PHYSICAL REVIEW E **89**, 062130 (2014)

Planck radiation law and Einstein coefficients reexamined in Kaniadakis κ statistics

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(Received 30 December 2013; published 24 June 2014)

Blackbody radiation is reconsidered using the counterpart of the Bose-Einstein distribution in the κ statistics arising from the Kaniadakis entropy. The generalized Planck radiation law is presented and compared to the usual law, to which it reduces in the limiting case $\kappa \to 0$. Effective Einstein's coefficients of emission and absorption are defined in terms of the Kaniadakis parameter κ. It is shown that the Kaniadakis statistics keeps unchanged the first Einstein coefficient A while the second coefficient B admits a generalized form within the present theoretical framework.

DOI: 10.1103/PhysRevE.89.062130 PACS number(s): 05.30.Jp, 05.90.+m, 05.20.−y

I. INTRODUCTION

Due to its historical significance and ability to describe experimentally realizable phenomena, blackbody radiation has been a favorite test for advances in statistical physics. A large number of contributions have focused on its reformulation in the framework of nonextensive q statistics [1–14] first recognized by Rényi [15] and subsequently proposed by Tsallis [16]. Several approximations have been involved and different aspects of the blackbody radiation have attracted a good deal of interest. The thermodynamical quantities have been reconsidered [12] and it has been shown that the basic thermodynamical relations (though remaining form invariant) are affected by the nonextensive parameter q. In Ref. [1], use was made of the generalized nonextensive Planck radiation law to interpret data from the cosmic microwave background radiation. Considering a simple model, Wang and Méhauté [7] showed that the nonextensivity prescription unexpectedly modifies the Einstein coefficients of emission and absorption. There are also some studies in connection with the Rényi entropy [12] and the Beck-Cohen superstatistics [14]. Such studies have been—partially—motivated by one fact: even though the usual Planck radiation law has been confirmed experimentally over and over again, small deviations from this law have been detected in the cosmic microwave radiation. A possible explanation is that these deviations could have arisen at the time of matter-radiation decoupling, due to the nonextensive statistics environment [1].

On the other hand, major developments in statistical physics have been motivated by the observation of distributions that do not arise from the usual Boltmann-Gibbs-Shannon (BGS) statistical mechanics. The latter can be obtained by a maximization of the BGS entropy, leading to exponential distributions, whereas some observations clearly indicate distributions that exhibit a power-law asymptotic behavior. Such distributions can originate from a deformation of the usual BGS entropy. The distributions arising from the Tsallis entropy (which is a one-parameter generalization of the BGS entropy) present a power-law behavior. The latter have shown a good agreement with observations and experimental measurements (distribution of cold atoms in dissipative optical lattices [17], spin-glass relaxation [18], velocity distributions in driven dissipative dusty plasma [19], etc.). However, other generalizations of the entropy also lead to power-law distributions which are different from those generated by the Tsallis entropy. Hence, it is of interest to study the implication of such alternative distributions and their ability to describe key phenomena.

In a very interesting and influential paper [20], Kaniadakis proposed a deformation of the entropy for relativistic systems: a deformation of the usual BGS entropy, involving one continuous parameter κ. The Kaniadakis entropy leads inherently to nonexponential distributions [21–25]. Those distributions also exhibit a power-law asymptotic behavior, according to the experimental evidence. Such power-law distributions have been observed in a variety of relativistic systems such as plasmas [26] and multiparticle production processes [27]. For instance, empirical nonexponential distributions with power-law tails have been systematically used to model high-energy plasmas. The κ-deformed distributions arising from the Kaniadakis entropy have been applied to cosmic rays [21], quark-gluon plasma formation [28], kinetics of interacting atoms and photons [29], nonlinear kinetics [30,31], etc.

In this paper, we propose to reconsider blackbody radiation within the theoretical framework of the κ statistics arising from the Kaniadakis entropy. To this end, we use the κ counterpart of the Bose-Einstein distribution and apply it to photons. This distribution has been successfully applied to generalize Bose-Einstein condensation and to interpret liquid ^{4}He behavior at low temperatures [32]. It may be of interest to see how different entropies leading both to power-law distributions can describe the same phenomenon. The generalized Planck radiation law in q statistics has already been employed in interpreting the cosmic microwave background radiation and has been compared to observations made by the FIRAS spectrographer in the Cosmic Background Explorer satellite [1,11].

This article is organized in the following fashion. In the next section, we derive the generalized Planck radiation law in the framework of the Kaniadakis κ statistic and compare it with the usual Plank law arising from BGS statistics. The low- (*Rayleigh-Jeans law*) and high- (*Wien shift law*) frequency limits are exposed. In Sec. III, we focus on the effect of Kaniadakis entropy on Einstein's coefficients. Working parallel to the treatment in Ref. [7] and considering a model of N atoms with two nondegenerate energy states, we define

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effective Einstein's coefficients within the κ statistics and compare them to the usual ones arising from the standard BGS theory. A summary of our findings and conclusions is given in Sec. IV.

II. GENERALIZED PLANCK RADIATION LAW

Recently, κ deformation of the Bose-Einstein statistical distribution was proposed [33]. The latter is given by

$$f(\varepsilon) = \frac{1}{\exp_\kappa\left(\frac{\varepsilon-\mu}{kT}\right) - 1}, \tag{1}$$

where μ, T, and k are, respectively, the chemical potential, the temperature, and the Boltzmann constant. Note that $\exp_\kappa$ stands for a one-parameter generalization of the exponential, given by

$$\exp_\kappa(x) = (\sqrt{1+\kappa^2x^2} + \kappa x)^{1/\kappa}. \tag{2}$$

In the limiting case $\kappa \to 0$, it reduces to the ordinary exponential function and the usual Bose-Einstein distribution is then recovered. Equation (1) can be used to describe photons, assuming the dilute gas approximation for $\varepsilon = h\nu$, where ν is the frequency of the photon and $\mu = 0$ since the photon gas is not a variable system state. The number of photons within a frequency range between ν and $\nu + d\nu$ is given by

$$dN_\nu = \frac{8\pi V}{c^3}\frac{\nu^2 d\nu}{\exp_\kappa\left(\frac{h\nu}{kT}\right) - 1} \tag{3}$$

with a corresponding energy

$$dE_\nu = \frac{8\pi V}{c^3}\frac{h\nu^3 d\nu}{\exp_\kappa\left(\frac{h\nu}{kT}\right) - 1} = \frac{8\pi hcV}{\lambda^5}\frac{-d\lambda}{\exp_\kappa\left(\frac{hc}{\lambda kT}\right) - 1}, \tag{4}$$

where $\lambda = c/\nu$ stands for the wavelength. Equation (4) is the Planck radiation law generalized in the framework of Kaniadakis κ statistics. In the limit $\kappa \to 0$, it reduces to the usual Planck radiation law. It is worth noting that (4) is different from its nonextensive counterpart, arising from the Tsallis statistical theory, which is given by

$$dE_\nu = \frac{8\pi V}{c^3}\frac{h\nu^3 d\nu}{\left[1+(q-1)\frac{\hbar\omega}{kT}\right]^{1/(q-1)} - 1}. \tag{5}$$

It may be useful to note that, for $q < 1$ and in contrast to (4), expression (5) exhibits a cutoff $\frac{\hbar\omega}{kT} < \frac{1}{1-q}$. In the limit of low frequencies ($h\nu \ll kT$), the denominator in Eq. (4) can be expanded to first order, leading to

$$dE_\nu = \frac{8\pi V}{c^3}kT\nu^2 d\nu. \tag{6}$$

Equation (6) is the classical Rayleigh-Jeans law. It is worthwhile to notice that it is identical to the usual Rayleigh-Jeans law. In fact and as mentioned by Kaniadakis [34], the effect of the κ deformation is appreciable only for large values of x [in fact, the first three terms in the Taylor expansion of $\exp_\kappa(x)$ are the same as for the ordinary exponential]. Since $\exp_\kappa(x)$ is a monotonically increasing function of x, the limit of high frequencies ($h\nu \gg kT$) gives

$$dE_\nu = \frac{8\pi V}{c^3}\frac{h\nu^3}{\exp_\kappa\left(\frac{h\nu}{kT}\right)}d\nu, \tag{7}$$

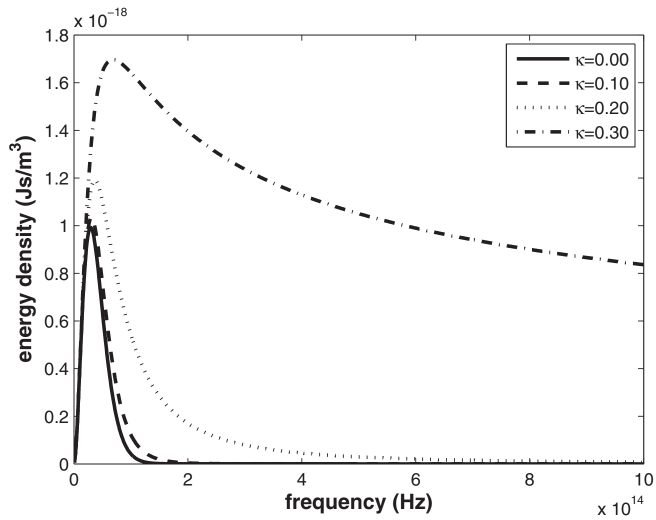

FIG. 1. Variation of the energy density with the frequency, for different values of the Kaniadakis parameter $\kappa = 0.00$ (solid line), 0.10 (dashed line), 0.20 (dotted line), and 0.30 (dash-dotted line), with $T = 500$ K.

which can be viewed as a generalization of the Wien shift law. In this limit, the effect of the Kaniadakis κ statistics is noticeable, since it shows an asymptotic power-law behavior instead of the exponentially decreasing law inherent in BGS statistics. It is worth noticing that the same power-law behavior is shown by the Tsallis q statistics for $q > 1$, with the correspondence $\kappa \to (q-1)$. However, in the $q < 1$ case, the behavior is very different since the q-generalized law exhibits a cutoff. In the limit $\kappa \to 0$, Eq. (7) reduces to the usual Wien shift law

$$dE_\nu = \frac{8\pi V}{c^3}\exp\left(-\frac{h\nu}{kT}\right)h\nu^3 d\nu. \tag{8}$$

From Eq. (4), one can obtain the following energy density:

$$\rho_\kappa(\nu,T) = \frac{8\pi h\nu^3}{c^3}\frac{1}{\exp_\kappa\left(\frac{h\nu}{kT}\right) - 1} \tag{9}$$

The frequency dependence of the energy density is then traced in Fig. 1 for different values of the Kaniadakis parameter κ and $T = 500$ K. Figure 1 shows that an increase of κ leads to an increase in the energy emitted by the blackbody, especially in the high-frequency region. This result is very similar to the one produced by nonextensive effects with $q > 1$. It is interpreted in the nonextensive q statistics as due to long-range correlations in the system energy [7]. Such an interpretation in the framework of κ statistics remains an open question.

III. GENERALIZED EINSTEIN COEFFICIENTS

Shortly before the development of quantum theory, Einstein postulated on thermodynamic grounds a relationship between the probabilities for spontaneous and stimulated emission. Later on, the proposed relationship was confirmed by quantum calculations. The so-called Einstein coefficients allow an understanding of several radiative processes such as the absorption and scattering of light and the amplification of light beams by lasers. In Ref. [7], it was shown that the q-generalized coefficients of stimulated emission and absorption are different from those already predicted by the BGS formalism. It becomes of interest to investigate whether the Kaniadakis κ statistics affect these coefficients and to what extent. Following Ref. [7], let us consider a cavity containing N identical atoms, each having two nondegenerate energy states E_1 and E_2. The number of photons in the cavity varies with absorption and emission by the atoms, with a frequency ν corresponding to $E_2 - E_1 = h\nu$. Let A_{21} be the probability of a spontaneous transition from E_2 to E_1, $\rho_\kappa(\nu,T)B_{21}$ the probability of stimulated emission from E_2 to E_1, and $\rho_\kappa(\nu,T)B_{12}$ the absorption probability from E_1 to E_2. Let N_1 and N_2 stand for the numbers of atoms in the energy states E_1 and E_2, respectively. It is easy to determine them in the canonical ensemble (since the number of atoms remains constant). The fraction of atoms in each level (which corresponds to the probability of finding the atom in the corresponding state) is given in the Kaniadakis κ statistics by

$$p_1 = \frac{N_1}{N} = \frac{1}{Z}\exp_\kappa(-E_1/kT) \quad \text{and}$$
$$p_2 = \frac{N_2}{N} = \frac{1}{Z}\exp_\kappa(-E_2/kT), \tag{10}$$

where Z stands for the partition function. In the limit $\kappa \to 0$, Eq. (10) reduces to the well-known Boltzmann weight $\exp(-E_i/kT)$. This expression can be determined by a maximization of the Kaniadakis entropy. Let us however derive them in a less orthodox way, considering an alternative method recently proposed by Oikonomou *et al.* [35]. From thermodynamical considerations, the authors showed that such an equilibrium probability distribution related to an entropic form[1] written as

$$S = \sum_i p_i \Lambda(1/p_i) \tag{11}$$

is given by

$$p_i = \frac{[\Lambda^{-1}(\beta E_i)]^{-1}}{Z}, \tag{12}$$

where β denotes the inverse temperature in energy units $\beta = 1/kT$. The Kaniadakis entropy is defined as the average of the function [20]

$$\Lambda(x) \equiv \ln_\kappa = \frac{x^\kappa - x^{-\kappa}}{2\kappa}, \tag{13}$$

which is a one-parameter generalization of the logarithm. In the limit $\kappa \to 0$, it reduces to the usual logarithm and the Kaniadakis entropy of course reduces to the usual BGS entropy. The inverse function of $\ln_\kappa$ is the κ exponential presented above (in fact, $\ln_\kappa[\exp_\kappa(x)] = x$). The equilibrium probability distributions are then given by (10). The conservation of the number of atoms leads to

$$\frac{dN_1}{dt} = -\frac{dN_2}{dt} = N_2A_{21} - N_1\rho_\kappa(\nu,T)B_{12} + N_2\rho_\kappa(\nu,T)B_{21}. \tag{14}$$

[1]The method considered here is valid for all trace-form entropies. Note that it apparently does not apply to non-trace-form entropies.

Considering the last equation at thermal equilibrium ($\frac{dN_1}{dt} = \frac{dN_2}{dt} = 0$), we obtain the energy density

$$\rho_\kappa(\nu,T) = \frac{A_{21}}{(N_1/N_2)B_{12} - B_{12}}. \tag{15}$$

Substituting Eq. (10) into Eq. (15), we obtain

$$\rho_\kappa(\nu,T) = \frac{A_{21}}{\{\exp_\kappa(-E_1/kT)/\exp_\kappa(-E_2/kT)\}B_{12} - B_{12}}$$
$$= \frac{A_{21}/B_{21}}{\{\exp_\kappa(-E_1/kT)/\exp_\kappa(-E_2/kT)\}B_{12}/B_{21} - 1}. \tag{16}$$

Equation (16) must be consistent with the κ-generalized Planck law (9). We derive therefore the following generalized Einstein's coefficients in the Kaniadakis κ statistics:

$$\frac{A_{21}}{B_{21}} = \frac{8\pi h\nu^3}{c^3}, \tag{17}$$

$$\frac{B_{21}}{B_{12}} = \frac{\exp_\kappa(-E_2/kT)}{\exp_\kappa(-E_1/kT)\exp_\kappa(-h\nu/kT)}. \tag{18}$$

The first ratio (17) is independent of the continuous parameter κ and is identical to the one arising from the BGS formalism [36]. It has been shown that it remains unchanged in the Tsallis q statistics also [7]. The second ratio is κ dependent. In the limit $\kappa \to 0$, the coefficients of stimulated emission and absorption are equal and their ratio is equal to unity. The fact that the latter Einstein coefficient is temperature dependent deserves discussion. In fact, the usual Einstein coefficients are properties of the system and thus depend only on its quantum mechanical nature. It is natural to ask why this ratio is temperature dependent while it is a constant in the usual BGS framework. We believe that the key is in the fundamental question: under what circumstances is this procedure justified? In Ref. [36], one can read a part of the answer: "The Einstein

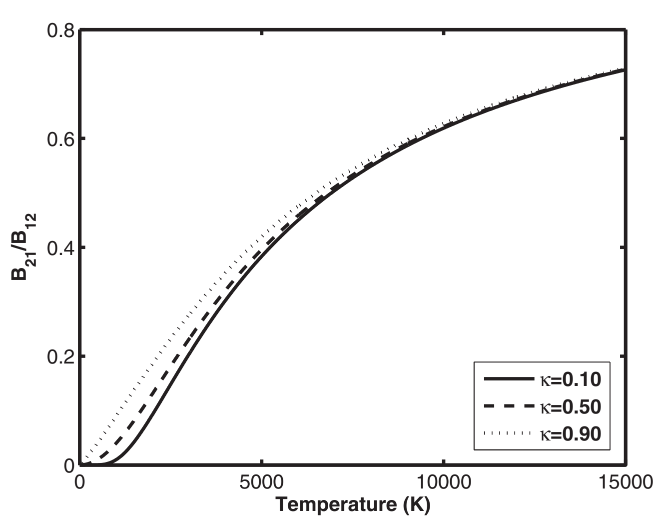

FIG. 2. Variation of the ratio B_{21}/B_{12} with the temperature, for different values of the Kaniadakis parameter $\kappa = 0.10$ (solid line), 0.50 (dashed line), and 0.90 (dotted line), with $\nu = 10^{14}$ Hz and a given E_1.

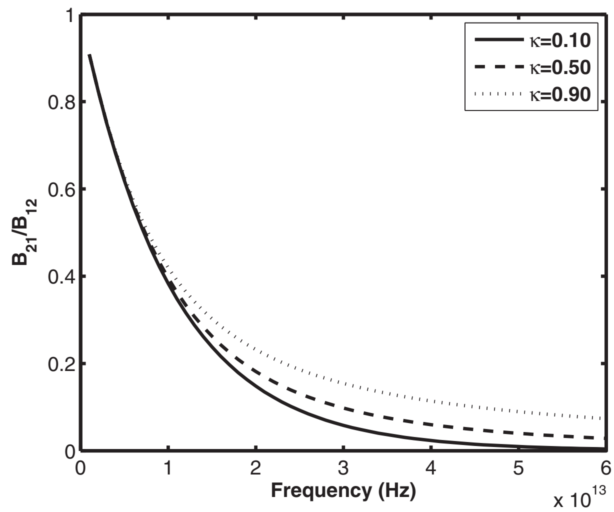

FIG. 3. Variation of the ratio B_{21}/B_{12} with the frequency, for different values of the Kaniadakis parameter $\kappa = 0.10$ (solid line), 0.50 (dashed line), and 0.90 (dotted line), with $T = 1000$ K and a given E_1.

coefficients are derived by consideration of a cavity in thermal equilibrium, where the radiative energy is homogeneous and isotropic in space. They hold generally for any spatially isotropic distribution of radiative energy density. However, the external light beams used in experiments do not usually have this property, as in the example of a parallel light beam." In Ref. [7], dealing with a Tsallis q-statistical approach of these coefficients, the authors relate this temperature dependence to a symmetry problem. They write: "This study would give us an opportunity to [$\cdots$] investigate the relation between this new theory (i.e., Tsallis q statistics) and physical systems in the non-Euclidean space-lime (e.g., fractal or mutifractal space-time)." Based upon these arguments, our interpretation is the following: If the radiative energy is homogeneous and isotropic in space, the rates of absorption and stimulated emission are the same ($B_{12} = B_{21} = 1$). Otherwise, the symmetry of the problem may produce a difference between them and favor one of the processes with respect to the other. Equation (18) tells us that this effect depends on the temperature of the cavity.

Figure 2 depicts the variation of the ratio (18) with temperature, for different values of κ, a constant frequency $\nu = 10^{14}$ Hz, and a given E_1. Figure 2 shows that an increase of κ leads to an increase of the ratio B_{21}/B_{12}, i.e., an increase (decrease) of the probability of emission (absorption). The effect of the Kaniadakis κ distribution is more marked for low temperatures and tends to vanish in the limit $T \to \infty$ (the emission and absorption coefficients tend to be equal). Note that in the zero-temperature limit, $B_{21}/B_{12} = 0$, notwithstanding the strength of the κ parameter. The coefficient of absorption is much superior to the emission coefficient (black-hole effect). This result is similar to the one observed for $q > 1$ in the q generalization of the Einstein coefficients [7]. In Fig. 3 is plotted the variation of the same ratio with frequency for different values of κ, with a given E_1 and a temperature $T = 1000$ K. It is found that in the limit of vanishingly small frequency ($\nu \to 0$), the ratio $B_{21}/B_{12} = 1$, a result that is consistent with the one obtained within the theoretical framework of q statistics [7]. The ratio then decreases with an increase of the frequency. The falloff of B_{21}/B_{12} becomes less rapid as κ increases (Fig. 3).

IV. CONCLUSION

To conclude, we have proposed an alternative generalization to the Planck law for blackbody radiation, using the counterpart of the Bose-Einstein distribution in the Kaniadakis κ statistics. A generalized Plank radiation law (which reduces to the usual one for $\kappa \to 0$) is derived. The Kaniadakis κ statistics appears to have a similar effect on the Plank law to the one produced by the nonextensive q statistics for $q > 1$. In fact, the κ effect enhances the energy emitted by the blackbody, particularly in the high-frequency region. The so-called Einstein coefficients for emission and absorption were also reconsidered within the theoretical frame of the Kaniadakis κ statistics. Interestingly, it is shown that the Kaniadakis statistics keeps unchanged the first Einstein coefficient while the second one admits a generalized form within the present theoretical framework. The Kaniadakis entropy (as has already been shown for the Tsallis entropy [7]) induces a difference between the two Einstein coefficients B_{12} and B_{21}. The ratio B_{21}/B_{12} reduces to unity in the limit $T \to \infty$, rendering the effect of the Kaniadakis κ statistics undistinguishable. In the zero–temperature limit, we observed an interesting black-hole effect. In the limit $\kappa \to 0$, the two coefficients become equal, as is well known from BGS statistics. We believe that it may be of interest to investigate the effect of the Kaniadakis κ statistics on the thermodynamical quantities of the blackbody (free energy, entropy, total radiation energy, specific heat, etc.). This issue appears as one of the interesting prospects that can be proposed as a continuation of the present work. In fact, as already shown for the Tsallis q statistics, the thermodynamical relations must be form invariant under κ generalization. In addition, a comparison with experimental data in the case of the microwave background radiation would be welcome.

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