

Experimental observation of a transition from amplitude to oscillation death in coupled oscillators

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(Received 4 April 2014; published 3 June 2014)

We report the experimental evidence of an important transition scenario, namely the transition from amplitude death (AD) to oscillation death (OD) state in coupled limit cycle oscillators. We consider two Van der Pol oscillators coupled through mean-field diffusion and show that this system exhibits a transition from AD to OD, which was earlier shown for Stuart-Landau oscillators under the same coupling scheme [T. Banerjee and D. Ghosh, *Phys. Rev. E* **89**, 052912 (2014)]. We show that the AD-OD transition is governed by the density of mean-field and beyond a critical value this transition is destroyed; further, we show the existence of a nontrivial AD state that coexists with OD. Next, we implement the system in an electronic circuit and experimentally confirm the transition from AD to OD state. We further characterize the experimental parameter zone where this transition occurs. The present study may stimulate the search for the practical systems where this important transition scenario can be observed experimentally.

DOI: [10.1103/PhysRevE.89.062902](https://doi.org/10.1103/PhysRevE.89.062902)

PACS number(s): 05.45.Xt

The suppression of oscillation has been attracting the attention of researchers due to its ubiquity in diverse fields like physics, biology, and engineering [1]. In coupled oscillators, there exists two distinct types of oscillation quenching processes, namely amplitude death (AD) and oscillation death (OD). Although AD and OD are two structurally different phenomena, their clear distinction has been made only recently in pioneering works reported in Refs. [1–3] (see Ref. [1] for an extensive review on OD). In AD, all the coupled oscillators populate a common stable steady state that was unstable otherwise and thus gives rise to a stable homogeneous steady state (HSS) [4]. But, in the case of OD, due to symmetry breaking bifurcation, the oscillators populate different coupling-dependent stable steady states, resulting in stable inhomogeneous steady states (IHSS). Thus, the stabilization of HSS gives rise to AD, and the stabilization of IHSS gives birth to OD.

In this context, Koseska *et al.* [2] first proved that, despite their different origins, AD and OD can simultaneously occur in a diffusively coupled system of oscillators; more significantly, they reported an important transition phenomenon, namely the transition from AD to OD. They established the analogy between this transition and the Turing-type bifurcation [5] in spatially extended systems. The AD-OD transition in identical Stuart-Landau oscillators is reported in Ref. [3] for the dynamic [6] and conjugate [7] coupling schemes. In Ref. [8], diverse routes to AD-OD transition have been shown in identical nonlinear oscillators that are coupled diffusively with an additional repulsive coupling link. Recently, the present authors have reported the AD-OD transition in Stuart-Landau oscillators coupled via mean-field diffusion [9]. In that paper, we have shown that the AD-OD transition is governed by the mean-field density parameter. Also, we have reported a novel nontrivial AD state that coexists with OD for a certain parameter zone and which is destroyed by the parameter mismatch.

Although OD and AD have been separately observed previously in a number of different dynamical systems [10–12], to the best of our knowledge, hitherto no experimental

confirmation of the predicted AD-OD transition has been reported.

In this paper, we report the experimental evidence of AD-OD transition in coupled oscillators. For this we consider two Van der Pol oscillators [13] in their stable oscillation mode coupled via mean-field diffusion. The paradigmatic Van der Pol oscillator is widely used for the demonstration and understanding of nonlinear dynamics; further, it has a rich connection with engineering and biological systems [14]. The choice of the mean-field coupling is motivated by the fact that it is one of the important coupling schemes owing to its presence in many natural systems [12,15,16]. Also, experimental observation of AD-OD transition in this coupling scheme is comparatively easy because, as we showed in Ref. [9], there exists a wide parameter region where OD is the only existing state, which is in contrast to the other diffusive coupling schemes where, in general, OD is accompanied by limit cycle oscillations [17]. We at first carry out theoretical and numerical analyses to explore the dynamical behaviors of the coupled Van der Pol oscillators and characterize the AD-OD transition. Next, the coupled system is implemented in electronic circuit to experimentally demonstrate the transition. Experimental results show the evidence of AD-OD transition for a wide range of parameter values.

We consider two Van der Pol (VdP) oscillators interacting through mean-field diffusive coupling; mathematical model of the coupled system is given by

$$\dot{x}_{1,2} = y_{1,2} + \epsilon(Q\bar{X} - x_{1,2}), \quad (1a)$$

$$\dot{y}_{1,2} = a_{1,2}(1 - x_{1,2}^2)y_{1,2} - x_{1,2}. \quad (1b)$$

Here $\bar{X} = \frac{1}{2} \sum_{i=1}^2 x_i$ is the mean field of the coupled system. The individual VdP oscillator shows a near sinusoidal oscillation for small $a_{1,2}$, and relaxation oscillation for large $a_{1,2}$. The coupling strength is given by ϵ ; Q is called the mean-field density parameter that determines the density of mean-field diffusion [12,16]; $0 \leq Q < 1$. As the limiting case we take two identical VdP oscillators: $a_{1,2} = a$. From Eq. (1) we can see that the system has the following fixed points: the origin $(0,0,0,0)$ is the trivial fixed point, and two additional coupling-dependent fixed points: (i) $(x_1^*, y_1^*, -x_1^*, -y_1^*)$

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where $x_1^* = \frac{y_1^*}{\epsilon}$ and $y_1^* = \sqrt{\epsilon^2 - \frac{\epsilon}{a}}$. (ii) $(x_1^\dagger, y_1^\dagger, x_1^\dagger, y_1^\dagger)$

where $x_1^\dagger = \frac{y_1^\dagger}{\epsilon(1-Q)}$ and $y_1^\dagger = \sqrt{\epsilon^2(1-Q)^2 - \frac{\epsilon(1-Q)}{a}}$.

The eigenvalues of the system at the origin are

$$\lambda_{1,2} = \frac{(a - \epsilon) \pm \sqrt{(a + \epsilon)^2 - 4}}{2}, \quad (2a)$$

$$\lambda_{3,4} = \frac{[a - \epsilon(1 - Q)] \pm \sqrt{[a + \epsilon(1 - Q)]^2 - 4}}{2}. \quad (2b)$$

From eigenvalue analysis we derive that the two pitchfork bifurcation (PB) points, PB1 and PB2, emerge at the following coupling strengths:

$$\epsilon_{PB1} = \frac{1}{a}, \quad (3a)$$

$$\epsilon_{PB2} = \frac{1}{a(1 - Q)}. \quad (3b)$$

The IHSS $(x_1^*, y_1^*, -x_1^*, -y_1^*)$ emerges at ϵ_{PB1} through a symmetry breaking pitchfork bifurcation. The other nontrivial fixed point $(x_1^\dagger, y_1^\dagger, x_1^\dagger, y_1^\dagger)$ comes into existence at ϵ_{PB2} , which gives rise to a unique *nontrivial HSS* (to be discussed later in this paper). From Eq. (2) we can see that no Hopf bifurcation of trivial fixed point occurs for $a > 1$; in that case, only pitchfork bifurcations exist. Thus, for $a > 1$ no AD, and AD-OD transition are possible. For any $a < 1$, equating the real part of $\lambda_{1,2}$ and $\lambda_{3,4}$ to zero, we get two Hopf bifurcation points at

$$\epsilon_{HB1} = a, \quad (4a)$$

$$\epsilon_{HB2} = \frac{a}{1 - Q}. \quad (4b)$$

We use XPPAUT package [18] to compute the bifurcation branches. Figure 1(a) shows the bifurcation diagram of $x_{1,2}$ with ϵ for $Q = 0.3$ and $a = 0.35$ [without any loss of generality, throughout this paper, we use $a = 0.35$ ($a < 1$)]. It is observed that at $\epsilon_{HB2} = 0.5$, AD is born through an inverse Hopf bifurcation; whether at $\epsilon_{HB1} = 0.35$, an unstable limit cycle is born. This AD (stable HSS) state becomes unstable through a supercritical pitchfork bifurcation (PB1) to give birth to OD (stable IHSS) at $\epsilon_{PB1} = 1/a = 2.857$. Now, with increasing Q value, ϵ_{HB2} moves toward ϵ_{PB1} , and at a particular Q value, say Q^* , HB2 collides with PB1: $Q^* = (1 - a^2)$. At $Q = Q^*$, the AD state, and thus the AD-OD transition, cease to take place. Figure 1(b) shows this scenario for $a = 0.35$, and $Q = Q^* = 0.8775$. Now, for $Q > Q^*$, $\epsilon_{HB2} > \epsilon_{PB1}$, i.e., HB2 point moves to the right hand side of PB1; subsequently, the IHSS now becomes stable at ϵ_{HBS} through a subcritical Hopf bifurcation, where

$$\epsilon_{HBS} = \frac{1}{\sqrt{(1 - Q)}}. \quad (5)$$

This is derived from the eigenvalues of the nontrivial fixed point $(x_1^*, y_1^*, -x_1^*, -y_1^*)$ given by

$$\lambda_{1,2} = \frac{-b_1^* \pm \sqrt{b_1^{*2} - 4c_1^*}}{2}, \quad (6a)$$

$$\lambda_{3,4} = \frac{-b_2^* \pm \sqrt{b_2^{*2} - 4c_2^*}}{2}, \quad (6b)$$

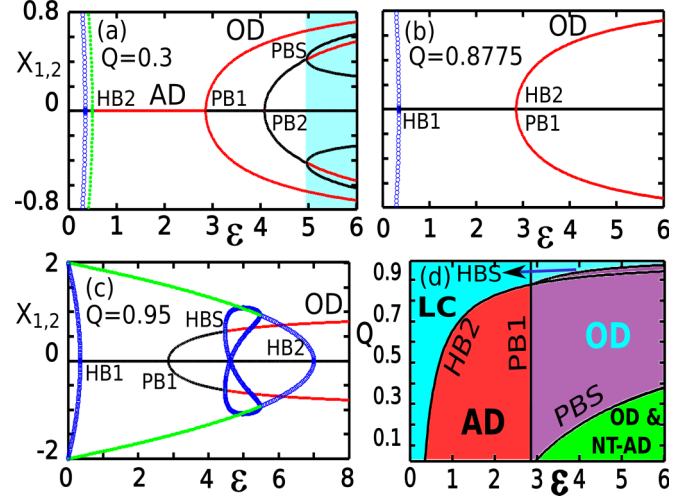


FIG. 1. (Color online) Bifurcation diagram (using XPPAUT) of two mean-field-coupled identical Van der Pol oscillators ($a = 0.35$). Gray (red) lines, stable fixed points; black lines, unstable fixed points; solid circle (green), stable limit cycle; open circle (blue), unstable limit cycle. HB1,2 and HBS are Hopf bifurcation points; PB1,2 and PBS are pitchfork bifurcation points. (a) $Q = 0.3$ ($< Q^*$): AD-OD transition takes place; coexistence of OD ($x_1 = -x_2$) and *nontrivial* AD (NT-AD) ($x_1 = x_2$) is also shown [shaded (cyan) region]. (b) $Q = 0.8775$ ($= Q^*$): AD vanishes, AD-OD transition just destroyed. (c) $Q = 0.95$ ($> Q^*$): no AD-OD transition, only OD. (d) Phase diagram in $\epsilon - Q$ space. With increasing Q , collision of HB2 and PB1 destroys the AD-OD transition scenario.

where $b_1^* = \epsilon - a(1 - x_1^{*2})$, $c_1^* = 1 + 2ax_1^*y_1^* - a\epsilon(1 - x_1^{*2})$, $b_2^* = \epsilon(1 - Q) - a(1 - x_1^{*2})$, $c_2^* = 1 + 2ax_1^*y_1^* - a\epsilon(1 - Q)(1 - x_1^{*2})$. Using Eq. (5), for $Q = 0.95$, we get $\epsilon_{HBS} \approx 4.472$ that matches with Fig. 1(c).

The second nontrivial fixed point $(x_1^\dagger, y_1^\dagger, x_1^\dagger, y_1^\dagger)$ that was created at ϵ_{PB2} becomes stable through a subcritical pitchfork bifurcation at ϵ_{PBS} :

$$\epsilon_{PBS} = \frac{2 - Q}{2a(1 - Q)^2}. \quad (7)$$

This is derived from the eigenvalues corresponding to $(x_1^\dagger, y_1^\dagger, x_1^\dagger, y_1^\dagger)$, which are same as Eq. (6) except now the “*” signs are replaced by “†” sign. From Fig. 1(a) we have, $\epsilon_{PBS} \approx 4.956$, that exactly matches with Eq. (7). For $\epsilon > \epsilon_{PBS}$, stable IHSS (OD) (i.e., $x_1^* = -x_2^*$) coexists with the *nontrivial* AD (NT-AD) state (i.e., $x_1^\dagger = x_2^\dagger$) [shaded (cyan) region in Fig. 1(a)]. The attribute *nontrivial* is used because it results from the stabilization of *nontrivial* HSSs (x^\dagger, y^\dagger) . Further, the observed NT-AD is *unique* because unlike conventional AD, the NT-AD state is born via a subcritical pitchfork bifurcation. Also, contrary to AD, the NT-AD state is completely destroyed by parameter mismatch [9]. The whole bifurcation scenario in the $\epsilon - Q$ parameter space is shown in Fig. 1(d) for $a = 0.35$. We can see that, with increasing Q , at $Q = 0.8775$, collision of HB2 with PB1 destroys the AD-OD transition. It also shows the coexisting region of NT-AD and OD.

Next, we implement the coupled system in electronic circuit. Figure 2 shows the electronic circuit diagram of two mean-field-coupled Van der Pol oscillators. Shaded (blue)

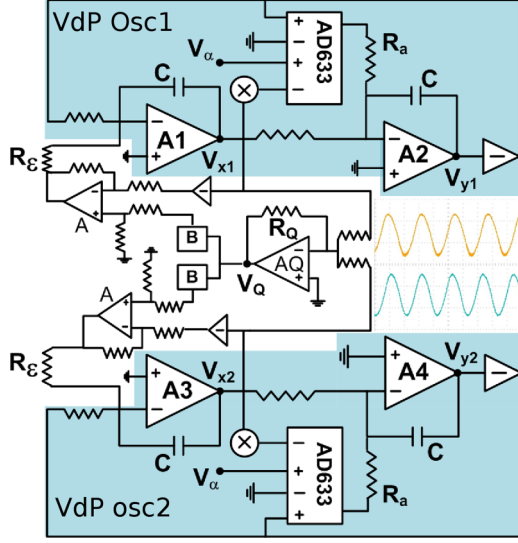


FIG. 2. (Color online) Experimental circuit diagram of the mean-field-coupled VdP oscillators. A, A1–A4, and AQ are realized with TL074 op-amps. All the unlabeled resistors have value $R = 10 \text{ k}\Omega$. $C = 10 \text{ nF}$, $R_a = 286 \Omega$, $V_\alpha = 0.1 \text{ v}$. Box denoted by “B” are op-amp-based buffers; inverters are realized with the unity gain inverting op-amps. \otimes sign indicates squarer using AD633. Inset (in the middle part) shows the oscillation from the uncoupled VdP oscillators: upper trace (yellow) V_{x1} , lower trace (cyan) V_{x2} (y axis: 10 v/div , x axis: $250 \mu\text{s/div}$).

regions in the upper and lower portions represent the individual VdP oscillators [19]. We use TL074 (quad JFET) op-amps and AD633 analog multiplier chips (having differential inputs); output of the multiplier is scaled by a factor of 0.1 . $\pm 15 \text{ v}$ power supplies are used; resistors (capacitors) have $\pm 5\%$ ($\pm 1\%$) tolerance. The unlabeled resistors have value $R = 10 \text{ k}\Omega$. The op-amp AQ is used to generate the mean field: $V_Q = -\frac{2R_Q}{R} \sum_{j=1}^2 \frac{V_{xj}}{2}$, which is subtracted from $V_{x1,2}$ using op-amps denoted by A. One can see that R_ϵ determines the coupling strength, and R_Q determines the mean-field density. The voltage equation of the circuit can be written as:

$$CR\dot{V}_{xi} = V_{yi} + \frac{R}{R_\epsilon} \left[\frac{2R_Q}{R} \sum_{j=1}^2 \frac{V_{xj}}{2} - V_{xi} \right], \quad (8a)$$

$$CR\dot{V}_{yi} = \frac{R}{R_a} \left(V_\alpha - \frac{V_{xi}^2}{10} \right) \frac{V_{yi}}{10} - V_{xi}. \quad (8b)$$

Here, $i = 1, 2$. Equation (8) is normalized with respect to CR , and thus now becomes equivalent to Eq. (1) for the following normalized parameters: $\dot{u} = \frac{du}{d\tau}$, $\tau = t/RC$, $\epsilon = \frac{R}{R_\epsilon}$, $Q = \frac{2R_Q}{R}$, $a = \frac{R}{100R_a}$, $10V_\alpha = 1$, $x_i = \frac{V_{xi}}{V_{\text{sat}}}$, and $y_i = \frac{V_{yi}}{V_{\text{sat}}}$. V_{sat} is the saturation voltage of the op-amp. In the experiment we take $V_\alpha = 0.1 \text{ v}$, and $C = 10 \text{ nF}$; we choose $a = 0.35$ by taking $R_a = 286 \Omega$ [using a precision potentiometer (POT)]. We vary the coupling strength ϵ and mean-field density Q by varying R_ϵ and R_Q , respectively (using precision POTs). For the uncoupled case, the individual oscillations have a frequency of 1.7 kHz , and are shown in Fig. 2 (inset). Next, at first we fixed $Q = 0.3$ by taking $R_Q = 1.5 \text{ k}\Omega$ and vary R_ϵ . With the increasing coupling strength (i.e., decreasing R_ϵ) we observed

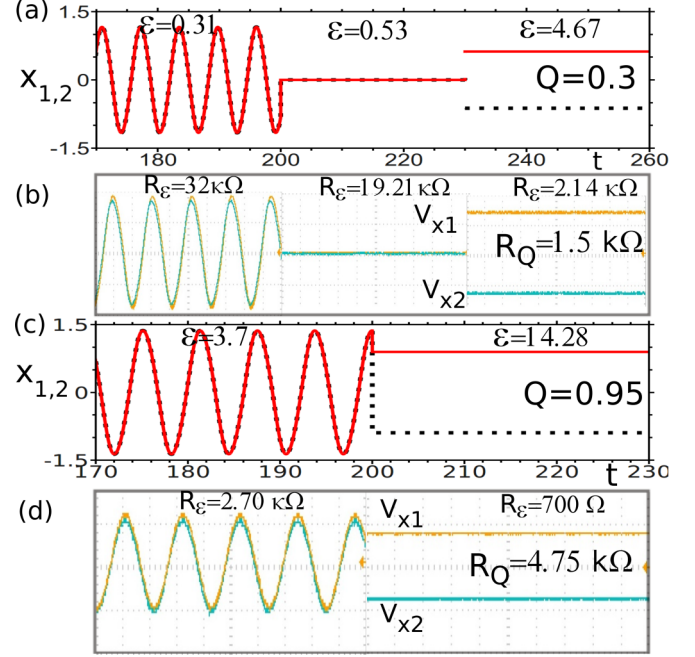


FIG. 3. (Color online) Experimental real time traces [(b) and (d)] of V_{x1} and V_{x2} along with the numerical time series plots [(a) and (c)] of x_1 and x_2 . (a) $Q = 0.3$, (b) $R_Q = 1.5 \text{ k}\Omega$: complete synchronized limit cycle at $R_\epsilon = 32 \text{ k}\Omega$ ($\epsilon = 0.31$), AD at $R_\epsilon = 19.2 \text{ k}\Omega$ ($\epsilon = 0.53$), and OD at $R_\epsilon = 2.14 \text{ k}\Omega$ ($\epsilon = 4.67$). (c) $Q = 0.95$, (d) $R_Q = 4.75 \text{ k}\Omega$: complete synchronized limit cycle at $R_\epsilon = 2.70 \text{ k}\Omega$ ($\epsilon = 3.70$) and OD state $R_\epsilon = 700 \Omega$ ($\epsilon = 14.28$). [(b), (d) y axis: 10 v/div ; x axis: $250 \mu\text{s/div}$].

the transition from limit cycle (complete synchronized) state to AD at $R_\epsilon = 30.9 \text{ k}\Omega$, and then a transition from AD to OD state at $R_\epsilon = 3.8 \text{ k}\Omega$. In Fig. 3(b), using the experimentally obtained snapshots of the waveforms [with a digital storage oscilloscope (Tektronix TDS2002B, 60 MHz , 1 GS/s)], we demonstrate different dynamical behaviors for the following parameter values: complete synchronized limit cycle at $R_\epsilon = 32 \text{ k}\Omega$, AD at $R_\epsilon = 19.21 \text{ k}\Omega$, and OD at $R_\epsilon = 2.14 \text{ k}\Omega$. For the comparison purpose, we also show the numerical results (using fourth order Runge-Kutta method with step-size 0.01) taking ϵ values that are equivalent to R_ϵ (note that $\epsilon = \frac{R}{R_\epsilon}$); Fig. 3(a) shows this with limit cycle (for $\epsilon = 0.31$), AD (for $\epsilon = 0.53$), and OD (for $\epsilon = 4.67$). It can be seen that the numerical and experimental results are in close agreement with each other. As we increase R_Q , the AD region reduces; for $R_Q \geq 4.32 \text{ k}\Omega$, no AD occurs and the limit cycle state directly transits into OD state beyond a certain coupling strength. This is in agreement to the theory that for $Q > Q^*$ ($=0.8775$) no AD-OD transition takes place. Note the close proximity between Q^* and experimental value of Q^* , i.e., $Q_{\text{expt}}^* = \frac{2R_Q^*}{R} = 0.864$. Next, we take $Q = 0.95$ ($> Q^*$) by taking $R_Q = 4.75 \text{ k}\Omega$. Here, in accordance with the theory, we observed direct transition from limit cycle (complete synchronized) to the OD state (at $R_\epsilon = 1.95 \text{ k}\Omega$); Fig. 3(d) shows this scenario: limit cycle (at $R_\epsilon = 2.7 \text{ k}\Omega$) and OD state (at $R_\epsilon = 700 \Omega$). Figure 3(c) shows the same in numerical simulation having limit cycle (at $\epsilon = 3.7$) and OD (at $\epsilon = 14.28$). We repeat the experiment for a large number of values of R_Q and note the R_ϵ values where

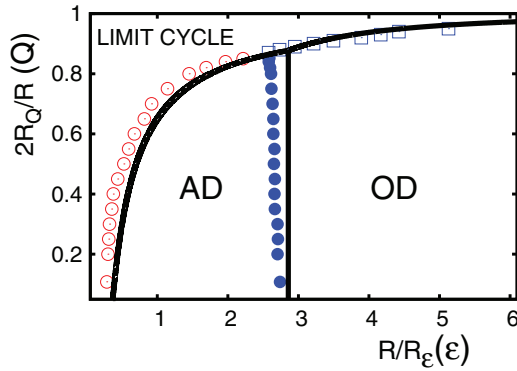


FIG. 4. (Color online) Experimental phase diagram in the $\frac{R}{R_\epsilon} - \frac{2R_Q}{R}(Q)$ space. Open circle, experimental transition points from limit cycle to AD; solid circle, experimental transition points from AD to OD; open square, experimental transition points from limit cycle to OD. Theoretical curves (line) are also shown, which closely match with the experimental points.

AD, OD, and AD-OD transition occur. To represent the whole experimental scenario, we plot the experimental phase diagram in the $\frac{R}{R_\epsilon} - \frac{2R_Q}{R}(Q)$ space (Fig. 4). Theoretically obtained curves are also plotted in the same graph. It is noteworthy that

the experimental points are in close proximity to the theoretical curves. The slight deviation from the theoretical result occurs due to the inherent parameter fluctuation in electronic circuit, and also the possible parameter mismatches present between the oscillators. We further note that, due to this inherent parameter mismatch, we could not observe the NT-AD state, which is in agreement with the findings of the authors of Ref. [9], that any parameter mismatch destroys the NT-AD state.

In conclusion, we have experimentally observed the transition from amplitude death to oscillation death state in mean-field-coupled limit cycle oscillators. We have chosen the paradigmatic Van der Pol oscillators coupled via mean-field diffusion and implemented the system in electronic circuit. By changing the coupling strength for a fixed mean-field parameter, we have experimentally observed the transition from AD to OD if the mean-field parameter has a value less than a threshold value. Beyond that threshold value, no AD occurs, and limit cycle oscillation directly transforms into an OD state. We have corroborated the experimental results by suitable theory and bifurcation analysis. We believe that apart from electronic circuits the AD-OD transition scenario can be observed experimentally in laser and neuronal systems that may reveal the practical application of this transition in those systems.

D.G. acknowledges DST, India for providing the INSPIRE fellowship.

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