Two-exponent Lavalette function: A generalization for the case of adherents to a religious movement

Marcel Ausloos

eHumanities Group, Royal Netherlands Academy of Arts and Sciences (KNAW), Joan Muyskenweg 25, 1096 CJ Amsterdam, The Netherlands
and Group of Researchers for Applications of Physics in Economy and Sociology (GRAPES), rue de la Belle Jardiniere, 483/0021,
B-4031 Liege, Wallonia-Brussels Federation, Belgium

The Lavalette function is generalized to a two-exponent function in order to represent data looking like a sigmoid on semilogarithmic plots. A Mandelbrot trick is suggested for further investigations, if more fit parameters are needed. The analyzed data is that of the number of adherents to the main religions in the 20th century.

I. INTRODUCTION

Recently, there have been several studies of activities related to society dynamics based on physics ideas [1–3]. This is part of an aim of physicists toward applying their concepts and methods in sociological fields [4–6]. Among these are aspects of religion and religious movements.

A few recent papers are mentioned [7–13] in order to set a frame and pretend that one aspect among others has not been examined. It is true that the evolution of adherents to religious movements has been studied through the (statistical) distribution of the number of adherents. However, one cannot (apparently) distinguish between various analytical forms, or say whether the distribution is stable. There is a report on some "heavy tail" in [10], but the discussion and analysis are quite reduced to the shortest arguments for some data snapshot.

One aim of this paper is to report another (simple) approach, the "least effort method" [14] about the rank-size law of religious movements, through the number of adherents for a set of religions, selected through published data. However, a second aim pertains to the findings suggesting that the discovered law might be more general than exemplified here, in fact, as being valid for many cases of sets measured through the number of members and agents. It is found that a Lavalette law [15], generalized in quite another field [16], is rather robust, and represents a fine data envelope, equivalent to a mean field approximation of the rank-size relationship asymptotic behavior. Thereby, hinting that such a law—even possibly further generalized, as suggested below—can serve to a huge array of data interpretation, models, etc., in many possible research fields—in physics and outside—and could be usefully examined on basic principles, outside the present aims, however.

In Sec. II, the (religious movement adherent) data is presented and analyzed along the conventional rank-size method, classically bearing upon Zipf-Mandelbrot-Pareto (ZMP) ideas. Moreover, introducing a Mandelbrot trick onto the generalized Lavalette function is possible and likely of interest. Such a generalization and subsequent successful applications seem to indicate that possible "supergeneralizations" (see Sec. III) have a wide, approximately universal, content. In fact, it is readily related to the Feller-Pareto distribution function (for a chosen set of exponents, see below). Such a finding suggests some conclusion, found in Sec. IV.

II. DATA AND ANALYSIS

Due to the unusual type of data hereby considered for a "scaling theory" like analysis, some introduction to the analyzed World Christian Encyclopedia (WCE) data [17] is given here. Indeed, a warning is necessary: It is unclear how much distinction was made in the WCE surveys concerning denominations and sects so-called adstrated to the main religions. Nevertheless, admitting some flexibility in the definition of religion and ambiguity in that of adhesion, the data are as equally valid as many of those reported about agent behavior in the recent sociophysics literature and examined along statistical mechanics approaches.

The data set is taken from the WCE [17]: It gives information on the number of adherents of the world's main religions and their main denominations (56 religions overall). I have desaggregated the data for the less general denominations, also removing data on so-called doubly affiliated disaffiliated, and doubly counted religionists, as listed in [17]. Moreover some data refers to *atheists* and *nonreligious* persons. Thereafter for conciseness, those three sets are also called "religions.'

All religions do not exist in every country. Countries can be ranked according to the number of religions which have local adherents. This leads to a so-called Pareto plot for the cumulative distribution of the (56) religious movements in the (258) countries (Fig. 1).

On this *log-log plot*, giving the relationship between the number of nations ranked according to the number of religions which can be found locally, it is obvious that the size-frequency distribution is *surely not* a power law (not even shown); visually, the data has a convex shape, and also hardly looks like a Poisson (exponential) form (green dashed lines) in the (two-parameter) exponential case (Exp2)

$$y(r) = be^{-\beta r}. \tag{1}$$

However, a fit by a generalized Lavalette function (Lav3) [16] (red dotted line), i.e., a decaying power law with a power-law cutoff,

$$y(r) = \Lambda \frac{[r]^{-\phi}}{[N-r+1]^{-\psi}}, \tag{2}$$

gives a quite good regression coefficient $R^2 \simeq 0.981$; the exponents have values $-\phi = 0.001$ and $\psi = 2.315$. For completeness, let it be recalled that Eq. (2) is a two-exponent (thus three-parameter) generalization of the basic two-parameter
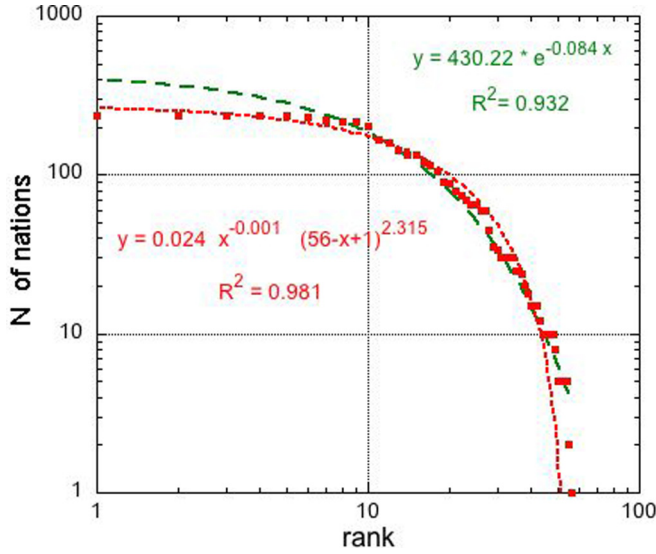
FIG. 1. (Color online) Cumulative distribution over the 20th century of the number of nations where a number of religions can be found; the number of countries is ranked in decreasing order of the number of "supported or available" religions.

(one-exponent) (Lav 2) function introduced by Lavalette [15], when discussing impact factor distributions:

$$
\begin{aligned}
y(r) &= \kappa [Nr/(N-r+1)]^{-\chi} \\
&\equiv \hat{\kappa}[r/(N-r+1)]^{-\chi} \\
&\equiv \hat{\kappa} r^{-\chi}(N-r+1)^{+\chi}.
\end{aligned} \tag{3}
$$

In the WCE data set [17], further information is given about changes in the number of adherents of each (main) religion from 1900 till 2000, measured over a five year span. In Fig. 2, the *1900 data* distribution is emphasized: There
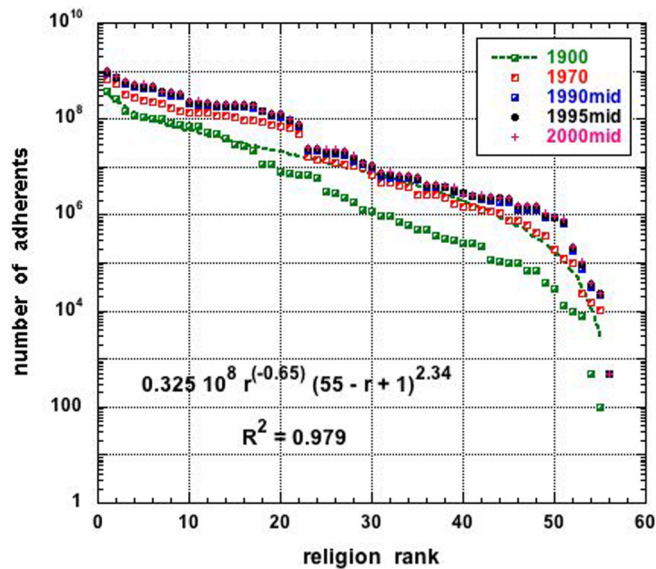

FIG. 2. (Color online) The *1900 data* distribution of adherents to the 55 major denominations (green half-filled squares) data, with the three-parameter free Lavalette function, evolving after $r = 18$, toward the recent (after 1970) data for $r > 24$ (see Fig. 3).

were 55 "major denominations," available to a few billion people. Nowadays there are 56 available major denominations for about twice as many possible adherents. The 56th, the "black muslims," came later in the century.

Observe that a fit of the 1900 (green half-filled squares) data with the three-parameter free Lavalette function evolves toward the recent (after 1970) data for $r > 24$. The fit with a Lavalette three-parameter free function is very stable as a function of time, and not only matches very well the data for the denominations with $r > 24$, but also extends toward and covers very finely the "high rank" regime.

An interesting point, confirming the warning about data precision, has to be made: It should be emphasized that the jump in N (the number of adherents) between $r = 24$ and 25 is reminiscent of the observation in [10] that it is *somewhat difficult* (author's emphasis) for major denominations to measure their true number of adherents, in fact (daring author's interpretation?) likely preferring to overestimate such an adhesion in order to appear "big." *A contrario*, it seems common sense to consider that "less popular" denominations have a better measurement of their number of adherents. The bigger-smaller border occurs at $\sim 10^7$ adherents, corresponding to $r \sim 24$.

The more recent data (1970–2000) is analyzed in two ways in Figs. 3 and 4, either within the Lavalette "model" (Lav 3) or within a mere exponential behavior (Exp2). It is simpler to display the parameters in a table, i.e., Table I, before further discussion. As expected the R^2 values are better for the Lav3 than for the exponential law. Let it be observed that the β decay rate is pretty stable (~ 0.173) in modern times. The amplitude b follows the evolution of the world population, but Λ decreases, indicating that people likely become less
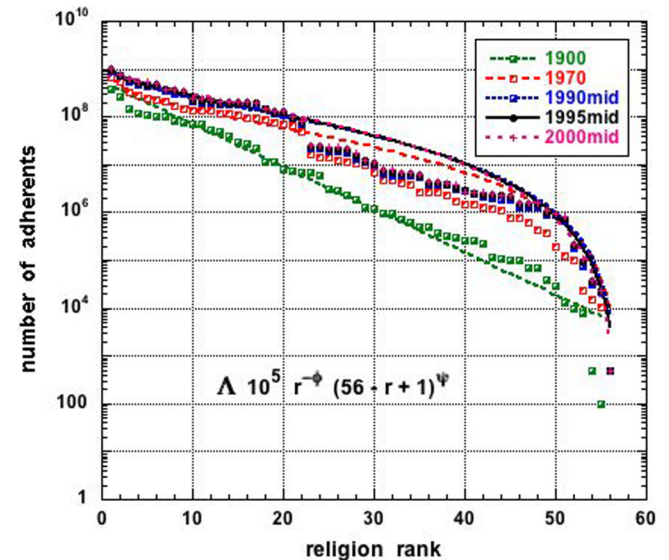

FIG. 3. (Color online) The more recent data (1970–2000) about the number of adherents to the 56 major denominations is analyzed within the Lavalette model (Lav3). Note the slight difference of the 1970 data with respect to the very recent data, quasioverlapping each other, including the fits of course. A mere exponential fit ($\sim e^{-0.21r}$) is drawn (green dashed line) through the 1900 data (green half-filled squares) for visual comparison.
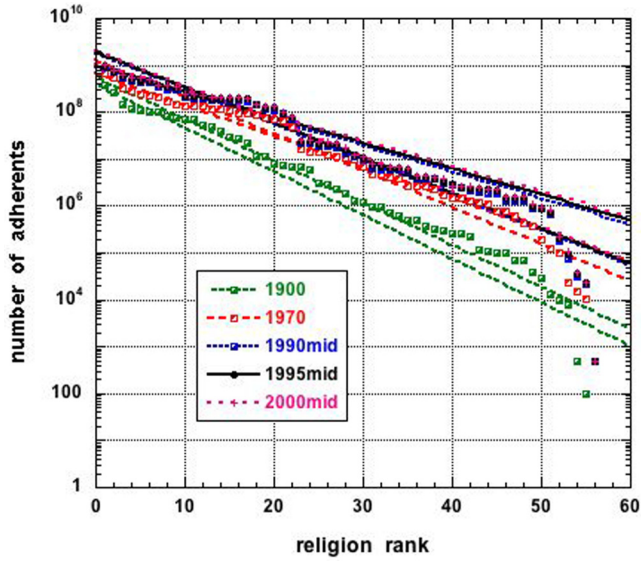
FIG. 4. (Color online) The whole data (1900–2000) is analyzed assuming a mere exponential behavior (Exp2); fit parameters and R^2 values are found in Table I.

adherent to (the main 56) religions, which can be an "artificial result" due to the fact that the number of small religious movements itself increases, but these are not considered in the WCE surveys. The power-law exponent ϕ, measuring the decay, steadily decreases. However, the exponent ϕ of the cutoff at high rank presents a minimum (at some time in the middle of the century) and increases nowadays. Note that $(\psi + \phi)$ measures the slope at half range on a semilogarithmic plot, and approximately follows the (increasing nowadays) behavior of ψ, i.e., the power-law cutoff. This indicates that proportionally the number of adherents to the main religions in fact increases. Another Matthew effect [18] in some sense is that the "winning" religions are always the same ones, and stay more at the top than others, even though the number of adherents in the population decreases.

TABLE I. Parameters of the Lavalette function, Eq. (3), and of the exponential, Eq. (1), fits to the number of adherents to religious movements in the 20th century according to WCE data at various years; the number of religions N_r is 55 in 1900, 56 otherwise; the regression coefficient R^2 is given.

	1900	1970	1990	1995	2000
Generalized Lavalette law (Lav3), Eq. (2)					
$\Lambda\ 10^{-5}$:	0.325	0.684	0.325	0.182	0.100
ϕ:	0.648	0.506	0.391	0.363	0.338
ψ:	2.341	2.289	2.554	2.713	2.874
R^2:	0.979	0.984	0.986	0.986	0.986
Poisson exponential law (Exp 2), Eq. (1)					
$b\ 10^{-8}$:	3.995	6.558	9.157	9.771	10.376
β:	0.214	0.153	0.129	0.126	0.125
R^2:	0.937	0.945	0.973	0.975	0.976

III. HYPERGENERALIZED LAVALETTE FUNCTION

The pure power-law distribution, known as the zeta distribution, or discrete Pareto distribution, is expressed as

$$p(k) = \frac{k^{-\gamma}}{\zeta(\gamma)}. \tag{4}$$

where k is a positive integer usually measuring some variable of interest, $p(k)$ is the probability of observing the value k, γ is the power-law exponent, and $\zeta(\gamma) \equiv \sum_{k=1}^{\infty} k^{-\gamma}$ is the Riemann zeta function. It is important to note, from this definition, that $\gamma > 1$ for the Riemann zeta function to be finite. Such a mere power law is of no use here. However, an adaptation of Eq. (4) was suggested for text appraisal by Mandelbrot,

$$J(r) = b/(\nu + r)^{\xi}, \tag{5}$$

in order to take into account a flattening of the data at low rank, on a log-log plot; it is known as the Zipf-Mandelbrot-Pareto three-parameter free function (ZMP3) for describing the "queen effect" [19].

Note that the role of r as an independent variable in Eq. (5) is taken by the ratio $r/(N - r + 1)$ between the descending and the ascending ranking numbers in Eqs. (2) and (3). Moreover, observe that, on a semilogarithmic graph, Lav2 and Lav3 follow a characteristic (flipped) sigmoidal S shape which by no means can be provided by a power law or by ZMP3 [20].

Moreover, note that N [as a factor of r, in Eq. (3)] is not really needed; in fact it can be usefully replaced by some simple factor having the order of magnitude of $y(r_M/2)$. This can be seen through other ways of writing the two-parameter Lavalette form, introducing the scaling factors $\hat{\kappa}$ or Λ.

Finally, if the data fits show such a need—this is not presently needed–an introduction of the Mandelbrot trick into the Lavalette function could be made in order to obtain a (six-parameter-free) hypergeneralized Lavalette function, i.e., (1) in its most generalized form, with a power-law cutoff,

$$y(x) = A(x + m_3)^{-m_1}(N + m_5 - x^{m_4})^{m_2}; \tag{6}$$

or (2) alternatively with an exponential cutoff,

$$y(x) = A(x + m_3)^{-m_1} e^{-m_2\,(x^{m_4} + m_5)}. \tag{7}$$

In the latter formula, m_5 is not officially needed; it can be incorporated into A. But, beyond such possibilities, some interpretation of the parameters remains to be had, thus there is much theoretical work ahead.

Nevertheless, in concluding this section, observe that the above generalized in (two-exponent) Lavalette function can find a root in the Feller-Pareto function:

$$y(r) = \Lambda[r]^{-\phi}[N + 1 - r]^{+\psi} \equiv \hat{\Lambda} u^{-\phi}\,(1 - u)^{+\psi}, \tag{8}$$

when $\phi > 0$ and $\psi < 0$; with $u \equiv r/(N + 1)$. To represent a true density function, Λ should then be the inverse of a B (beta) function, for normalization purposes. The Feller-Pareto function is known to describe random walk distributions, one-dimensional electronic density of states, moving average intersections, etc. However, the Feller-Pareto function has neither the flexibility of the generalized nor the hypergeneralized Lavalette functions [20].

IV. CONCLUSION

When employing combined functions, an excellent fit of rank-frequency distributions is usually ensured. However, improving the quality of the approximation at the expense of an increase in the number of fitting parameters does not provide the best solution toward inventing an appropriate model. Having a large number of fitting parameters causes difficulties in the extraction of their implication (meaning) and increases their dependency on each other, in worst cases. The present work, devoted to developing a fitting function that provides (i) an optimal approximation of rank-probability distributions and (ii) a correlation of its fitting parameters with features of an interesting modern data base,

has emphasized that the Lavalette function is of interest in order to represent a sigmoid function with few parameters on semilogarithmic plots. Yet, the basic Lavalette function does not provide enough flexibility for fitting curves in the range of small rank values. Therefore a two-exponent function has been used, and a Mandelbrot trick suggested for further investigations.

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