Synchronization in a mechanical resonator array coupled quadratically to a common electromagnetic field mode

G. León Aveleyra, C. A. Holmes, and G. J. Milburn
School of Mathematical and Physical Sciences, The University of Queensland, St. Lucia, Queensland 4072, Australia
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Optomechanical systems are based on the nonlinear coupling between the electromagnetic (EM) field in a resonator and one or more bulk mechanical resonators such that the frequency of the EM field resonator depends on the displacement coordinates of each of the mechanical resonators. In this paper we consider the case of multiple mechanical resonators interacting with a common field for which the frequency of the EM resonance is tuned to depend quadratically (to lowest order) on the displacement of the resonators. By using the method of amplitude equations around a critical point, it is shown that groups of near-identical bulk mechanical resonators with low driving fail to synchronize unless their natural frequencies are identical, in which case the resulting system can exhibit multistability.

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I. INTRODUCTION

The theory of synchronization in all-to-all coupled limit cycle oscillators [1–3] has established itself as a fundamental mechanism for emergent collective behavior in complex systems. However, the coupled oscillators that arise from arrays of bulk mechanical resonators interacting via a common electromagnetic field mode are a special case that, as we shall show, do not fit this standard model. Nevertheless, understanding whether synchronization arises in high frequency mechanical resonators coupled strongly to one or more modes of the electromagnetic field in a resonant cavity is important experimentally [4,5].

Recent experimental progress in this area has been driven by a desire to explore the deep quantum domain in which coupled mechanical resonators are prepared at or near their vibrational ground state [6,7]. The first step to understand these systems is to understand their classical dynamics, which can be complex, as the interactions are nonlinear. The common feature in these systems is the so-called radiation pressure coupling, whereby the displacement of each mechanical resonator independently changes the resonance frequency of a common electromagnetic resonator, the *cavity field*, by an amount dependent on the displacement of each mechanical resonator. Typically, this dependence is predominantly linear [8], but in optomechanical scenarios such as that of a "membrane in the middle" [9–11], the optical cavity may be tuned so that this dependence is quadratic in the resonator's displacement. This is the case considered in this paper.

One of the advantages of tuning the optical cavity to be quadratic to lowest order in the membrane's displacement is that the system then experiences extremely low optical loss [12]. In both the linear and quadratic cases the effective conservative force acting on each mechanical resonator is proportional to the circulating power in the electromagnetic cavity. If the cavity is externally driven, this interaction mediates an indirect all-to-all coupling between each of the mechanical resonators, and typically, resonators with similar frequencies synchronize [8]. However, we find that if the optical cavity is tuned to be quadratic to lowest order in the membrane's displacement, even the slightest discrepancy in their natural frequencies will lead to a loss of synchronization and, in fact, amplitude death in all but one of the resonators.

Most of the work to date on all-to-all coupled oscillators has concentrated on sets of nonlinear oscillators which are coupled linearly [1,2]. In this paper, a system of resonators that are coupled together via a common cavity field is considered, so that although they experience all-to-all coupling, it is first necessary to derive amplitude equations to make that coupling explicit. Once they are derived, it is obvious that the resulting coupled oscillator systems are weakly damped linear systems which experience nonlinear coupling. Therefore the usual intuitions do not apply. Since all the nonlinearity is in the coupling, one would expect that increasing the strength of the coupling would destabilize the rest solution of the resonators, which it typically does [8], but when the resonators are placed at the antinodes of the intracavity field, it is found that the stability of the rest solution of the resonators does not change with the level of external driving (for moderate levels of external driving) and that the driving can result in multistable behavior.

The derivation and analysis of the amplitude equations of the nonidentical and identical resonators form a major part of this paper. However, a small section is also included on why the adiabatic approximation, here and more generally, fails to see the synchronized oscillatory motion. In the second section the equations of motion for the n resonators and the cavity [effectively a $(2n+2)$-dimensional system] are derived, and it is shown that in the identical resonator case these can be recast in terms of a set of five collective variables. The third section is devoted to the adiabatic case, and in the fourth section the coupled amplitude equations are derived and analyzed. The experimental physical realization of the dynamics we describe is given in the appendix, but there are other realizations. The same equations of motion govern the dynamics of n nanomechanical resonators capacitively coupled to a superconducting microwave cavity [8].

II. THE MODEL

The dynamics of a single electromagnetic (EM) resonator is described as a highly underdamped single harmonic oscillator with a frequency equal to the cavity resonance frequency. The canonical degrees of freedom in this case correspond to the in-phase x and quadrature phase y amplitudes of the EM field in the cavity. However, we will usually describe the dynamics in terms of a complex amplitude $\alpha = x + iy$. We use units such that the EM energy stored in the cavity is $\mathcal{E}_o = \hbar\Omega_c|\alpha|^2$, where Ω_c is the bare (i.e., without mechanical coupling) cavity resonance frequency. The energy damping rate for the cavity field is given in terms of its linewidth κ. We will use this parameter to scale all physical frequencies in the model so that we can fix $\kappa = 1$, and thus in dimensionless units the cavity resonance frequency is $\omega_c = \Omega_c/\kappa$.

The mechanical degrees of freedom are each described by a simple (underdamped) harmonic oscillator with one degree of freedom corresponding to the displacement of single bulk flexural mode of an elastic mechanical resonator. This can be arranged by suitable fabrication of a bulk mechanical resonator of which we give an example in Appendix A. We designate the canonical coordinates for the mechanical resonators as the dimensionless variables q_i, p_i, where $i = 1,2,\ldots,N$, corresponding to a mechanical resonant frequency of $\Omega_{m,i}$. The corresponding dimensioned displacement and momentum are given as $Q_i = \bar{x}x_i, P_i = \bar{p}p_i$, with $\bar{x} = \bar{p}/m\kappa$, with the momentum scale fixed in terms of the (elastic deformation) energy scale E_0 as $\bar{p} = \sqrt{mE_0}$. The Hamiltonian for the mechanical degrees of freedom is then given by

$$H = E_0 \sum_{i=1}^{N} \left(\frac{p_i^2}{2} + \omega_{m,i}^2 \frac{x_i^2}{2} \right), \tag{1}$$

where

$$\omega_{m,i} = \frac{\Omega_{m,i}}{\kappa} \tag{2}$$

is a dimensionless frequency. We also need to specify the rate Γ_i at which mechanical energy is dissipated in each resonator. This will be given by the dimensionless parameter $\gamma_i = \Gamma_i/\kappa$.

In a wide variety of physically relevant models, the coupling between the EM field and the mechanical resonators is specified by giving the resonance frequency as a function of the mechanical displacements,

$$\omega_c = \omega_c(x_i), \tag{3}$$

which effectively makes the detuning parameter $\Delta = \Delta(x_i)$ a function of the mechanical displacements.

In Appendix A we describe the physical model in which it is possible to engineer the EM resonator frequency to be a symmetric function of the mechanical displacements so that to lowest order $\Delta(q_j) = \delta + \frac{1}{2}\frac{d^2\Delta(0)}{dx_j{}^2}x_j^2$. The equations of motion, which are derived in Appendix A, are

$$\frac{d\alpha}{dt} = -\left(1 + i\delta + i\sum_{j=1}^{n} g_j x_j^2\right)\alpha - i\epsilon, \tag{4}$$

$$\frac{d^2x_j}{dt^2} = -\omega_{m,j}^2 x_j - 2\gamma_j\frac{dx_j}{dt} - g_j\omega_{m,j}|\alpha|^2 x_j, \tag{5}$$

where the units of time are chosen such that the EM resonator energy decay rate is unity and the bare detuning between the EM resonance and the carrier frequency of the driving is $\delta = \Delta(0)/\kappa$, the coupling constant is given by

$$g_j = \frac{1}{2\kappa}\frac{d^2\Delta(0)}{dx_j{}^2}, \tag{6}$$

and the amplitude of the driving field is described by the dimensionless parameter ϵ.

We have used the decay rate of the electromagnetic energy to define a natural time scale in the dynamics. Typically, the decay of the mechanical energy is much slower than this, so that γ_i is very small.

Critical points and bifurcations

For all values of the parameters there is a critical point where all the resonators are stationary:

$$(x_j, \dot{x}_j, \alpha) = \left(0, 0, -\frac{i\epsilon}{1 + i\delta}\right). \tag{7}$$

A linear stability analysis shows that this is stable for $g_j \geqslant 0$ and also stable if $g_j < 0$, provided $|\epsilon| < \epsilon_{bp}$, where

$$\epsilon_{bp} = \min_j \sqrt{\frac{\omega(1+\delta^2)}{-g_j}}.$$

When the resonators are identical ($g_j = g$, $\omega_{m,j} = \omega$), so that $\epsilon_{bp} = \sqrt{\frac{\omega(1+\delta^2)}{-g}}$, there are additional "spheres" of critical points that exist for $g < 0$ and $|\epsilon| < \epsilon_{bp}$, given by

$$|\alpha|^2 = \frac{\omega}{-g},\ g\sum_{i=1}^{n} x_i^2 = \delta \pm \sqrt{\frac{-g\epsilon^2}{\omega} - 1}, \tag{8}$$

and it is useful to think in terms of collective variables. Three collective variables can be used to describe the motion of the resonators: $X \equiv \sum_{i=1}^n x_i^2$, $Y \equiv \sum_{i=1}^n x_i \frac{dx_i}{dt}$, and $Z \equiv \sum_{i=1}^n \frac{1}{\omega^2}(\frac{dx_i}{dt})^2$. The resulting system is five-dimensional:

$$\begin{aligned} \frac{dX}{dt} &= 2Y, \\ \frac{dY}{dt} &= \omega^2 Z - 2\gamma Y - (\omega^2 + g\omega|\alpha|^2)X, \\ \frac{dZ}{dt} &= -4\gamma Z - 2Y\left(1 + \frac{g}{\omega}|\alpha|^2\right), \\ \frac{d\alpha}{dt} &= -i\delta\alpha - ig\alpha X - i\epsilon - \alpha. \end{aligned} \tag{9}$$

A stability analysis of the spheres of critical points (8), which become critical points in the collective variables, shows that they are unstable. The characteristic equation gives one zero solution plus a quartic that can be shown to have a real positive solution or a complex pair of solutions with real parts. The details are given in Appendix B. The characteristic equation for the full system of n resonators has an additional $(n-1)$ zero roots and n stable roots, so the individual oscillators are neutrally stable with respect to each other. Although neither critical point (7) nor (8) undergoes Hopf bifurcations, there are parameter regions where periodic orbits exist, in particular for $g > 0, \delta < 0$, and sufficiently large ϵ [see Fig. 1(a)] and for $g < 0$ and sufficiently large ϵ [see Fig. 1(b)]. These are created via saddle-node bifurcations of limit cycles,

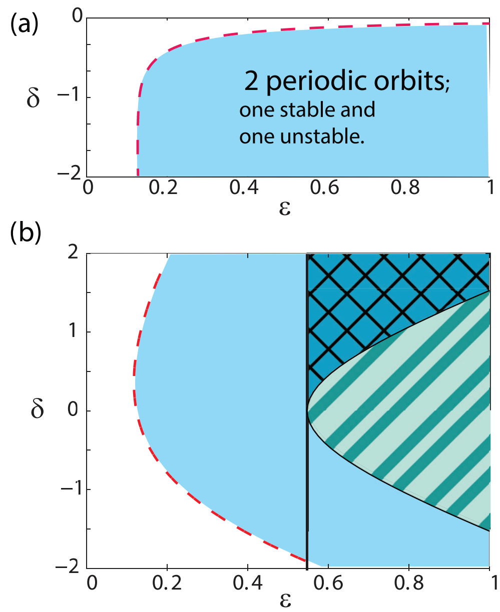

FIG. 1. (Color online) Bifurcations in the (ϵ, δ) parameter space for the identical resonator case. (a) shows the region where there are periodic orbits (shaded) for $g = 1$, $\omega = 0.3$, and $\delta < 0$. The critical point (7) is stable for all $g > 0$, so multistability is also present in the shaded region. Saddle-node bifurcations of limit cycles occur on the boundary (bold red dashed line), which creates both a stable limit cycle and an unstable limit cycle. [This curve was calculated using the amplitude equations; however it has been cross checked with MATCONT simulations of the system in collective variables (9).] In (b), where $g = -1$ ($\omega = 0.3$), the critical point (7) is only stable outside the striped region, which is bounded by the curve $\epsilon = \epsilon_{bp}$. In terms of the collective variables this is a branch point bifurcation. In the striped region there is one "sphere" of unstable critical points (8) (associated with the plus sign before the square root). In the hatched region critical point (7) is still stable, but in addition there are two spheres of unstable critical points (8). Saddle-node bifurcations of limit cycles (bold red dashed line) occur for fairly small ϵ, and everywhere to the left of this curve a stable limit cycle and an unstable limit cycle exist.

and we consider them in some depth in the section on amplitude equations.

III. THE ADIABATIC APPROXIMATION

The case where the amplitude decay rate of the common cavity mode is much larger than the other parameters present is often treated adiabatically. Here the amplitude decay rate of the common cavity mode has been scaled to 1 so that the other parameters ω_i, δ, ϵ, and g_i are assumed to be small compared to 1 with the result that the field is slaved to the mechanical resonators and the cavity amplitude can be approximated by

$$\alpha = \frac{-i\epsilon}{1 + i\left(\delta + \sum_{i=1}^{n} g_i x_i^2\right)}. \tag{10}$$

The evolution of the resonators is then given by

$$\frac{d^2 x_i}{dt^2} + \omega_i^2 x_i = -2\gamma_i \frac{dx_i}{dt} - g_i \omega_i \epsilon^2 \frac{x_i}{1 + \left(\delta + \sum_{i=1}^{n} g_i x_i^2\right)^2}. \tag{11}$$

Without mechanical damping ($\gamma_i = 0\ \forall\ i$) and assuming that $g_i = g$, the resonators evolve conservatively, with the integral of the motion given by

$$H = \frac{1}{2}\sum_{i=1}^{n}\left[\left(\frac{dx_i}{dt}\right)^2 + \omega_i^2 x_i^2\right] + \frac{\epsilon^2\omega}{2}\arctan\left(\delta + g\sum_{i=1}^{n} x_i^2\right). \tag{12}$$

For a single resonator there are three qualitatively different cases with the possibility of one, two, or three wells: If $g > 0$ or $g < 0$ and small, that is, $|g| < \frac{\omega}{\epsilon^2}$, which will typically be the case, there is just one critical point, which is at a minimum of the energy. If $g < 0$ and not small, that is, $|g| > \frac{\omega}{\epsilon^2}$, there may be additional critical points, as shown in Fig. 1, but the fact that in the adiabatic approximation the other critical points are stable is an artifact of this approximation. This and the fact that the conditions for the existence of the other critical points require large coupling lead us to concentrate on the case of a single well.

If two resonators with slightly different frequencies are considered and mechanical damping is neglected, $\gamma_i = 0\ \forall\ i$, the presence of a resonance implies that a stable resonant or synchronized solution exists. But when damping is included, this synchronized solution no longer exists in the adiabatic approximation. This can be verified by constructing amplitude equations since, if the γ_i and ϵ are small, the equation of motion (11) describes a weakly forced oscillator. Defining a slow time which is proportional to the weak damping ($\tau = \gamma t$), assuming that $\epsilon = \sqrt{\gamma}\,\bar{\epsilon}$ and that the frequencies of the resonators are given by $\omega_i = \omega + \gamma\Delta\omega_i$, Eq. (11) implies that the position of a resonator is given by

$$x_i = \hat{A}_i(\tau)e^{i\omega t} + \hat{A}_i^*(\tau)e^{-i\omega t} \tag{13}$$

for some complex function of slow time $\hat{A}_i(\tau)$. This is the same scaling of time that is used in the next section. The resulting amplitude equations are then of the following form:

$$\frac{d\hat{A}_i}{d\tau} = (-1 + i\Delta\omega_i)\hat{A}_i + i\hat{A}_i U_a(\hat{R},\hat{S}) + \hat{A}_i^*\left(\sum_{j=1}^{n}\hat{A}_j^2\right)H_a(\hat{R},\hat{S}), \tag{14}$$

where $U_a(\hat{R},\hat{S})$ is a real function and $H_a(\hat{R},\hat{S})$ is a pure imaginary function of the collective variables $\hat{R}(\tau), \hat{S}(\tau)$ given by

$$\hat{R}(\tau)e^{2i\hat{\phi}(\tau)} \equiv \sum_{i=1}^{n}\hat{A}_i^2(\tau), \qquad \hat{S}(\tau) \equiv \sum_{i=1}^{n}|\hat{A}_i(\tau)|^2. \tag{15}$$

This means that $\frac{d|\sum_{i=1}^{n}\hat{A}_i(\tau)^2|}{d\tau} = -|\sum_{i=1}^{n}\hat{A}_i(\tau)^2|$, so this approximation predicts that the amplitude of the synchronized solution decays to zero on the time scale of the slow time. In fact, this last result can be shown to hold equally well for the case where the the optical cavity is tuned so that this dependence is linear in the resonator's displacement [8]. However, as is shown in the next section and has been shown in [8], for the more general case where ω_i and δ are assumed to be order 1, synchronized solutions persist in the presence of mechanical damping.

IV. AMPLITUDE EQUATIONS

If the mechanical damping and forcing are assumed to be small and either the coupling strength or the driving is small, the resonators will still act as oscillators with slowly varying amplitude and phase, and we can capture the effect of the interaction between the mechanical damping and the coupling. This slow variation is proportional to the mechanical damping, so, as in the previous section, there are two time scales: a "fast" one, t, and a "slow" one, $\tau = \gamma t$. It is then consistent to assume that the resonators are almost identical apart from small differences in frequency; $g_i = g$, $\gamma_i = \gamma\ \forall\ i$ and $\omega_i = \omega + \gamma\Delta\omega_i$.

In fact, the equation of motion of the resonators, given by (4), implies that $x_i = \hat{A}_i(\tau)e^{i\omega t} + \hat{A}_i^*(\tau)e^{-i\omega t}$, provided $g|\alpha|^2$ is also order γ, which is ensured if the driving is tuned to be of order $\sqrt{\gamma}$ ($\epsilon \to \sqrt{\gamma}\,\epsilon$) or if the coupling g is order γ. Substituting this into the equation for the cavity (5), which can be solved in series form, and then solving for $\hat{A}_i(\tau)$ from (4) give the following amplitude equation (see Appendix C for more details):

$$\frac{d\hat{A}_i}{d\tau} = -\{1 - i[\Delta\omega_i + U(\hat{R},\hat{S})]\}\hat{A}_i + \hat{A}_i^*\hat{R}e^{2i\hat{\phi}}H(\hat{R},\hat{S}), \tag{16}$$

where $U(\hat{R},\hat{S})$ is a real function, which does not affect the dynamics and can be removed by letting $A_i \equiv \hat{A}_i e^{-i\int U(\hat{R},\hat{S})d\tau}$, but $H(\hat{R},\hat{S})$ is complex, with both real and imaginary parts (in contrast to the adiabatic case).

Noting that $\hat{R} = R$ and $\hat{S} = S$, the simplified amplitude equations are

$$\frac{dA_i}{d\tau} = (-1 + i\Delta\omega_i)A_i + A_i^* R e^{2i\phi}H(R,S). \tag{17}$$

A. Identical resonators

If the resonators have identical frequencies $\Delta\omega_i = 0$, the amplitude equations simplify further,

$$\frac{dA_i}{d\tau} = -A_i + A_i^* R e^{2i\phi}[H_r(R,S) + iH_i(R,S)], \tag{18}$$

and their dynamics is determined by the dynamics of the collective variables $R(\tau)$, $S(\tau)$, and $\phi(\tau)$:

$$\begin{aligned} R' &= -2R + 2RSH_r(R,S),\\ S' &= -2S + 2R^2H_r(R,S),\\ \phi' &= SH_i(R,S). \end{aligned} \tag{19}$$

Note that the R-S subsystem is independent of ϕ and has critical points, which are periodic orbits in the full (R, S, ϕ) space, given by

$$S = R, \quad RH_r(R) = 1, \tag{20}$$

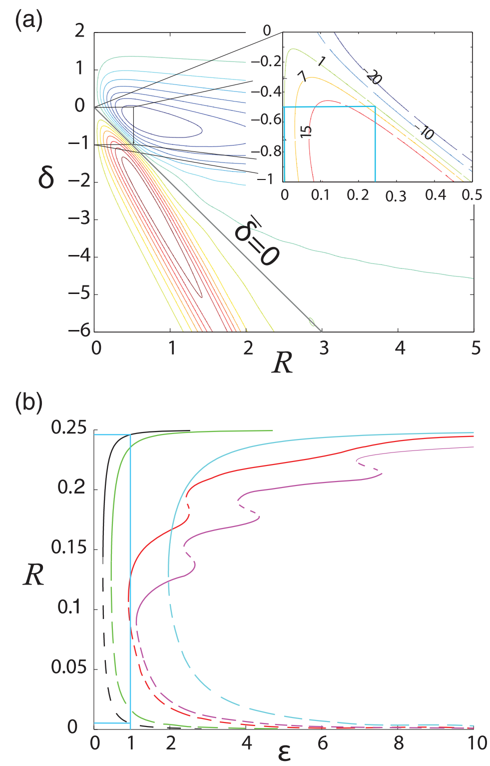

FIG. 2. (Color online) The case of $g = 1$. (a) is the contour diagram of $RH_r(R)$ for $g = 1$, $\gamma = 0.001$, and $\omega = 0.3$ in (R, δ) space; the gray line $\bar{\delta} = 0$ divides negative (blue) and positive (red) values of $RH_r(R)$; purple lines indicate the rough value of R where $RH_r(R) = 1$. (b) is a plot of the periodic orbits given by the amplitude equations' (ϵ, R) space (dashed lines are unstable periodic orbits, and solid lines are stable ones) for $\delta = -0.5$ with various values of ω: black is 0.3, red is 0.013, magenta is 0.01, green is 1, and cyan is 3. The stability of the periodic orbits simply depends on the slope of $RH_r(R)$. They are stable if $\frac{d}{dR}[RH_r(R)] < 0$.

where $H_r(R) \equiv H_r(R,R)$. Within the R-S subsystem the invariant subspace $R = S$ is stable since

$$\frac{d(R^2 - S^2)}{d\tau} = -4(R^2 - S^2),$$

which implies that $R^2 - S^2$ decays exponentially. Further, on that stable subspace, the critical point is stable if $\frac{d}{dR}[RH_r(R)] < 0$.

To gain some insight into the amplitude and stability of these periodic orbits for different values of δ it is useful to plot the contours of $RH_r(R)$ in (R, δ) space. From these plots we see that the types of periodic solutions depend critically on whether δ and g have the same sign. These two distinct cases are illustrated in Fig. 2, where $g = 1$, and Fig. 3, where $g = -1$. For both cases, notice that the critical point at the origin is always stable because the coupling function has no linear part and $\gamma > 0$. This means that when stable periodic motion does occur, there is multistability.

When δ and g have different signs, $\bar{\delta} = \delta + 2gR$ can be zero, and since H_r is proportional to $\bar{\delta}$, the line $R = -\frac{\delta}{2g}$ is a zero contour of $RH_r(R)$. In fact, the line $R = -\frac{\delta}{2g}$ divides the (R, δ) space into a region where periodic orbits can exist

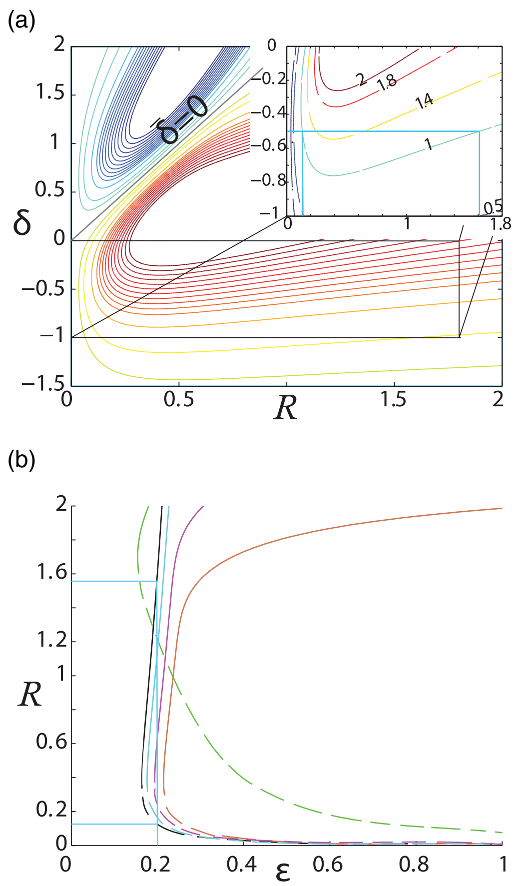

FIG. 3. (Color online) The case of $g = -1$. (a) is the contour diagram of $RH_r(R)$ for $\gamma = 0.001$ and $\omega = 0.3$ in (R, δ) space; the gray line $\bar{\delta} = 0$ divides negative (blue) and positive (red) values of $RH_r(R)$; purple lines indicate the rough value of R where $RH_r(R) = 1$. (b) is a plot of the periodic orbits given by the amplitude equations' (ϵ, R) space (dashed lines are unstable periodic orbits, and solid lines are stable ones) for $\delta = -0.5$ with various values of ω; black is 0.3, red is 0.013, magenta is 0.01, green is 1, and cyan is 3.

and a region where no periodic orbits can exist. For $g > 0$ [$RH_r(R) > 0$ for $R < -\frac{\delta}{2g}$] periodic orbits only exist for $\delta < 0$ [Fig. 2(a)].

Figures 2 and 3(b) show the position and stability of the periodic orbits for $\delta = -0.5$ in terms of the fixed values of the collective variable R as a function of ϵ for various values of ω. Notice that if $g > 0$, periodic orbits only exist for $R < -\frac{\delta}{2g}$ [Fig. 2(b)], whereas for $g < 0$ there is no such asymptote. Further the range of values of ϵ where multistable behavior is exhibited is much smaller for the case where $g < 0$.

In Figs. 4 and 5 we compare the position of the periodic orbits given by the amplitude equations with direct simulations of the equations of motion for the two cases $g = 1$ and $g = -1$, using the package MATCONT. The amplitude equations give their best fit for small ϵ. In Fig. 5 we have also plotted the result given by the amplitude equation for the case where $g > 0$ to show that for low values of R the unstable periodic orbits have similar amplitudes. [This is because the leading order expansion of $H_r(R, R)$ in R is proportional to g^2.]

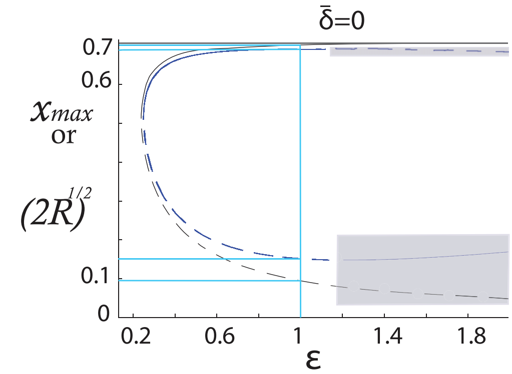

FIG. 4. (Color online) Comparison of the amplitude of the collective motion as calculated from the amplitude equations with numerical calculations from the equations of motion using the package MATCONT for the case with $g = 1$, $\delta = -0.5$, $\omega = 0.3$, $\gamma = 0.001$, and $\kappa = 1$.

So far we have just considered the collective motion. In fact, the individual resonators can be shown to synchronize to the stable periodic orbits of the collective system. To see this let $A_i = r_i e^{i\theta_i}$ and $RH(R,S) = \mathcal{H}(R,S)e^{i\eta(R,S)}$, where R and S are constants of the collective motion. Then the equation of motion for θ_i decouples from that for r_i:

$$\frac{d\theta_i}{d\tau} = \mathcal{H}(R,S)\sin[2(\phi - \theta_i) + \eta(R,S)]. \tag{21}$$

Letting $\bar{\theta}_i = \theta_i - \phi$, we have

$$\frac{d\bar{\theta}_i}{d\tau} = -\mathcal{H}\sin(\eta) + \mathcal{H}\sin(\eta - 2\bar{\theta}_i). \tag{22}$$

Now assuming that the solution to the collective motion lies on a periodic orbit [$S = R$ and $RH_r(R) = 1$] so that both $\mathcal{H}$ and η are constants, stable critical points of this system always exist, which implies that the individual resonators synchronize to the collective phase variable ϕ. However, they may synchronize in phase or out of phase depending on the initial phases of the oscillators.

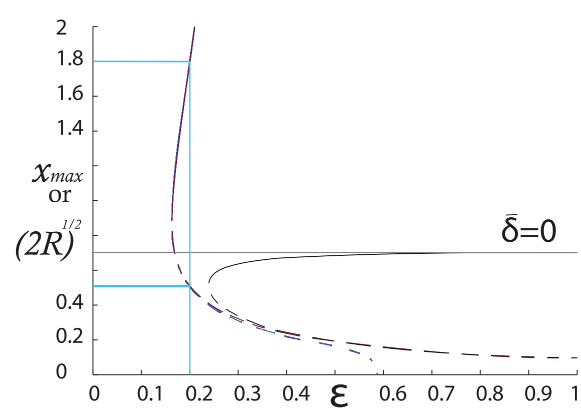

FIG. 5. (Color online) Comparison of the amplitude of the collective motion as calculated from the amplitude equations with numerical calculations from the equations of motion using the package MATCONT for the case with $g = -1$, $\delta = -1$, $\omega = 0.3$, $\gamma = 0.001$, and $\kappa = 1$.

Also if we consider the equation of motion for r_i, with $\dot{\bar{\theta}}_i = 0$,

$$\frac{dr_i}{d\tau} = r_i[-1 + \mathcal{H}\cos(2\bar{\theta}_i + \eta)], \tag{23}$$

since $\mathcal{H}\cos(\eta) = RH_r(R,S)$, the right hand side of this equation is zero irrespective of the value of r_i. This means that the synchronized solution is neutrally stable and the individual radii at which the different resonators synchronize depends on their initial condition.

B. Nonidentical resonators

If the resonators have different natural frequencies ($\Delta\omega_i \neq \Delta\omega_j$), then we can show that synchronization does not occur. From (17) the equations of motion of r_i and θ_i for the nonidentical resonator case are

$$\begin{aligned}\frac{dr_i}{d\tau} &= r_i\{-1 + \mathcal{H}\cos[\eta + 2(\phi - \theta_i)]\},\\ \frac{d\theta_i}{d\tau} &= -\Delta\omega_i + \mathcal{H}\sin[\eta + 2(\phi - \theta_i)],\end{aligned} \tag{24}$$

where both $\mathcal{H}$ and η are functions of the collective variables R and S and the equations of motion for the collective variables R and ϕ now involve extra sums.

If synchronization were to occur, then $\frac{d\theta_i - \theta_j}{d\tau} = 0$ for all $i \neq j$. But this equals

$$\Delta\omega_{j,i} + 2\mathcal{H}\cos[\eta + 2\phi - (\theta_i + \theta_j)]\sin(\theta_i - \theta_j),$$

where $\Delta\omega_{j,i} = \Delta\omega_j - \Delta\omega_i$, so this cannot occur if $\theta_i = \theta_j$ with $\Delta\omega_j \neq \Delta\omega_i$. Assuming that $\theta_i \neq \theta_j$, this means that $\theta_i - \theta_j$ is constant as is

$$2\mathcal{H}\cos[\eta + 2\phi - (\theta_i + \theta_j)] = \frac{\Delta\omega_j - \Delta\omega_i}{\sin(\theta_i - \theta_j)}.$$

But this implies that $r_i r_j$ grows or decays exponentially, as $\frac{d\ln|r_i r_j|}{d\tau}$ is

$$\begin{aligned}&-2 + 2\mathcal{H}\cos[\eta + 2\phi - (\theta_i + \theta_j)]\cos(\theta_i - \theta_j)\\ &= -2 + (\Delta\omega_j - \Delta\omega_i)\cot(\theta_i - \theta_j).\end{aligned}$$

So bounded synchronized motion, with r_i and $r_j \neq 0$, is not possible unless $\Delta\omega_j = \Delta\omega_i$ and $\theta_i = \theta_j$. This result, although it does not depend on the specific details of $\mathcal{H}$ and η, other than that they are functions of the collective variables, is specific to the form of the coupling function. In fact, because amplitude death is a direct result of the synchronization condition, it is a further indication of the difference between this model and a phase only model.

When a set of, say, seven different frequency groups are coupled together in a δ,ω,g,ϵ parameter region where synchronization may occur, all but one frequency group decays to the origin. This is shown in Fig. 6.

V. LINEAR AND NONLINEAR COUPLING

The motivation for the model considered in this paper is provided by the experimental results showing that the optical cavity may be tuned so that the resonance frequency of a common electromagnetic resonator depends quadratically on the resonator's displacement [9–11]. However, the result that nonidentical oscillators cannot synchronize if this dependence is quadratic makes one wonder what would actually occur experimentally if nonidentical oscillators were tuned in this way. To gain some insight into how synchronization may be lost in this tuning process we have considered a system which includes both linear and quadratic coupling terms in the simplest way:

$$\begin{aligned}\frac{d^2x_i}{dt'^2} &= -(\omega + \Delta\omega_i)^2 x_i - 2\gamma_i\frac{dx_i}{dt'}\\ &\quad - (\omega + \Delta\omega_i)|\alpha|^2\left(\frac{g_1}{2} + g_2 x_i\right),\\ \frac{d\alpha}{dt'} &= -\left\{1 + i\left[\delta + \sum_{i=1}^{n}\left(g_1 x_i + g_2 x_i^2\right)\right]\right\}\alpha - i\epsilon.\end{aligned} \tag{25}$$

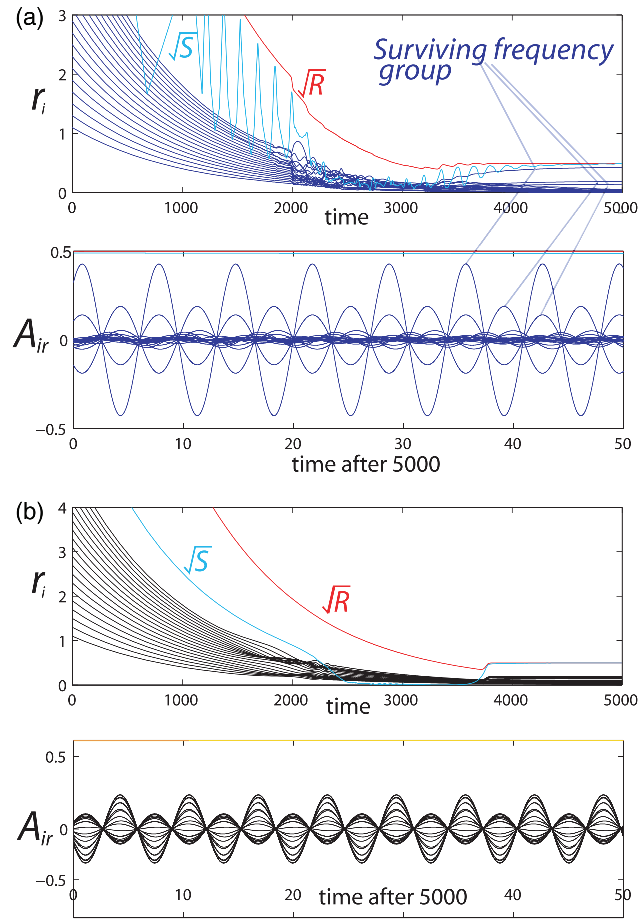

FIG. 6. (Color online) Nonidentical oscillators vs identical oscillators. Both (a) and (b) show the amplitudes $r_i(t)$ of 20 oscillators for $t = (0,5000)$, followed by the real part of the complex amplitude for $t = (5000,5050)$. But in (a) there are seven frequency groups: $\Delta\omega_1 = 1, \Delta\omega_{2,3,4} = 0.98, \Delta\omega_{5,6,7} = 0.96$, etc., whereas in (b) there is only one, $\Delta\omega_i = 1$, $\forall i$. After some time all but one frequency group has nonzero amplitude. In (a) a group of three oscillators synchronizes, with one out of phase with the other two. Also plotted are the collective variables S and R. In both cases the parameter values are $g = 1$, $\delta = -0.5$, $\omega = 0.3$, $\gamma = 0.001$, $\epsilon = 1$, and $\kappa = 1$.

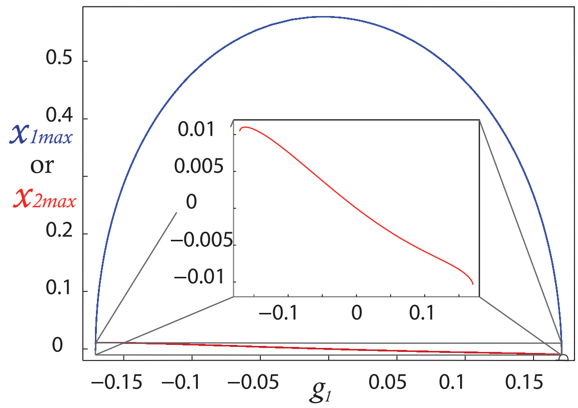

FIG. 7. (Color online) The amplitude of the synchronized motion in each individual oscillator (x_1 and x_2) as a function of the coefficient of the linear coupling term g_1 with the other parameters fixed as $g_2 = 1$, $\gamma = 0.001$, $\Delta\omega = 0.2$, $\kappa = 1$, and $\delta = -1.5$ using the package MATCONT. As the limit $g_1 \to 0$ is approached, $x_{2\max} \to 0$ (amplitude death), while $x_{1\max}$ reaches a maximum.

The system considered in this paper can be regained as a special case of this system where g_1 is set to zero and $g_2 = g$. Further the system considered in an earlier paper [8] where it was found that synchronization occurred for $\Delta\omega_i \neq \Delta\omega_j$ corresponds to setting $g_2 = 0$, albeit at different radii.

Considering just two resonators, we followed the periodic orbit of (25) numerically as $g_1 \to 0$. As Fig. 7 shows, taking this limit resulted in amplitude death in one (x_2) of the resonators and not in the other. In practice, as the cavity is tuned to be quadratic in the resonator's displacement, one would see one resonator's amplitude decay to zero.

VI. DISCUSSION AND CONCLUSIONS

This paper shows that deriving and analyzing the full nonlinear amplitude equations are an effective way to investigate synchronization in optomechanical scenarios such as that of a membrane in the middle [9–11]. Here we have considered the case where the optical cavity frequency may be tuned so that its dependence is quadratic. (In a previous paper we considered the case where this dependence is linear [8].)

We also show that the adiabatic approximation, for which we can also derive amplitude equations, fails to capture the full dynamics of the system and, in particular, synchronization because the phenomenon relies on the exchange of energies between the cavity and the mechanics.

The form of the coupled amplitude equations for the individual resonators reveals the first two most striking properties of their dynamics: From the radial equation,

$$\frac{dr_j}{d\tau} = -r_j\{1 - R[H_r(R,S)\cos(2\phi - 2\theta_j)] + [H_i(R,S)\sin(2\phi - 2\theta_j)]\}, \quad (26)$$

we can see that even when the mechanical damping on the resonators is extremely weak compared to the cavity damping (because the coupling term has no linear part), oscillator death is a stable solution for any resonator. Also, because the coupling term is nonlinear, we expect there to be additional (nonzero) solutions resulting in multistability. By looking further we show that synchronized oscillatory motion is only possible for groups of resonators that have identical natural frequencies. This situation is in stark contrast to that considered in [8], where the resonance frequency of a common electromagnetic cavity field depends only on the linear displacement of each mechanical resonator and we find that similar resonators synchronize to a mean oscillation at different, but nonzero, amplitudes. In fact, if one imagines changing the length of the cavity slightly, so as to tune the optical cavity away from a situation where the resonance frequency of a common electromagnetic cavity field depends on the linear displacement of each mechanical resonator to one where it depends quadratically on the displacement of each mechanical resonator, then the amplitudes of all but one resonator or one frequency group of resonators will tend to zero. This might be useful for applications that involve optomechanical resonators as sensitive transducers.

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APPENDIX A: THE PHYSICAL SYSTEM

We give here a physical implementation of the dynamical system we have discussed in this paper based on an optomechanical system comprising a Fabry-Pérot cavity containing n vibrating dielectric membranes, each with similar reflectivity, mass, and mechanical frequency $\omega_{m,j}$. The cavity is driven with a coherent source at carrier frequency ω_d. With a suitable choice of the equilibrium positions of the membranes the (classical) optomechanical Hamiltonian for one cavity mode can be taken as [10,13–15]

$$H = \sum_{j=1}^{n} \frac{P_j^2}{2m} + \frac{m\Omega_{m,i}}{2} Q_j^2 + \hbar\Omega_c(Q_j)|\alpha|^2 + \hbar\sqrt{\kappa}(\varepsilon^* e^{i\Omega_d t}\alpha + \varepsilon e^{-i\Omega_d t}\alpha^\dagger), \quad (A1)$$

where P_j and Q_j, $j = 1,\ldots,n$, denote the momentum and position of the membranes relative to their equilibrium values, m is the effective mass of the bulk mechanical mode of interest, κ is the linewidth of the optical resonance, ε is the amplitude of the driving laser in units such that $|\varepsilon|^2$ has units of s^{-1} (a photon flux), and it is assumed that the membranes interact with just one resonant mode of the cavity. The first term describes the mechanical energy of the oscillating membranes, the second describes the energy of the optical mode, and the last describes the driving of the cavity mode. The optical mode is described by a complex amplitudes α,α^*. The Hamiltonian dynamics of these variables is determined by the Poisson bracket

$$\{\alpha,\alpha^*\} = -i. \quad (A2)$$

This bracket is inherited from the canonical Poisson bracket for the quadrature phase amplitudes x,y of the cavity field defined by $\alpha = (x + iy)\sqrt{2}$. Moving to an interaction that rotates at the frequency of the driving laser, we make the canonical change of variable,

$$\alpha \rightarrow \alpha e^{-i\Omega_d t}, \tag{A3}$$

and fixing the phase reference for the driving laser so that ε is real, we get the interaction picture Hamiltonian,

$$H = \sum_{j=1}^{n} \frac{P_j^2}{2m} + \frac{m\Omega_{m,i}^2}{2} Q_j^2 + \hbar\Delta(Q_j)|\alpha|^2 + \hbar\kappa\epsilon(\alpha + \alpha^*), \tag{A4}$$

where $\epsilon = \frac{\varepsilon}{\sqrt{\kappa}}$ is a dimensionless driving field amplitude and $\Delta(Q_j) = \Omega_c(Q_j) - \Omega_d$ is a known nonlinear oscillatory function of Q_j dependent on the reflectivity of the membrane and the length of the cavity [10]. Engineering the system appropriately, we can choose the equilibrium position of each membrane to ensure that the mode frequency is quadratic (to lowest order),

$$\Delta(Q_j) = \Delta + \frac{1}{2}\frac{d^2\Delta(0)}{dQ_j^2} Q_j^2, \tag{A5}$$

in the membrane's displacement. The Hamiltonian becomes

$$H = \sum_{j=1}^{n} \frac{P_j^2}{2m} + \frac{m\Omega_{m,i}^2}{2} Q_j^2 + \hbar\big(\Delta + G_j Q_j^2\big)|\alpha|^2 + \hbar\kappa\epsilon(\alpha + \alpha^*), \tag{A6}$$

where $\Delta = \Omega_c - \Omega_d$ is the detuning between the empty cavity resonance and the driving laser carrier frequency and $G_j = \frac{1}{2}\frac{d^2\Delta(0)}{dQ_j^2} Q_j^2$ is the optomechanical coupling constant.

In optomechanics the objective is to control the quantum dynamics of mechanical resonators that have been cooled close to their quantum ground state. This requires that $\omega_m \geqslant \kappa$, a condition known as the resolved sideband regime. In this case the convenient position and momentum scales are of the order of the rms fluctuations in these quantities in the ground state so that we choose $P_j = \bar{p} p_j, Q_j = \bar{x} x_j$, with

$$\bar{x} = \sqrt{\frac{\hbar}{m\kappa}}, \tag{A7}$$

$$\bar{p} = \sqrt{\hbar m\kappa} \tag{A8}$$

for each mechanical resonator. This fixes $E_0 = \hbar\kappa$, and the Hamiltonian can be written as

$$H = \hbar\kappa \sum_{j=1}^{n} \frac{p_j^2}{2} + \frac{\omega_{m,j}^2}{2} x_j^2 + \hbar\kappa\left(\delta + \frac{g_j}{2} x_j^2\right)|\alpha|^2 + \hbar\kappa\epsilon(\alpha + \alpha^*), \tag{A9}$$

where $\omega_{m,j} = \Omega_{m,j}/\kappa$, $\delta = \Delta/\kappa$, and $g_j = \frac{1}{\kappa}\frac{d^2\Delta(0)}{dx_j^2}$. We can now rescale the energy in units of $\hbar\kappa$ to finally arrive at the Hamiltonian

$$H = \sum_{j=1}^{n} \frac{p_j^2}{2} + \frac{\omega_{m,j}^2}{2} x_j^2 + \left(\delta + \frac{g_j}{2} x_j^2\right)|\alpha|^2 + \epsilon(\alpha + \alpha^*). \tag{A10}$$

Adding cavity decay and mechanical dissipation to the Hamiltonian equations of motion gives the model equations of motion:

$$\begin{aligned} \frac{d\alpha}{dt} &= -\left(\kappa + i\delta + i\sum_{j=1}^{N} g_j x_j^2\right)\alpha - i\epsilon, \\ \frac{d^2 x_j}{dt^2} &= -\omega_{m,j}^2 x_j - 2\gamma_j \frac{dx_j}{dt} - g_j\omega_{m,j} x_j |\alpha|^2, \end{aligned} \tag{A11}$$

where $\kappa \equiv 1$ is the cavity decay rate and γ is the mechanical damping. We will ignore the fluctuation terms that would accompany the frictional damping of the mechanical resonator.

APPENDIX B: THE CHARACTERISTIC EQUATION OF CRITICAL POINT (8)

Taking the linearized matrix about the critical points (8) (for the system in collective variables), $(-g)X_\pm = \delta \pm B$, where $B = \sqrt{\frac{-g\epsilon^2}{\omega} - 1}$, gives one zero eigenvalue and four eigenvalues given by the roots of the following quartic:

$$\lambda(\lambda + 2\gamma)[(\lambda + 1)^2 + B^2] = \pm 4\omega B(\delta \pm B). \tag{B1}$$

Now this quartic has either one real positive root or two complex roots with a positive real part.

First, consider the case where $\gamma = 0$. If $X = X_+$ the right hand side of this equation is positive, and there is one positive real root for λ because for $\lambda > 0$ the quartic $\lambda^2[(\lambda + 1)^2 + B^2] > 0$. If $X = X_-$, so that the right hand side of this equation is negative, then there is no positive real root, but there must be at least two complex conjugate roots $(a \pm ib)$. This is because the quartic (B1) has no linear term in λ, which implies that, if the other roots are $\lambda_{1,2}$,

$$2a\lambda_1\lambda_2 + (a^2 + b^2)(\lambda_1 + \lambda_2) = 0. \tag{B2}$$

Now if either $\lambda_{1,2}$ are real and negative or $\lambda_{1,2}$ are complex and have negative real parts, then (B2) implies that $a > 0$, and the real part of the root is positive. So both critical points are unstable if $\gamma = 0$.

Now consider how these roots change as γ is increased away from zero. If either critical point were to become stable, there must exist a value of γ, in the case of X_+, where there is a zero root and, in the case of X_-, where there are pure imaginary roots. Neither of these are possible for $\gamma > 0$. So both critical points are unstable if $\gamma > 0$ as well.

APPENDIX C: DERIVATION OF THE AMPLITUDE EQUATIONS

For slowly varying oscillations about the critical point $(x_i, y_i, |\alpha|^2) = (0, 0, \frac{\epsilon^2}{1+\delta^2})$, let $x_i = \hat{A}_i(\tau)e^{i\omega t} + \hat{A}_i^*(\tau)e^{-i\omega t}$. The equation for the cavity then becomes

$$\dot{\alpha} = -[1 + i(\delta + 2g\{\hat{R}\cos[2(\omega t + \hat{\phi})] + \hat{S}\})]\alpha - i\epsilon, \tag{C1}$$

which can be solved in series form by letting $\alpha = e^{i\psi}\sum_m B_m e^{2i\omega m t}$, so that

$$\dot{\alpha} = i\dot{\psi}\alpha + e^{i\psi}\sum_m 2im\omega B_m e^{2im\omega t}. \tag{C2}$$

If we choose ψ such that

$$i\alpha\dot{\psi} = -2i\alpha g\hat{R}\cos[2(\omega t + \hat{\phi})], \tag{C3}$$

which means that

$$\psi = -\frac{g\hat{R}}{\omega}\sin[2(\omega t + \hat{\phi})], \tag{C4}$$

and if we use the Jacobi-Anger expansion $e^{iz\sin y} = \sum_n J_n(z)e^{iny}$, then Eq. (C1) becomes

$$\sum_m (2im\omega + 1 + i\delta + 2ig\hat{S})B_m e^{2im\omega t} = -i\epsilon e^{-i\psi}$$

$$= -i\epsilon\sum_m J_m\left(\frac{g\hat{R}}{\omega}\right)e^{2im\omega t}e^{2im\hat{\phi}}, \tag{C5}$$

so that

$$B_m = \frac{-i\epsilon J_m\left(\frac{g\hat{R}}{\omega}\right)e^{2im\hat{\phi}}}{2im\omega + 1 + i\delta + 2ig\hat{S}}. \tag{C6}$$

Using this in the expression for α and substituting it into Eq. (5) and then equating terms rotating at $e^{i\omega t}$ and $e^{-i\omega t}$, respectively, it is possible to obtain equations of motion in τ for the amplitudes $\hat{A}_i$ and their complex conjugates $\hat{A}_i^*$.

$$\frac{d\hat{A}_i}{d\tau} = -\hat{A}_i + i[\Delta\omega_i + U(\hat{R},\hat{S})]\hat{A}_i + \hat{A}_i^*\hat{R}e^{2i\hat{\phi}}H(\hat{R},\hat{S}), \tag{C7}$$

where $U(\hat{R},\hat{S})$ is a real function,

$$U(\hat{R},\hat{S}) \equiv \frac{g}{2}\sum_m \frac{\bar{\epsilon}^2 J_m^2\left(\frac{g\hat{R}}{\omega}\right)}{1 + (2m\omega + \delta + 2g\hat{S})^2}, \tag{C8}$$

but $H(\hat{R},\hat{S})$ is complex with both real and imaginary parts, which are given by

$$H_r(\hat{R},\hat{S}) = \frac{-8g\bar{\delta}\bar{\epsilon}^2\omega^2}{\hat{R}}\sum_{m=1}^{\infty}\frac{J_m\left(\frac{g\hat{R}}{\omega}\right)J_{m-1}\left(\frac{g\hat{R}}{\omega}\right)(2m-1)[1+\bar{\delta}^2+4m(m-1)\omega^2]}{|u_m(\hat{S})|^2|u_{m-1}(\hat{S})|^2|u_{-m}(\hat{S})|^2|u_{-m+1}(\hat{S})|^2}, \tag{C9}$$

$$H_i(\hat{R},\hat{S}) = \frac{-g\bar{\epsilon}^2\bar{\delta}\omega}{\hat{R}}\sum_{m=1}^{\infty}\frac{J_m\left(\frac{g\hat{R}}{\omega}\right)J_{m-1}\left(\frac{g\hat{R}}{\omega}\right)(2m-1)v_m(\hat{S})}{|u_m(\hat{S})|^2|u_{m-1}(\hat{S})|^2|u_{-m}(\hat{S})|^2|u_{-m+1}(\hat{S})|^2}, \tag{C10}$$

and u_m and v_m depend on the collective variable $\hat{S}$ via $\bar{\delta} = \delta + 2g\hat{S}$:

$$u_m(\hat{S}) = [1 + i(2m\omega + \bar{\delta})], \tag{C11}$$

$$v_m(\hat{S}) = 2(1 + \bar{\delta}^2 - 4m^2\omega^2)[1 + \bar{\delta}^2 - 4(m-1)^2\omega^2] + 32m(m-1)\omega^2. \tag{C12}$$

[1] Y. Kuromoto, in *International Symposium on Mathematical Problems in Theoretical Physics*, edited by H. Araki, Lecture Notes in Physics, Vol. 39 (Springer, New York, 1975), pp. 420–422.
[2] S. Strogatz, Phys. D **143**, 1 (2000).
[3] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, 2003).
[4] M. Zhang, G. S. Wiederhecker, S. Manipatruni, A. Barnard, P. McEuen, and M. Lipson, Phys. Rev. Lett. **109**, 233906 (2012).
[5] G. J. Milburn and M. J. Woolley, Contemp. Phys. **49**, 413 (2008).
[6] A. D. O'Connell, M. Hofheinz, M. Ansmann, R. C. Bialczak, M. Lenander, E. Lucero, M. Neeley, D. Sank, H. Wang, M. Weides *et al.*, Nature (London) **464**, 697 (2010).
[7] M. Aspelmeyer, T. J. Kippenberg, and F. Marquardt, arXiv:1303.0733.
[8] C. A. Holmes, C. Meany, and G. J. Milburn, Phys. Rev. E **85**, 066203 (2012).
[9] J. D. Thompson, B. M. Zwickl, A. M. Jayich, F. Marquardt, S. M. Girvin, and J. G. E. Harris, Nature (London) **452**, 72 (2008).
[10] A. M. Jayich, J. C. Sankey, B. M. Zwickl, C. Yang, J. D. Thompson, S. M. Girvin, A. A. Clerk, F. Marquardt, and J. G. E. Harris, New J. Phys. **10**, 095008 (2008).
[11] H. Seok, L. F. Buchmann, S. Singh, S. K. Steinke, and P. Meystre, Phys. Rev. A **85**, 033822 (2012).
[12] J. C. Sankey, C. Yang, B. M. Zwickl, A. M. Jayich, and J. G. E. Harris, Nature Physics **6**, 707 (2010).
[13] H. Seok, L. F. Buchmann, S. Singh, and P. Meystre, Phys. Rev. A **86**, 063829 (2012).
[14] M. J. Hartmann and M. B. Plenio, Phys. Rev. Lett. **101**, 200503 (2008).
[15] M. Bhattachararya, H. Uys, and P. Meystre, Phys. Rev. A **77**, 033819 (2008).