Performance optimization of minimally nonlinear irreversible heat engines and refrigerators under a trade-off figure of merit

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A performance optimization for minimally nonlinear heat engines and refrigerators is conducted under an optimization criterion of Ω. The results show that under tight-coupling conditions, the efficiency and coefficient of performance (COP) bounds in asymmetric dissipation limits are the same as those obtained by de Tomas *et al.* [Phys. Rev. E **87**, 012105 (2013)] for low dissipation heat devices. The efficiency bounds for heat engines under nontight-coupling conditions are also analyzed and the experimental results lie between theoretical results obtained under different coupling strengths. For refrigerators, the theoretical results are also in good agreement with some observed results. The efficiency and COP bounds under the Ω criterion are refined, which are closer to real heat engines and refrigerators.

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I. INTRODUCTION

Conditioned on energy saving and fuel depletion, the optimization of real heat devices has attracted much attention. Carnot efficiency $\eta_C = 1 - T_c/T_h$ defines the maximum energy conversion rate of real heat engines [1], as does the Carnot coefficient of performance $\varepsilon_C = T_c/(T_h - T_c)$ of real refrigerators, where T_h and T_c denote the temperatures of the hot and cold reservoirs, respectively. All processes in Carnot heat devices are quasistatic. This means that the time duration for completing a cycle is infinitely long, which leads to zero power extracted for heat engines or a zero cooling load rate for refrigerators. Therefore, Carnot devices must be speeded up to meet the actual demands. By considering finite durations of the heat transfer processes between the heat reservoirs and working fluid, Curzon-Ahlborn (CA) proposed the upper bound of efficiency ($\eta_{\mathrm{CA}} = 1 - \sqrt{T_c/T_h}$) for heat engines working at maximum power output conditions [2]. This opened a new chapter of thermodynamics, i.e., finite time thermodynamics, to which much effort has been devoted [3,4].

For heat engines, the efficiency at maximum power (EMP) is often adopted as the criterion for optimization. Taking into account the entropy generation in isothermal processes, which are treated as inversed functions of process time duration, i.e., $\sum_h/\tau_h$ and $\sum_c/\tau_c$ in heat absorbing and releasing processes, respectively, Esposito *et al.* [5] proposed the low dissipation model, and then obtained the lower and upper bounds of the efficiency at EMP under asymmetric dissipation limits. In addition, under the symmetric dissipation condition, the CA efficiency is recovered. Later, further research on the low dissipation model by considering either the time duration and irreversibility in the adiabatic processes, or by treating the entropy generation in the isothermal processes as a quadratic form of heat exchange rate between the working media and reservoirs under the EMP criterion, is also conducted [6–8]. Furthermore, provided that the temperature difference between the hot and cold reservoirs is very small ($T_h \approx T_c$), Van den Broeck [9] investigated linear irreversible heat engines, which can be described by the following Onsager relations [10]:

$$J_1 = L_{11}X_1 + L_{12}X_2, \tag{1}$$

$$J_2 = L_{21}X_1 + L_{22}X_2, \tag{2}$$

where $X_1 = F/T$ under an external force F, $J_1 = \dot{x}$ is the derivative of the conjugate variable x of F, $X_2 \approx \Delta T/T^2$, $J_2 = \dot{Q}_h$, and the Onsager coefficients (L_{ij}) with the reciprocity $L_{12} = L_{21}$ satisfy the relations $L_{11} \geqslant 0$, $L_{22} \geqslant 0$, and $L_{11}L_{22} - L_{12}L_{21} \geqslant 0$. In his paper, the CA efficiency is retrieved. By adding a nonlinear term to the linear relations to consider the power dissipation, Izumida and Okuda [11] obtained the same upper bound as that in Ref. [5]. Moving further, they demonstrated that the model could describe the low dissipation models for both heat engines and refrigerators [11,12]. However, the arbitrariness in selecting the thermal flux may result in fatal consequences of the optimization for refrigerators, and Sheng and Tu considered a more general model with weighted reciprocal of temperature and thermal flux [13,14].

For refrigerators, the EMP criterion is not an appropriate figure of merit for optimization [15]. Yan and Chen [16] proposed a new criterion $\varepsilon\dot{Q}_c$, where ε is the coefficient of performance and $\dot{Q}_c$ is the cooling load rate. Later, de Tomas [17] developed this criterion and defined a unified optimization figure of merit $\chi = zQ_{\mathrm{in}}/t_{\mathrm{cycle}}$, which is appropriate for both heat engines and refrigerators, where z is the converter efficiency (η for heat engines and ε for refrigerators), Q_{in} is the heat absorbed by the system, and t_{cycle} denotes the time duration for a cycle. χ becomes the EMP and $\varepsilon\dot{Q}_c$ the figure of merits for heat engines and refrigerators, respectively. Wang *et al.* [15] selected χ as the figure of merit and obtained the lower and upper bounds of the COP for the maximum figure of merit for low dissipation refrigerators, i.e., $0 < \varepsilon_{\max\chi} < (\sqrt{9 + 8\varepsilon_C} - 3)/2$. They declared that their theoretical prediction agreed well with some real refrigerators.

Actual heat engines or refrigerators may not work at their maximum power output or maximum cooling load rate, but might work under a compromise between energy benefits and losses. Hernández *et al.* [18] proposed a new figure of merit Ω, accounting for both the energy benefits and

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losses. Using the Ω criterion, de Tomas *et al.* [19] obtained the efficiency and COP of low dissipation heat engines and refrigerators under asymmetric limits. The COP of low dissipation refrigerators with irreversibility in the adiabatic processes was also considered by Hu *et al.* [20] under the Ω criterion. In addition, by comparing the bounds and efficiencies of heat engines described by different models under the EMP criterion and Ω criterion, Sánchez-Salas *et al.* [21] showed the maximum Ω regime was more efficient. Furthermore, Aperter *et al.* [22] declared that the real refrigerators do not operate under the maximum cooling power condition but under the trade-off between the cooling power and the COP. However, under the trade-off figure of merit (Ω), the proposed lower bounds are very close to the upper bounds and are not in accord with observed efficiencies or COPs for heat engines and refrigerators, respectively. In the present paper, we systemically discuss the efficiency and COP for minimally nonlinear heat engines and refrigerators under the figure of merit Ω in Secs. II and III, respectively. Our results are compared with experimental ones, both for heat engines and refrigerators. Finally, in Sec. IV some important conclusions are drawn.

II. HEAT ENGINES AND THE Ω CRITERION

For heat engines, a certain amount of heat $\dot{Q}_h$ is absorbed from the hot reservoir (T_h), some of which ($\dot{Q}_c$) is evacuated to the cold reservoir (T_c) at the end of a cycle. Meanwhile, the work is produced. After a cycle, the working fluid in the heat engine returns to its initial state; therefore, its entropy change per cycle is zero. The total entropy production rate $\dot{\sigma}$ of the heat engine can be written as

$$\dot{\sigma} = -\frac{\dot{Q}_h}{T_h} + \frac{\dot{Q}_c}{T_c} = -\frac{\dot{W}}{T_c} + \dot{Q}_h\left(\frac{1}{T_c} - \frac{1}{T_h}\right), \tag{3}$$

where the dot denotes the quantity per unit time for simultaneous heat engines or the quantity divided by the cycle time duration for sequential heat engines. We assume the system performs work W against an external force F with its conjugate variable x. The corresponding thermodynamic force and its conjugate flux can be defined as $X_1 = F/T_c$ and $J_1 = \dot{x}$, respectively. The power can be rewritten as $\dot{W} = -F\dot{x} = -J_1X_1T_c$; the other thermodynamic force and its conjugate flux can be also defined as $X_2 = 1/T_c - 1/T_h$ and $J_2 = \dot{Q}_h$, respectively. Furthermore, extended Onsager relations are adopted to describe the heat engines [11]:

$$J_1 = L_{11}X_1 + L_{12}X_2, \tag{4}$$

$$J_2 = L_{21}X_1 + L_{22}X_2 - \gamma_h J_1^2. \tag{5}$$

The characteristics of the Onsager coefficients in Eqs. (1) and (2) are assumed to hold in Eqs. (4) and (5). The nonlinear term $-\gamma_h J_1^2$ is interpreted as the power dissipation due to the hot reservoir, where γ_h denotes its strength ($\gamma_h > 0$). The heat evacuated to the cold reservoir can be calculated as

$$\dot{Q}_c = \dot{Q}_h - \dot{W} = \dot{Q}_h + J_1X_1T_c \equiv J_3. \tag{6}$$

According to Eq. (4), Eqs. (5) and (6) can be rewritten as

$$J_2 = \frac{L_{21}}{L_{11}}J_1 + L_{22}(1-q^2)X_2 - \gamma_h J_1^2, \tag{7}$$

$$J_3 = \frac{L_{21}}{L_{11}}\frac{T_c}{T_h}J_1 + L_{22}(1-q^2)X_2 + \gamma_c J_1^2, \tag{8}$$

where $q = L_{12}/\sqrt{L_{11}L_{22}}$ is the dimensionless coupling strength ($|q| \leqslant 1$ [9]). The nonlinear term $\gamma_c J_1^2$ characterizes the power dissipation due to the cold reservoir, where $\gamma_c = T_c/L_{11} + \gamma_h$ and denotes its strength ($\gamma_c > 0$). The existence of the two nonlinear terms results in the decrease of the heat absorbed by the working fluid, which can be transformed into work, and the increase of the heat released to the cold reservoir.

The power can be rewritten as

$$\dot{W} = \frac{L_{21}}{L_{11}}\eta_C J_1 - \frac{T_c}{L_{11}}J_1^2 \tag{9}$$

and the efficiency is given by

$$\eta = \frac{\dot{W}}{\dot{Q}_h} = \frac{\frac{L_{21}}{L_{11}}\eta_C J_1 - \frac{T_c}{L_{11}}J_1^2}{\frac{L_{21}}{L_{11}}J_1 + L_{22}(1-q^2)X_2 - \gamma_h J_1^2}. \tag{10}$$

The Ω criterion, taking into account both the maximum work extracted and the losses, is defined as $\Omega = (2\eta - \eta_{\max})Q_h$ [18]. Then, the target function $\dot{\Omega} = (2\eta - \eta_{\max})\dot{Q}_h$ is expressed as

$$\dot{\Omega} = 2\dot{W} - \eta_C J_2 = -\left(2\frac{T_c}{L_{11}} - \eta_C\gamma_h\right)J_1^2 + \frac{L_{21}}{L_{11}}\eta_C J_1 - \eta_C L_{22}(1-q^2)X_2. \tag{11}$$

By taking the derivative $\dot{\Omega}$ with respect to J_1, we let $\partial\dot{\Omega}/\partial J_1 = 0$. Furthermore, the second derivative of $\partial\dot{\Omega}/\partial J_1$ satisfies $\partial^2\dot{\Omega}/\partial J_1^2 < 0$, which means $\dot{\Omega}$ achieves its maximum value at $J_{1,\max\dot{\Omega}}$. Then, we have

$$J_{1,\max\dot{\Omega}} = \frac{\frac{L_{21}}{L_{11}}\eta_C}{4\frac{T_c}{L_{11}} - 2\eta_C\gamma_h}. \tag{12}$$

Substituting Eq. (12) into Eq. (10), we obtain the general efficiency under the maximum $\dot{\Omega}$ criterion:

$$\eta_{\max\dot{\Omega}} = \eta_C\frac{3\beta - 2\eta_C}{4\beta - 3\eta_C + \frac{1}{\beta}\left(\frac{1}{q^2} - 1\right)(4\beta - 2\eta_C)^2}, \tag{13}$$

where $\beta = \frac{T_c}{L_{11}\gamma_h} = \frac{\gamma_c}{\gamma_h} + 1$, and γ_c/γ_h denotes the ratio of power dissipation due to the cold reservoir and that by the hot reservoir. According to $\gamma_c = T_c/L_{11} - \gamma_h > 0$, $1 < \beta < \infty$.

Therefore, we can obtain the lower and upper bounds of $\eta_{\max\dot{\Omega}}$ by considering the asymmetrical dissipation limits $\beta \to \infty$ and $\beta \to 1$, respectively.

$$\frac{3}{4 + 16\left(\frac{1}{q^2} - 1\right)}\eta_C \leqslant \eta_{\max\dot{\Omega}} \leqslant \frac{3 - 2\eta_C}{4\left(\frac{1}{q^2} - 1\right)\eta_C^2 - \left[3 + 16\left(\frac{1}{q^2} - 1\right)\right]\eta_C + 4 + 16\left(\frac{1}{q^2} - 1\right)}\eta_C. \tag{14}$$

The upper and lower bounds monotonously decrease with the decrease of the square of the coupling strength q^2 and achieve their maximum values under the coupling strength limits $|q| \to 1$.

$$\frac{3}{4}\eta_C \leqslant \eta_{\max\dot{\Omega}} \leqslant \frac{3-2\eta_C}{4-3\eta_C}\eta_C \equiv \eta^{+}_{\max\dot{\Omega}} \tag{15}$$

The result in Eq. (15) is the same as that in Ref. [19], which is obtained by the low dissipation model. Although the left term in Eq. (15) has the same expression as the lower bound proposed in Ref. [19], in this paper the lower bound will be discussed further under the nontight-coupling strengths. Furthermore, under the tight-coupling condition ($|q| = 1$), according to Eq. (13), the efficiency under the maximum trade-off criterion under symmetric dissipation ($\beta = 2$) is

$$\eta^{\rm sym}_{\max\dot{\Omega}} = \eta_C\frac{6-2\eta_C}{8-3\eta_C} = \frac{3}{4}\eta_C + \frac{1}{32}\eta_C^2 + \frac{3}{256}\eta_C^3 + O(\eta_C^4), \tag{16}$$

while the efficiency under the maximum trade-off criterion in the endoreversible model [18] is

$$\eta^{\rm endo}_{\max\dot{\Omega}} = 1 - \sqrt{\frac{(1-\eta_C)(2-\eta_C)}{2}} = \frac{3}{4}\eta_C + \frac{1}{32}\eta_C^2 + \frac{3}{128}\eta_C^3 + O(\eta_C^4). \tag{17}$$

Comparing Eqs. (16) and (17), under the symmetric dissipation condition, the efficiency at maximum trade-off criterion is equivalent to that obtained through the endoreversible model to accuracy of the second order of η_C. This means the model in this paper could describe the endoreversible Carnot heat engines. Furthermore, Eq. (17) is also obtained through the low dissipation model under the symmetric dissipation condition [19]. According to Ref. [11], the low dissipation Carnot heat engine can be described by the minimally nonlinear irreversible model with tight-coupling strength. Substituting Eq. (12) into Eq. (11), we have

$$\dot{\Omega} = \frac{\left(\frac{L_{21}}{L_{11}}\eta_C\right)^2}{8\frac{T_c}{L_{11}} - 4\eta_C\gamma_h}. \tag{18}$$

Combining Eqs. (30), (32), and (34) in Ref. [11] and maximizing Eq. (18) with respect to $\lambda = \tau_c/\tau_h$, then the optimal $\lambda_{\rm opt}$ is obtained below:

$$\lambda_{\rm opt} = \sqrt{\frac{2\sum_c(1-\eta_C)}{\sum_h(2-\eta_C)}}. \tag{19}$$

Furthermore, $\beta_{\rm opt} = (1-\eta_C)/\lambda_{\rm opt} + 1$. Substituting it into Eq. (13), the same expression as Eq. (13) in Ref. [19] can be recovered. However, due to the different optimization space, the expressions of efficiencies for the two models under the corresponding symmetric dissipation condition are not the same. By maximizing $\dot{\Omega}(\lambda)$ with respect to λ, the equivalence can be established. As mentioned before, those two efficiencies are equal to the second order of η_C. This means for the minimally nonlinear irreversible model under the symmetric dissipation condition, the time duration ratio has little impact on the efficiency of the heat engines.

To step further, for the minimally nonlinear irreversible model under the symmetric dissipation condition ($\gamma_h = \gamma_c$), the irreversible entropy production ratio in the heat exchanging process is

$$\frac{\Sigma_c}{\Sigma_h} = \frac{2}{(1-\eta_C)(2-\eta_C)}, \tag{20}$$

while $\Sigma_c/\Sigma_h = 1$ for the low dissipation model under the symmetric condition. It can explain the little difference between those two efficiencies under the relevant symmetric conditions.

For nontight-coupling heat engines, where $|q| < 1$, the second terms in Eqs. (7) and (8) are the heat leak from the hot reservoir to the cold one. They do not have any impact on the power output, but decrease the corresponding efficiency. Furthermore, as the Ω criterion representing a compromise between heat benefits and losses, the heat leak increases heat loss and should have a significant impact on the efficiency under that criterion. Therefore the efficiency should be much smaller. As seen in Eq. (14), the lower bound is determined by the coupling strength, which is hard to measure, and depends on the design and manufacture of the heat engines and operation conditions. The experimental data and theoretical results for heat engines are compared and plotted in Fig. 1. All the observed efficiencies are under the upper bound $\eta^{+}_{\max\dot{\Omega}}$ and located between the theoretical efficiencies calculated at $q^2 = 1$ and $q^2 = 0.8$. That means real heat engines usually operate between the above two different coupling strengths in the Ω perspective. However, the lower bound proposed by de Tomas *et al.* [19] using the low dissipation model, is much higher than the experimental results. Therefore, they declared that many heat engines are designed to work at higher velocity, instead of levels of efficiency and energy saving. Actually, energy saving and efficiency should both be taken into consideration in Diesel and Otto engines in automobiles and trucks. The efficiency bounds proposed by our model fit well with the experimental ones by considering different coupling strengths. The results obtained in this paper

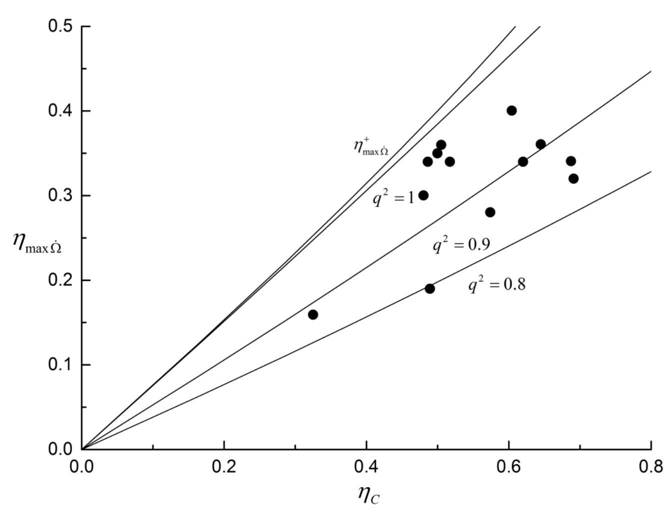

FIG. 1. Comparison between experimental results (dots) [19] and theoretical results for heat engines. The results are calculated under the symmetric dissipation condition ($\beta = 2$) and different coupling strengths ($q^2 = 0.8$, 0.9, and 1). Furthermore, the upper bound $\eta^{+}_{\max\dot{\Omega}}$ is also plotted.

could offer a more insightful perspective to study real-life heat engines.

III. REFRIGERATORS AND THE Ω CRITERION

For refrigerators, the cooling load ($\dot{Q}_c$) is absorbed from the cold reservoir (T_c) and a certain amount of heat ($\dot{Q}_h$) is evacuated to the hot reservoir (T_h) at the end of a cycle. After a cycle, the working fluid in the heat engine returns to its initial state; therefore, its entropy change per cycle is zero. The total entropy production rate $\dot{\sigma}$ of the heat engine can be written as

$$\dot{\sigma} = \frac{\dot{Q}_h}{T_h} - \frac{\dot{Q}_c}{T_c} = \frac{\dot{W}}{T_h} + \dot{Q}_c\left(\frac{1}{T_h} - \frac{1}{T_c}\right), \tag{21}$$

where the dot denotes the quantity per unit time for simultaneous refrigerators or the quantity divided by the cycle time duration for sequential refrigerators. The first thermodynamic force and its conjugate flux can be defined as $X_1 = F/T_c$ and $J_1 = \dot{x}$, respectively. The input power can be rewritten as $\dot{W} = F\dot{x} = J_1 X_1 T_h$; the other thermodynamic force and its conjugate flux can also be defined as $X_2 = 1/T_h - 1/T_c$ and $J_2 = \dot{Q}_c$, respectively. Furthermore, extended Onsager relations are adopted to illustrate the heat engines [12]:

$$J_1 = L_{11}X_1 + L_{12}X_2, \tag{22}$$

$$J_2 = L_{21}X_1 + L_{22}X_2 - \gamma_c J_1^2. \tag{23}$$

The characteristics of the Onsager coefficients in Eqs. (1) and (2) are assumed to hold in Eqs. (22) and (23). The nonlinear term $-\gamma_c J_1^2$ is interpreted as the power dissipation into the cold reservoir, where γ_c denotes its strength ($\gamma_c > 0$). The heat evacuated to the hot reservoir can be calculated as

$$\dot{Q}_h = \dot{Q}_c + \dot{W} = \dot{Q}_c + J_1 X_1 T_h \equiv J_3. \tag{24}$$

According to Eq. (22), Eqs. (23) and (24) can be rewritten as

$$J_2 = \frac{L_{21}}{L_{11}}J_1 + L_{22}(1-q^2)X_2 - \gamma_c J_1^2, \tag{25}$$

$$J_3 = \frac{L_{21}}{L_{11}}\frac{T_h}{T_c}J_1 + L_{22}(1-q^2)X_2 + \gamma_h J_1^2. \tag{26}$$

The nonlinear term $\gamma_h J_1^2$ characterizes the power dissipation into the hot reservoir, where $\gamma_h = T_h/L_{11} - \gamma_c$ denotes its strength ($\gamma_h > 0$). The existence of the two nonlinear terms results in the decrease of the heat absorbed by the working fluid and the increase of the heat released to the heat reservoir.

The input power can be rewritten as

$$\dot{W} = \frac{L_{21}}{L_{11}\varepsilon_C}J_1 + \frac{T_h}{L_{11}}J_1^2 \tag{27}$$

and the coefficient of performance is given by

$$\varepsilon = \frac{\dot{Q}_c}{\dot{W}} = \frac{\frac{L_{21}}{L_{11}}J_1 + L_{22}(1-q^2)X_2 - \gamma_c J_1^2}{\frac{L_{21}}{L_{11}\varepsilon_C}J_1 + \frac{T_h}{L_{11}}J_1^2}. \tag{28}$$

The Ω criterion is defined as $\Omega = (2\varepsilon - \varepsilon_{\max})W$ in Ref. [18]. Then, the target function $\dot{\Omega} = (2\varepsilon - \varepsilon_{\max})\dot{W}$ can be expressed as

$$\dot{\Omega} = 2J_2 - \varepsilon_C\dot{W} = \frac{L_{21}}{L_{11}}J_1 + 2L_{22}(1-q^2)X_2 - \left(2\gamma_c + \varepsilon_C\frac{T_h}{L_{11}}\right)J_1^2. \tag{29}$$

By taking the derivative of $\dot{\Omega}$ with respect to J_1, we let $\partial\dot{\Omega}/\partial J_1 = 0$. Furthermore, the second derivative of $\partial\dot{\Omega}/\partial J_1$ satisfies $\partial^2\dot{\Omega}/\partial J_1^2 < 0$, which means $\dot{\Omega}$ achieves its maximum value at $J_{1,\max\dot{\Omega}}$. Then, we have

$$J_{1,\max\dot{\Omega}} = \frac{\frac{L_{21}}{L_{11}}}{4\gamma_c + 2\varepsilon_C\frac{T_h}{L_{11}}}. \tag{30}$$

Substituting Eq. (30) into Eq. (28), we obtain the general efficiency under the maximum $\dot{\Omega}$ criterion.

$$\varepsilon_{\max\dot{\Omega}} = \varepsilon_C\frac{3 + 2\varepsilon_C\alpha - \frac{1}{\varepsilon_C\alpha}\left(\frac{1}{q^2} - 1\right)(4 + 2\varepsilon_C\alpha)^2}{4 + 3\varepsilon_C\alpha}, \tag{31}$$

where $\alpha = \frac{T_h}{L_{11}\gamma_c} = \gamma_h/\gamma_c + 1$ and γ_h/γ_c denotes the ratio of power dissipation into the hot reservoir and that into the cold reservoir. According to $\gamma_h = T_h/L_{11} - \gamma_c > 0$, we obtain $1 < \alpha < \infty$.

Under the tight-coupling condition $|q| = 1$ Eq. (31) can be rewritten as

$$\varepsilon_{\max\dot{\Omega}} = \frac{3 + 2\varepsilon_C\alpha}{4 + 3\varepsilon_C\alpha}\varepsilon_C. \tag{32}$$

We can also obtain the lower and upper bounds of $\varepsilon_{\max\dot{\Omega}}$ under the tight-coupling condition by considering the asymmetrical dissipation limits $\alpha \to \infty$ and $\alpha \to 1$, respectively.

$$\frac{2}{3}\varepsilon_C \leqslant \varepsilon_{\max\dot{\Omega}} \leqslant \frac{3 + 2\varepsilon_C}{4 + 3\varepsilon_C}\varepsilon_C \equiv \varepsilon^+_{\max\dot{\Omega}}. \tag{33}$$

This expression is in accord with that in Refs. [19] and [20] for low dissipation refrigerators. But the lower bound is not practical, which will be discussed further in the following.

Under the symmetric dissipation condition ($\alpha = 2$), Eq. (32) becomes

$$\varepsilon^{\text{sym}}_{\max\dot{\Omega}} = \frac{3 + 4\varepsilon_C}{4 + 6\varepsilon_C}\varepsilon_C = \frac{2}{3}\frac{1}{\frac{1}{\varepsilon_C} - \frac{1}{12\varepsilon_C^2} + \frac{1}{16\varepsilon_C^3} + O\left(1/\varepsilon_C^4\right)}. \tag{34}$$

The COP under the maximum trade-off criterion in the endoreversible model [18] is

$$\varepsilon^{\text{endo}}_{\max\dot{\Omega}} = \frac{\varepsilon_C}{\sqrt{(1+\varepsilon_C)(2+\varepsilon_C)} - \varepsilon_C} = \frac{2}{3}\frac{1}{\frac{1}{\varepsilon_C} - \frac{1}{12\varepsilon_C^2} + \frac{1}{8\varepsilon_C^3} + O\left(1/\varepsilon_C^4\right)}. \tag{35}$$

From Eqs. (34) and (35), the COP at maximum trade-off criterion is equivalent with that obtained through the endoreversible model to the second order of $1/\varepsilon_C$. Therefore, this model in this paper could also apply to the endoreversible Carnot refrigerators. Furthermore, Eq. (35) is also obtained through the low dissipation model under the symmetric dissipation condition [19]. According to Ref. [12], the low dissipation Carnot refrigerator can be described by the minimally nonlinear irreversible model with the tight-coupling strength. Substituting Eq. (30) into Eq. (29), we have

$$\dot{\Omega} = \frac{\left(\frac{L_{21}}{L_{11}}\right)^2}{8\gamma_c + 4\varepsilon_C \frac{T_h}{L_{11}}}. \tag{36}$$

According to Ref. [12] and maximizing Eq. (36) with respect to λ, the optimal $\lambda_{\rm opt}$ is obtained below:

$$\lambda_{\rm opt} = \sqrt{\frac{\sum_c (2+\varepsilon_C)}{\sum_h (1+\varepsilon_C)}}. \tag{37}$$

Furthermore, $\alpha_{\rm opt} = \lambda_{{\rm op}t}(1 + 1/\varepsilon_C) + 1$. Substituting it into Eq. (31), the same COP expression as Eq. (26) in Ref. [19] can be recovered. However, due to the different optimization space, the COPs under the symmetric dissipation condition for the two models are not the same. By maximizing $\dot{\Omega}(\lambda)$ with respect to λ, the equivalence is achieved. As mentioned before, those two COPs are equal to the second order of $1/\varepsilon_C$. This means for the minimally nonlinear irreversible refrigerator model under the symmetric dissipation condition, the time duration ratio also has little impact on the COP of the refrigerators.

To step further, for the minimally nonlinear irreversible model under the symmetric dissipation condition ($\gamma_h = \gamma_c$), the irreversible entropy production ratio in the heat exchanging processes is

$$\frac{\Sigma_c}{\Sigma_h} = \frac{(\varepsilon_C + 1)(\varepsilon_C + 2)}{\varepsilon_C^2}, \tag{38}$$

while $\Sigma_c/\Sigma_h = 1$ for the low dissipation model under the symmetric condition. It can explain the little difference between those two COPs under the relevant symmetric conditions.

Under nontight-coupling conditions $|q| < 1$, the second terms in Eqs. (25) and (26) indicate the coupling effects between the heat reservoirs. For actual refrigerators, they cannot be eliminated. As seen in Eq. (31), the COP is monstrously increasing function with respect to q^2 at fixed dissipation ratio. When $|q| \to 0$, the COP under the Ω criterion approximates negative infinite, while the negative values are of no physical meaning. The lower bound should be zero. That is to say, the COP bounds under the maximum trade-off criterion are

$$\varepsilon^-_{\max\dot{\Omega}} \equiv 0 \leqslant \varepsilon_{\max\dot{\Omega}} \leqslant \frac{3+2\varepsilon_C}{4+3\varepsilon_C}\varepsilon_C \equiv \varepsilon^+_{\max\dot{\Omega}}. \tag{39}$$

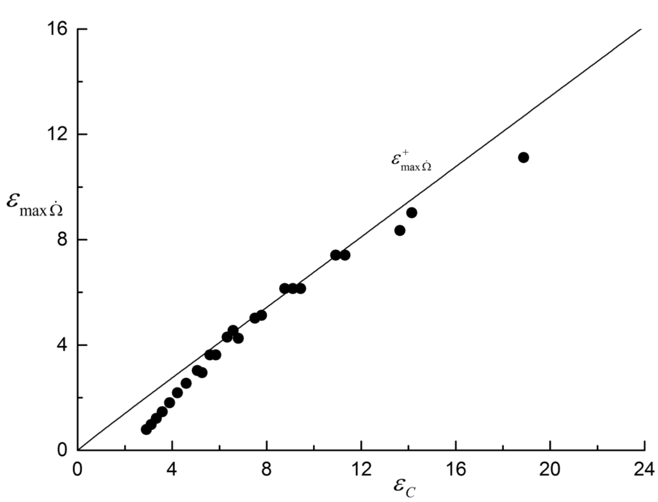

FIG. 2. Comparison between the experimental data of a nominal 1038-kW screw-compressor chiller [20] with theoretical results for refrigerators.

As shown in Fig. 2, good agreement between the experimental data and theoretical upper bound also exists for a kind of refrigerator working at a certain condition. This is a compromise between the energy benefits and losses. The lower bound proposed in Refs. [18] and [19] is also higher than the observed efficiency. However, according to our model the lower bound of COP is zero. That could offer a more insightful perspective to study real-life refrigerators.

IV. CONLUSIONS

The Ω criterion presents a compromise between the effective available energy and the losses. By applying this optimization criterion, the efficiency and COP are deduced for minimally nonlinear heat engines and refrigerators, respectively. Under the tight-coupling conditions, the lower and upper bonds of the efficiency and COP in asymmetric dissipation limits perfectly agree with those obtained through the low dissipation models for heat engines and refrigerators, respectively [19]. Due to the different optimization space, the efficiencies and COPs under the corresponding symmetric dissipation condition for the two models are not the same. Furthermore, the equivalence of the efficiencies and COPs under the trade-off criteria obtained from present models ($|q| = 1$) and through the low dissipation models can be achieved by further maximization of $\dot{\Omega}(\lambda)$ with respect to λ. The low dissipation model is just a special case of the minimally nonlinear irreversible models. The present model under the tight-coupling condition $|q| = 1$ is a sufficient rather than a necessary condition for recovering the low dissipation model, which specifies the the irreversible entropy production in each heat exchanging processes while the present model does not. Therefore, even under the tight-coupling conditions, the present model is more general and universal than the low dissipation one. In addition, for real-life heat engines and refrigerators, the coupling effects between the heat reservoirs should be considered. However the coupling strength is hard to be quantified. In this paper, some discussions on the nontight-coupling conditions have been conducted. The results are compared with experimental data, and good accordance exists both for heat engines and refrigerators, respectively. This paper could offer a more insightful perspective to study real-life heat engines and refrigerators.

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