H = 0.5$, respectively. The plot shows the Hurst exponent H as a function of the parameter p measured with 5×10^5 independent realizations. The circle symbols and the square symbols represent the data for the pLMEM and the nLMEM, respectively. The solid line is $H = (1+\alpha)/2$ and the dashed line is $H = (1-\alpha)/2$. It is shown that the data are in a good agreement with the lines, which indicates the LMEMs can well describe the all anomalous diffusions including superdiffusion and subdiffusions. The case of $\alpha = 1$ for the nLEME is shown in the inset (b) which shows the marginal behavior, $\langle x_t^2\rangle - \langle x_t\rangle^2$, increases logarithmically.

III. MODELS WITH MEMORY ENHANCEMENT

We suggest the following non-Markovian stochastic model where, for $t > 1$, σ_{t+1} is given by

$$\sigma_{t+1} = \begin{cases} \sigma_t, & \text{with probability} \quad 1 - 1/\mathrm{t}^{\alpha}, \\ 1 \text{ or } -1, & \text{with probability} \quad 1/\mathrm{t}^{\alpha} \end{cases} \tag{13}$$

and the walker starts at origin and moves to the right or left with equal probability at time $t = 1$. Over time, the probability of taking the same direction with the latest step increases and the larger value of parameter α is, the much faster the probability grows with time. That is, in this model only the latest step is remembered unlike the above perfect memory model, and the persistence with the previous step is enhanced with time, the degree of which is controlled by the parameter α. When $\alpha = 0$ it reduced to the original random walk. We shall refer to this model as the positive latest memory enhancement model (pLMEM). Meanwhile, in Eq. (13), if the rule $\sigma_{t+1} = -\sigma_t$ is taken, the correlation between two successive steps is negative and thus let's call this the negative latest memory enhancement model (nLMEM).

The computer simulations were run for these two LMEMs. Figure 2 shows the Hurst exponent H versus the parameter α for the pLMEM (circles) and for the nLMEM (squares). The solid line represents that the Hurst exponent H relates to the parameter α as $H = (1+\alpha)/2$ for $0 \leqslant \alpha \leqslant 1$ for the pLMEM. For the case of $\alpha > 1$, the probability $p(t) = 1 - 1/t^{\alpha}$ approaches one faster than the case of $\alpha = 1$ as time becomes large, so that it also shows ballistic motions resulting in $H = 1$. While the dashed line represents that $H = (1-\alpha)/2$ for the nLMEM. For the case $\alpha = 1$ of the nLMEM it shows the marginal behavior showing $\langle x_t^2\rangle - \langle x_t\rangle^2 \sim \ln t$ [the inset (b) in Fig. 2]. Thus the LMEMs well brought about both superdiffusions and subdiffusions with a single origin, although considering only for the latest memory. It is comparable that a walk process just depending on short-term memory at each time is reduced into the original random process. Therefore, it can be regarded as a different nonstationary microscopic mechanism describing anomalous diffusions.

IV. TIME-VARYING CORRELATIONS

In order to study in detail how the memory enhancement affects to the correlations between steps we consider the correlation function $C(t,\Delta)$ defined as

$$C(t,\Delta) = \langle \sigma_t \sigma_{t+\Delta}\rangle - \langle \sigma_t\rangle\langle\sigma_{t+\Delta}\rangle, \tag{14}$$

where, when $\Delta = 1$, $C(t,1) \sim 1 - t^{-\alpha}$ may be given by Eq. (13) and $\langle \cdots \rangle$ is the average for independent realizations. For convenience, we considered a function $g(t,\Delta) \equiv 1 - C(t,\Delta)$ and measured it for different values of Δ. Figure 3(a) shows the function $g(t,\Delta)$ versus time t for various values of

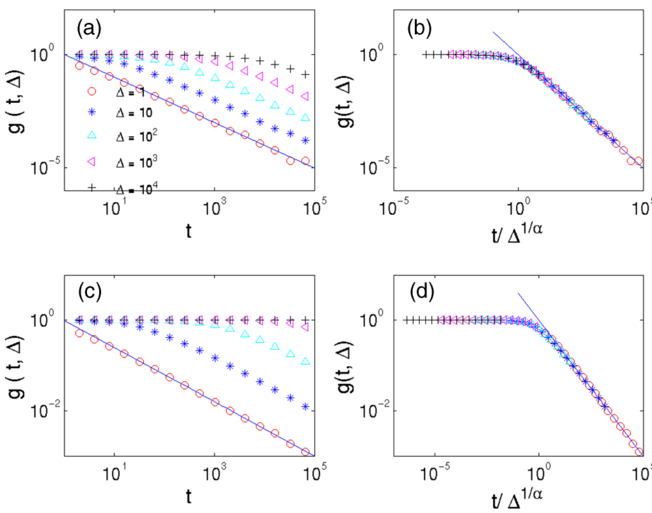

FIG. 3. (Color online) (a) Function $g(t,\Delta)$ is shown as a function of time t for various intervals $\Delta = 1$, 10, 10^2, 10^3, and 10^4 for $\alpha = 1$. The solid line represents $g(t,1) \sim t^{-\alpha}$. (b) The functions $g(t,\Delta)$ fall into a single curve by scaling time as $t/\Delta^{1/\alpha}$. The solid line represents $g(t,\Delta) \sim \Delta/t^{\alpha}$. (c) The function $g(t,\Delta)$ for $\alpha = 0.6$. (d) The data are well collapsed with the critical time $t_c = t/\Delta^{1/\alpha}$ for $\alpha = 0.6$.

Δ for $\alpha = 1$. The solid line represents that $g(t,1) \sim t^{-\alpha}$ for $t > 1$ with $\alpha = 1$ as expected. Meanwhile, it shows that, for $\Delta > 1$, the function g becomes 1 for $t \ll t_c$ and $g(t) \sim t^{-\alpha}$ for $t \gg t_c$. The data collapse into a single curve very well with $t_c = \Delta^{1/\alpha}$ as shown in Fig. 3(b). Figures 3(c) and 3(d) show the same results for $\alpha = 0.6$. Thus the correlation function $C(t,\Delta)$ scales as

$$C(t,\Delta) \sim \begin{cases} 0, & \text{for} \quad t \ll \Delta^{1/\alpha}, \\ 1 - \Delta/t^{\alpha}, & \text{for} \quad t \gg \Delta^{1/\alpha}. \end{cases} \tag{15}$$

At the critical time after which the correlations appear, the persistent probability to follow the last step is $p(t_c) = 1 - 1/\Delta$. Although the present step just depends on only one preceding step, it generates the correlations between steps far away from each other when the persistent probability is larger than the critical probability $p(t_c)$. That is, the shortest-term memory increasing with time can induce the long-range correlations in enough long-time limits. Also it has to be addressed that unlike the stationary series of the fBM in which the correlation does not change with time and only depends on a time interval such as $C(\Delta) \sim \Delta^{-2(1-H)}$, this process is nonstationary and the correlations depend on the time interval as well as time t. In the fBM the correlations decrease as the interval increases depending on the Hurst exponent H, while the correlation in the pLMEM decreases linearly with the interval regardless of H and increases over time. The larger the value of α, the much faster the correlation increase and so the more superdiffusive behaviors appear.

These properties of the correlation function are distinguished from those of the perfect memory model in which the correlation function decreases when time goes on. As shown in Fig. 4(a), for $p = 0.2$ the correlation function $C(t,\Delta)$ does not depend on time t as well as interval Δ and their averages for time become zero, which represents the normal diffusion. For $p = 0.8$ [Fig. 4(b)] the steps are positively correlated

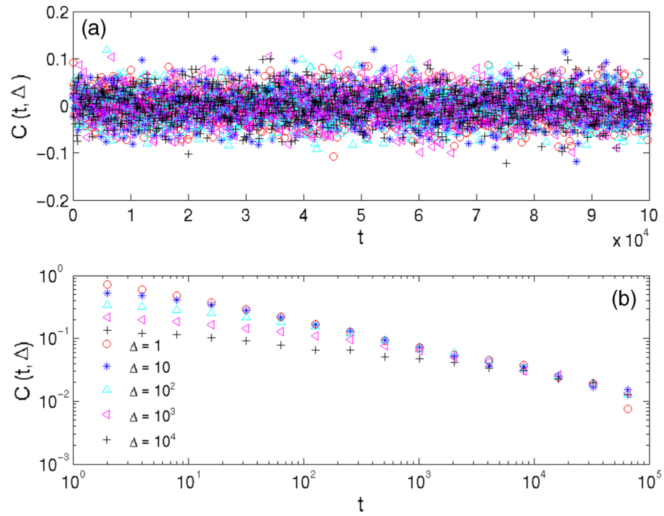

FIG. 4. (Color online) (a) Correlation function $C(t,\Delta)$ as a function of time t for various intervals $\Delta = 1$, 10, 10^2, 10^3, and 10^4 for $p = 0.2$ of the perfect memory model. In this case, the normal diffusion is shown and thus there are no correlations between steps. (b) The correlation function $C(t,\Delta)$ for $p = 0.8$ of the perfect memory model. The correlations between steps are positive for all measured times and intervals as expected in superdiffusions. However, they show the dependence on the time as well as the interval unlike the stationary process like the fBM.

as expected to make the superdiffusions. The correlation functions decrease when the time interval Δ becomes large at the same time and the difference lessens with time. That is, the process is also a nonstationary process and the correlation function depends on the time; however, it decreases with time unlike the pLMEM. Thus the perfect memory of whole history

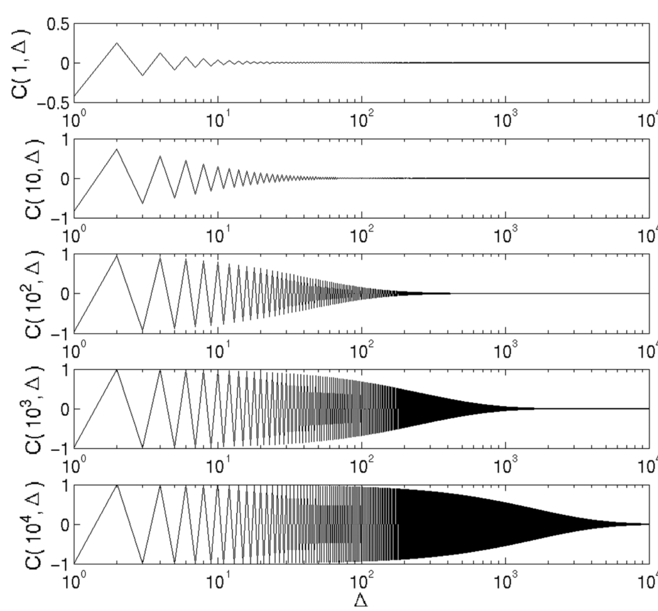

FIG. 5. Semilog plot of the correlation functions $C(t_{\times},\Delta)$ as a function of interval Δ at different times $t = 1$, 10, 10^2, 10^3, and 10^4 for $\alpha = 0.8$ of the nLMEM.

and the latest memory increasing with time are two different origins resulting in superdiffusive behaviors.

For the nLMEM, the time dependency of the absolute correlation function is the same as the pLEME, because the probability following the latest step is same for two LMEMs irrespective of the given sign, positive or negative correlation. For the nLMEM the correlation function as a function of interval Δ may be negative or oscillatory due to the subdiffusive nature. Considering nonstationary behaviors of the process, we measured the correlation function at fixed time $t_\times$ given by $C(t_\times,\Delta) = \langle \sigma_{t_\times}\sigma_{t_\times+\Delta}\rangle - \langle\sigma_{t_\times}\rangle\langle\sigma_{t_\times+\Delta}\rangle$. Figure 5 shows the correlation function as a function of interval Δ at different times. $C(t_\times,\Delta)$ oscillates totally in the nonzero regimes, which is distinguished from the subdiffusions of the fBM with negative correlation such as $C(\Delta) \sim -\Delta^{-2(1-H)}$. Also, when $t_\times$ becomes large, the oscillatory range is more longer. That is, like Eq. (15), $C(t_\times,\Delta)$ becomes

$$|C(t_\times,\Delta)| \sim \begin{cases} 1 - \Delta/t_\times^\alpha, & \text{for} \quad \Delta \ll t_\times^\alpha, \\ 0, & \text{for} \quad \Delta \gg t_\times^\alpha. \end{cases} \tag{16}$$

It indicates that the longer time, the more anticorrelation is persistent.

V. CONCLUSION

In conclusion, the microscopic nonstationary mechanisms of anomalous diffusions have been studied through the simple models with memory effects of previous walk processes. In the models, Markovian and non-Markovian processes were controlled by the probability parameter, and anomalous diffusions were induced with the Hurst exponent related to the parameter. The perfect memory of whole history invokes superdiffusive behaviors with $H = p$ for $p > 0.5$ in which the regime of the non-Markovian nature prevails, while subdiffusions are not invoked. The anomalous diffusive behaviors involving both superdiffusions and subdiffusions could be described in the mechanism where the latest memory increases with time. The persistent behaviors with the latest memory enhancement induced the superdiffusions with $H = (1+\alpha)/2$, while taking the opposite direction to the latest step brought about the subdiffusive behaviors with $H = (1-\alpha)/2$. The perfect memory resulted in the long-range step correlation decreasing with time, while even though the memory is restricted to the latest step, the memory enhancement resulted in the long-range correlations increasing with time above the critical time which increases with the interval. Thus the enhancement of memory may be a key origin describing all anomalous diffusions and we expect that these time-varying features will be measured in various real systems showing anomalous diffusions and these simple models can serve as basic models in studying other various aspects of anomalous diffusive phenomena.

Meanwhile, we need to consider the ergodicity breaking in the processes. It is known that nonstationary stochastic processes are generally not ergodic, that is, the means as ensemble averages are different from those as time averages [32–36]. These processes are nonstationary due to the memory effect and all the analysis in this study was made on the basis of nonequilibrium ensemble averages, so that it does not mean that when time average is run it gives the same Hurst exponent as the above provided. We are going to deal with the ergodicity breaking in these models elsewhere.

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