PHYSICAL REVIEW E **89**, 062107 (2014)

Grassmannian representation of the two-dimensional monomer-dimer model

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(Received 24 February 2014; revised manuscript received 14 May 2014; published 4 June 2014)

We present an application of the Grassmann algebra to the problem of the monomer-dimer statistics on a two-dimensional square lattice. The exact partition function, or total number of possible configurations, of a system of dimers with a finite set of n monomers at fixed positions can be expressed via a quadratic fermionic theory. We give an answer in terms of a product of two pfaffians and the solution is closely related to the Kasteleyn result of the pure dimer problem. Correlation functions are in agreement with previous results, both for monomers on the boundary, where a simple exact expression is available in the discrete and continuous case, and in the bulk where the expression is evaluated numerically.

DOI: 10.1103/PhysRevE.89.062107 PACS number(s): 05.20.−y, 02.10.Yn, 05.50.+q

The study of the classical dimer model has a very long history in physics and mathematics. This model is interesting as a direct physical representation, e.g., diatomic molecules on a two-dimensional substrate [1]. From the mathematical point of view, this model on bipartite lattice—known as a special case of perfect matching problem [2]—is a famous and active problem of combinatorics and graph theory [3]. The partition function of the 2D dimer model was solved independently using pfaffian methods [4–6], resulting in the exact calculation of correlation functions of two monomers along a row (or a column) [7] or along a diagonal [8,9] in the infinite square lattice limit using Toeplitz determinants. For the general case of an arbitrary orientation, exact results are given in terms of the pair correlations of the 2D square lattice Ising model at the critical point using recurrence relations [10,11].

For the general dimer problem where an arbitrary number of monomers are present—the lattice sites that are not covered by the dimers are regarded as occupied by monomers—there is no exact solution except in 1D where the solution can be expressed in terms of Chebyshev polynomials [12], on the complete graph and on locally tree-like graphs [13]. We can also mention that the matrix transfer method was used to express the general monomer-dimer problem [14] (monomer density is not fixed), here the partition function, in terms of the maximum eigenvalue instead of a pfaffian. In particular, a very efficient method based on variational corner transfer matrix has been found by Baxter [15], leading to a precise approximation of thermodynamic quantities, such as the average dimer density, which can be evaluated accurately as function of the dimer*activity*. For 3D lattices, no exact solution exists for the pure close-packed dimer problem. Recent advances concern the analytic solution of the problem where there is a single monomer on the boundary of a 2D lattice [16,17], correlation functions for monomers located on the boundary [18,19], and localization phenomena of a monomer in the bulk [20,21].

The field of analytical solutions in the monomer-dimer model is still uncharted, but many rigorous results exist, e.g., location of the zeros of the partition function [22,23], series expansions of the partition function [24], and exact recursion relation [25]. This lack of exact solution has been formalized in the context of computer science [26]. The importance of the dimer model in theoretical physics and combinatorics also comes from the direct mapping between the square lattice Ising model without magnetic field and the dimer model on a decorated lattice [4–6,27] and oppositely from the mapping of the square lattice dimer model to a eight-vertex model [28,29]. Furthermore, the Ising model in a magnetic field can be mapped to the general monomer-dimer model [23].

Here we present a Grassmannian or fermionic formulation of the monomer-dimer model, which possesses an exact solution in terms of the product of two explicit pfaffians. We study the close-packed model, where an allowed dimer configuration has the property that each site of the lattice is paired with exactly one of its nearest neighbors, creating a dimer. In the simplest form, the number of dimers is the same in all the configurations, and the partition function is given by the equally weighted average over all possible dimer configurations. In the following, we will include unequal fugacities, so that the average to be taken then includes nontrivial weighting factors.

An early representation of the dimer model was introduced using Grassmann techniques [30,32]. A pair of these variables is attached to each site, preventing double occupancy of a site by two dimers. This leads to a direct representation of the partition function in terms of a fermionic integral over a quartic action, from which diagrammatic expansions can be carried out. We first review a very simple noncombinatorial interpretation of the 2D dimer model based on the integration over Grassmann variables [31–33], and factorization principles for the density matrix [34,35]. A dimer model can be described with Boltzmann weights t_x and t_y of some coupling energy along the two directions. For example, a magnetic field along one direction implies nonidentical weight values. The partition function for a lattice of size $(L \times L)$ with L even can directly be written as

$$\mathcal{Q}_0 = \int \prod_{m,n} d\eta_{mn}(1 + t_x \eta_{mn}\eta_{m+1n})(1 + t_y \eta_{mn}\eta_{mn+1}), \quad (1)$$

where η_{mn} are nilpotent and commuting variables satisfying [36] $\eta_{mn}^2 = 0$, $\int d\eta_{mn}\eta_{mn} = 1$, and $\int d\eta_{mn} = 0$. The integrals can be performed if we introduce, following closely Hayn and Plechko [34], a set of Grassmann variables $\{a_{mn}, \bar{a}_{mn}, b_{mn}, \bar{b}_{mn}\}$,

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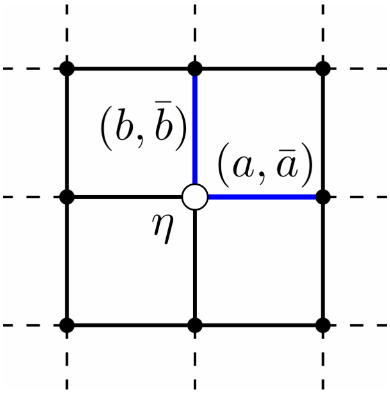

FIG. 1. (Color online) Variable configuration on site and links. At each site is associated a nilpotent variable η, such that $\eta^2=0$, and two pairs of Grassmann variables $(a,\bar{a})$ and $(b,\bar{b})$, one for each of the two directions.

see Fig. 1, such that

$$\begin{aligned}1+t_x\eta_{mn}\eta_{m+1n} &= \int d\bar{a}_{mn}da_{mn}e^{a_{mn}\bar{a}_{mn}}\\ &\quad\times(1+a_{mn}\eta_{mn})(1+t_x\bar{a}_{mn}\eta_{m+1n}),\\ 1+t_y\eta_{mn}\eta_{mn+1} &= \int d\bar{b}_{mn}db_{mn}e^{b_{mn}\bar{b}_{mn}}\\ &\quad\times(1+b_{mn}\eta_{mn})(1+t_y\bar{b}_{mn}\eta_{mn+1}).\end{aligned}\tag{2}$$

This decomposition allows for an integration over the Grassmann variables η_{mn}, after rearranging the different link variables $A_{mn}=1+a_{mn}\eta_{mn}$, $\bar{A}_{m+1n}=1+t_x\bar{a}_{mn}\eta_{m+1n}$, $B_{mn}=1+b_{mn}\eta_{mn}$, and $\bar{B}_{mn+1}=1+t_y\bar{b}_{mn}\eta_{mn+1}$. Then the partition function becomes

$$\mathcal{Q}_0=\mathrm{Tr}_{\{a,\bar{a},b,\bar{b},\eta\}}\prod_{m,n}(A_{mn}\bar{A}_{m+1n})(B_{mn}\bar{B}_{mn+1}),\tag{3}$$

where we use the integration measure $\mathrm{Tr}_{\{.\}}$ for the different Grassmannian and nilpotent variables with the adequate weights.

The noncommuting link variables are then moved through the product in such a way that each η_{mn} is isolated and can be integrated directly. This rearrangement is possible in two dimensions thanks to the mirror symmetry introduced by Plechko [37] for the 2D Ising model. This also imposes the conditions $\bar{A}_{1n}=1$, $\bar{A}_{L+1n}=1$, $\bar{B}_{m1}=1$, and $\bar{B}_{mL+1}=1$, or $\bar{a}_{0n}=\bar{a}_{Ln}=\bar{b}_{m0}=\bar{b}_{mL}=0$ for open boundary conditions. One finally obtains the following exact expression:

$$\mathcal{Q}_0=\mathrm{Tr}_{\{a,\bar{a},b,\bar{b},\eta\}}\overrightarrow{\prod_n}\left(\overleftarrow{\prod_m}\bar{B}_{mn}\overrightarrow{\prod_m}\bar{A}_{mn}B_{mn}A_{mn}\right).\tag{4}$$

The integration over the η_{mn} variables is performed recursively from $m=1$ to $m=L$ for each n. Each integration leads to a Grassmann quantity $L_{mn}=a_{mn}+b_{mn}+t_x\bar{a}_{m-1n}+(-1)^{m+1}t_y\bar{b}_{mn-1}$, which is moved to the left of the products over m in Eq. (4), hence a minus sign is needed in front of each $\bar{b}$ crossed by L_{mn} that is moved through the product of the $\bar{B}$ terms. Finally, the result $\mathcal{Q}_0=\mathrm{Tr}_{\{a,\bar{a},b,\bar{b}\}}\prod_{m,n}L_{mn}$ can be further rewritten by introducing additional Grassmann variables c_{mn}, such that $L_{mn}=\int dc_{mn}\exp(c_{mn}L_{mn})$. This expresses $\mathcal{Q}_0$ as a Gaussian integral over variables $\{a,\bar{a},b,\bar{b},c\}$. The integration over variables $\{a,\bar{a},b,\bar{b}\}$ can then be performed and, after antisymmetrization of the expression, one obtains explicitly

$$\begin{aligned}\mathcal{Q}_0 &= \int\prod_{m,n}dc_{mn}\exp\sum_{m,n}\Big[\frac{1}{2}t_x(c_{m+1n}c_{mn}-c_{m-1n}c_{mn})\\ &\quad+\frac{1}{2}t_y(-1)^{m+1}(c_{mn+1}c_{mn}-c_{mn-1}c_{mn})\Big]\\ &= \int\prod_{m,n}dc_{mn}\exp\mathcal{S}_0.\end{aligned}\tag{5}$$

Boundary conditions are now $c_{0,n}=c_{m,0}=c_{L+1,n}=c_{m,L+1}=0$. We consider a Fourier transformation satisfying open boundary conditions [34], $c_{mn}=i^{m+n}\sum_{p,q=1}^{L}c_{pq}f_m(p)f_n(q)$, where $f_n(p)=\sqrt{\frac{2}{L+1}}\sin\frac{\pi pn}{L+1}$ form an orthonormal set of functions $\sum_m f_m(p)f_m(q)=\delta_{pq}$. This leads to a block representation of the action in the momentum space, for momenta inside the reduced sector $1\leqslant p,q\leqslant L/2$. We note vectors $\mathbf{c}_\alpha={}^t(c_{pq},c_{-pq},c_{p-q},c_{-p-q})$, where $-p$ is meant for $L+1-p$ and label $\alpha=\{p,q\}$. The four components of these vectors will be written c_α^μ with $\mu=1\cdots4$. Then $\mathcal{S}_0=\frac{i}{2}c_\alpha^\mu M_\alpha^{\mu\nu}c_\alpha^\nu$, where the antisymmetric matrix M is defined by

$$M_\alpha=\begin{pmatrix}0 & 0 & -a_y(q) & -a_x(p)\\ 0 & 0 & a_x(p) & -a_y(q)\\ a_y(q) & -a_x(p) & 0 & 0\\ a_x(p) & a_y(q) & 0 & 0\end{pmatrix},$$

with $a_x(p)=2t_x\cos\frac{\pi p}{L+1}$ and $a_y(q)=2t_y\cos\frac{\pi q}{L+1}$. The factor i can be absorbed in a redefinition of the c's variables. One simply obtains a product of cosine functions [34] as found by Kasteleyn, Temperley, and Fischer [4–6], since the pfaffian of $\prod_\alpha M_\alpha$ is the product $\prod_{p,q}[a_x(p)^2+a_y(q)^2]$ in the reduced sector of momenta, or

$$\mathcal{Q}_0=\prod_{p,q=1}^{L/2}\left[4t_x^2\cos^2\frac{\pi p}{L+1}+4t_y^2\cos^2\frac{\pi q}{L+1}\right].\tag{6}$$

The matrix M_α is deeply related to the Kasteleyn [4] orientation matrix K since $\mathcal{Q}_0=\mathrm{Pf}(\prod_\alpha M_\alpha)=\mathrm{Pf}(K)$.

We consider now the case where an even number n of monomers are present in the lattice at different fixed positions $\mathbf{r}_i=(m_i,n_i)$ with $i=1,\cdots,n$; see Fig. 2. The partition

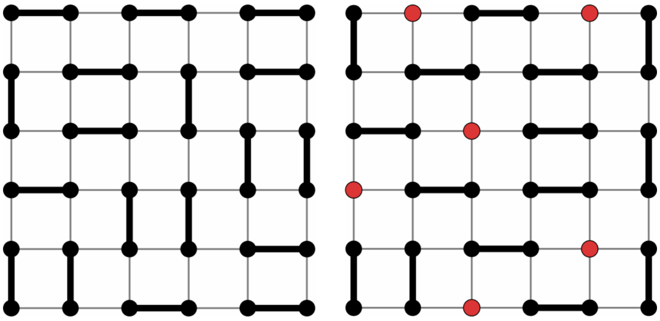

FIG. 2. (Color online) Typical dimer configuration for a square lattice of size 6×6 without monomer (left) and with six monomers (right, red dots).

function $\mathcal{Q}_n(\{\mathbf{r}_i\})$, which we define as a correlation function between monomers after summing up over all dimer configurations, is the number of all possible dimer configurations with the constraint imposed by fixing the given monomer positions. This quantity is evaluated by inserting $\eta_{m_i n_i}$ in Eq. (1) at each monomer location, which prevents dimers from occupying these sites. It is useful to introduce additional Grassmann variables h_i, such that $\eta_{m_i n_i} = \int dh_i \exp(h_i \eta_{m_i n_i})$. These insertions are performed at point $\mathbf{r}_i$ in Eq. (4), and the integration over $\eta_{m_i n_i}$ modifies $L_{m_i n_i} \to L_{m_i n_i} + h_i$. However, by moving the anticommuting variables dh_i to the left of the remaining ordered product, a minus sign is introduced in front of each $\bar{b}_{mn_i-1}$ or t_y coupling in $\bar{B}_{mn_i}$ for all $m > m_i$. We can replace more generally $\bar{b}_{mn-1}$ by $\epsilon_{mn}\bar{b}_{mn-1}$, such that $\epsilon_{mn_i} = -1$ for $m > m_i$, and $\epsilon_{mn} = 1$ otherwise. The integration is then performed on the remaining $\{a,\bar{a},b,\bar{b}\}$ variables as usual, so that $\mathcal{Q}_n(\{\mathbf{r}_i\})$ can be expressed as a Gaussian form, with a sum of counter-terms corresponding to the monomer insertions, or

$$\mathcal{Q}_n(\{\mathbf{r}_i\}) = \mathrm{Tr}_{\{c,h\}} e^{\mathcal{S}_0 + \sum_{\{\mathbf{r}_i\}} c_{m_i n_i} h_i + \mathcal{S}_I},$$

$$\mathcal{S}_I = 2t_y \sum_{\{\mathbf{r}_i\}} \sum_{m=m_i+1}^{L} (-1)^{m+1} c_{mn_i-1} c_{mn_i}. \tag{7}$$

The contribution $\mathcal{S}_I$ corresponds to a line of defects, as shown in Fig. 3. The addition of monomers is therefore equivalent to inserting a magnetic field h_i at points $\mathbf{r}_i$, as well as a line of defect $c_{mn_i-1}c_{mn_i}$ running from the monomer position to the right boundary $m = L$. If two monomers have the same ordinate $n_i = n_j$, the line of defects will only run between the two mononers and will not reach the boundary. This can be viewed as an operator acting on the links crossed by the line and running from a point on the dual lattice to the boundary on the right-hand side. More specifically, we can express the correlation functions, after integration over the fermionic magnetic fields h_i, as an average over composite fields

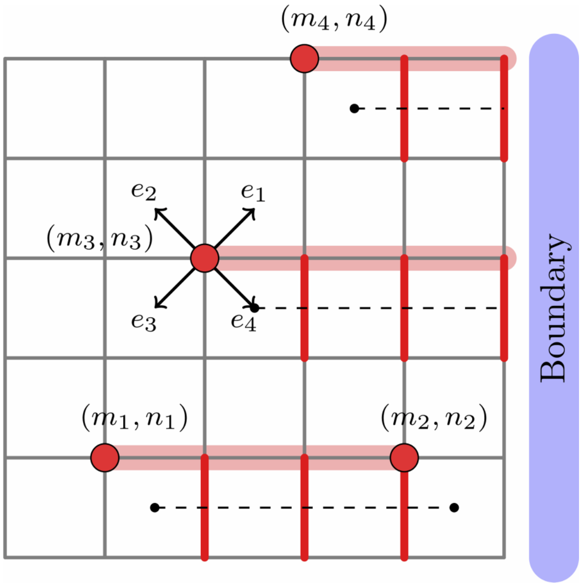

FIG. 3. (Color online) Typical configuration of the system with four monomers. The sign of the couplings t_y are reversed (red links) along the black-dashed line (or disorder operator, see text) that arises from moving the Grassmann fields conjugated to the defects toward the right boundary. Elementary vectors $\mathbf{e}_{i=1\dots4}$ are represented, and $\mathbf{e}_4$ indicates the starting location of the line of defects for the disorder operator.

$$\begin{aligned}\frac{\mathcal{Q}_n(\{\mathbf{r}_i\})}{\mathcal{Q}_0} &= \left\langle \prod_{\{\mathbf{r}_i\}} c_{m_i n_i} \exp\left(2t_y \sum_{m=m_i+1}^{L} (-1)^{m+1} c_{mn_{i-1}} c_{mn_i}\right)\right\rangle_0 \\ &= \left\langle \prod_{\{\mathbf{r}_i\}} c_{m_i n_i} \mu(\mathbf{r}_i + \mathbf{e}_4)\right\rangle_0 = \left\langle \prod_{\{\mathbf{r}_i\}} \Psi_4(\mathbf{r}_i)\right\rangle_0, \end{aligned} \tag{8}$$

where $\mu(\mathbf{r} + \mathbf{e}_i)$ is a disorder operator whose role is to change the sign of the vertical links across its path starting from vector $\mathbf{r} + \mathbf{e}_i$ on the dual lattice toward the right-hand side; see Fig. 3. The integration $\langle \cdots \rangle_0$ is performed relatively to the action $\mathcal{S}_0$. Elementary vectors $\mathbf{e}_i$ define a four-component fermionic field $\Psi_\mu(\mathbf{r}) = c_{mn}\mu(\mathbf{r} + \mathbf{e}_\mu)$, which is the fermionic counterpart of the scalar field introduced for the Ising-spin model [38,39]. In the latter case, a linear differential equation can be simply found for $\Psi_\mu(\mathbf{r}) = \sigma(\mathbf{r})\mu(\mathbf{r} + \mathbf{e}_\mu)$, with $\sigma(\mathbf{r}) = \pm 1$, leading to a Dirac equation. Here the general correlator between monomers is directly mapped onto the correlator between these fermionic composite fields. If we go back to Eq. (7), the part of the field interaction can be Fourier transformed such that $\sum_{\{\mathbf{r}_i\}} c_{m_i n_i} h_i = \sum_{p,q=1}^{L} c_{pq} H_{pq} = \sum_{\alpha,\mu} c_\alpha^\mu H_\alpha^\mu$. The term $\mathcal{S}_I$ in the action can be written as $\frac{i}{2} c_\alpha^\mu V_{\alpha\beta}^{\mu\nu} c_\beta^\nu$, with the perturbative matrix $V_{\alpha\beta}$ given by

$$\begin{aligned} V_{\alpha,\beta} = V_{pq,p'q'} = \sum_{\{\mathbf{r}_i\}} 2t_y (-1)^{n_i} \left\{ \sum_{m=m_i+1}^{L} f_m(p) f_m(p') \right\} \\ \times (f_{n_i-1}(q) f_{n_i}(q') - f_{n_i-1}(q') f_{n_i}(q)). \end{aligned} \tag{9}$$

The different components $V_{\alpha\beta}^{\mu\nu}$ are given explicitly, for the first terms, by $V_{\alpha\beta}^{11} = V_{pq,p'q'}$, $V_{\alpha\beta}^{12} = V_{pq,-p'q'}$, $V_{\alpha\beta}^{21} = V_{-pq,p'q'}$, and so on. Then the full fermionic action is $\mathcal{S} = \frac{i}{2} c_\alpha^\mu W_\alpha^{\mu\nu} c_\alpha^\nu + c_\alpha^\mu H_\alpha^\mu$ with antisymmetric matrix $W_{\alpha\beta}^{\mu\nu} = \delta_{\alpha\beta} M_\alpha^{\mu\nu} + V_{\alpha\beta}^{\mu\nu}$ satisfying $W_{\alpha\beta}^{\mu\nu} = -W_{\beta\alpha}^{\nu\mu}$. By construction, this matrix can be represented as a block matrix of global size $(L^2 \times L^2)$

$$W = \left(\underbrace{\begin{matrix} M_{\alpha=(1,1)} & V_{(1,1),(1,2)} & V_{(1,1),(1,3)} & \cdots \\ -V_{(1,1),(1,2)} & M_{(1,2)} & V_{(1,2),(1,3)} & \cdots \\ -V_{(1,1),(1,3)} & -V_{(1,2),(1,3)} & M_{(1,3)} & \cdots \\ \cdots & & & \end{matrix}}_{L^2/4 \text{ blocks}} \right),$$

where each of the $(L^2/4) \times (L^2/4)$ blocks is a (4×4) matrix. Labels α are ordered with increasing momentum $(1,1),(1,2) \cdots (1,L/2),(2,1) \cdots$. Then $\mathcal{Q}_n(\{\mathbf{r}_i\})$ can formally be written as $\mathcal{Q}_n(\{\mathbf{r}_i\}) = \mathrm{Tr}_{\{c,h\}} \exp(\frac{i}{2} c_\alpha^\mu W_{\alpha\beta}^{\mu\nu} c_\beta^\nu + c_\alpha^\mu H_\alpha^\mu)$. The linear terms in c_α^μ can be removed using a translation $c_\alpha^\mu \to c_\alpha^\mu + g_\alpha^\mu$, with $g_\alpha^\mu = i(W^{-1})_{\alpha\beta}^{\mu\nu} H_\beta^\nu$. After a further rescaling of variables $c_\alpha \to i^{-1/2} c_\alpha$, one obtains

$$\mathcal{Q}_n(\{\mathbf{r}_i\}) = \mathrm{Pf}(W) \mathrm{Tr}_{\{h\}} \exp\left[-\frac{i}{2} (W^{-1})_{\alpha\beta}^{\mu\nu} H_\alpha^\mu H_\beta^\nu \right].$$

The fields H^{μ}_{α} depend on h_i through the identity $H^{\mu}_{\alpha}=\sum_{i=1}^{n}\Lambda^{\mu}_{i,\alpha}h_i$, where coefficients $\Lambda^{\mu}_{i,\alpha}$ are expressed using a four-dimensional vector $\mathbf{\Lambda}_{i,\alpha}=f_{m_i}(p)f_{n_i}(q)\mathbf{\Lambda}_i$. The components of momentum-independent vector Λ^{μ}_{i} are $(i^{m_i+n_i}, -i^{-m_i+n_i}, -i^{m_i-n_i}, i^{-m_i-n_i})$. Its role is to fix whether the configuration of the monomers is allowed or not (in this case the correlator is zero). The final and compact expression for $\mathcal{Q}_n(\{\mathbf{r}_i\})$ is then

$$\mathcal{Q}_n(\{\mathbf{r}_i\})=\mathrm{Pf}(W)\mathrm{Pf}(C), \tag{10}$$

where C is a real $(n\times n)$ antisymmetric matrix with elements $C_{ij}=-i\Lambda^{\mu}_{i,\alpha}(W^{-1})^{\mu\nu}_{\alpha\beta}\Lambda^{\nu}_{j,\beta}$. The antisymmetry can be easily verified using the antisymmetry property of W or W^{-1}. $\mathcal{Q}_n$ is therefore a product of two pfaffians where the positions of the monomers are specified in both matrices W and C. The factorization Eq. (10) can generally be viewed as the product of a bulk and, by analogy, a *boundary* contribution. This can be found, for example, when a nonhomogeneous magnetic field is applied at the surface of a 2D Ising model [40], by using Grassmann techniques as well. Here the term Pf(C) is due to the contribution of monomers in the bulk leading to a corrective factor in the free energy of order of the number n of monomers, similar to a surface perturbation. Since the monomers are in the bulk, they contribute as well to the term Pf(W), which would otherwise, were the monomers located on the surface, be equal to $\mathcal{Q}_0$. It is worth noting that a similar factorization was found for the correlation function between two monomers in terms of the product of two spin-spin correlation functions of the Ising model at criticality [8,10], due to the analogy of the dimer model with two Ising models (or a complex fermionic field theory); see Appendix for precise details. It is, however, not obvious here to have such a direct identification with this result since the two pfaffians in Eq. (10) are of different nature. We can also mention that factorization of the correlation function exists in other models such as the one-dimensional XY chain [41]. Matrix V can be rewritten using additional matrices after considering the different components (μ,ν). We can indeed express V using four functions $u_k^{s=0,1}(\alpha,\beta)$, and $v_k^{s=0,1}(\alpha,\beta)$, for each monomer at location $\mathbf{r}_k=(m_k,n_k)$, with $m_k<L$, and such that $V_{\alpha\beta}=-2t_y\sum_{\mathbf{r}_k}\sum_{s,s'=0,1}u_k^s(\alpha,\beta)\Gamma_{ss'}v_k^{s'}(\alpha,\beta)$, with

$$\Gamma_{01}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},\quad \Gamma_{11}=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{pmatrix},$$

$$\Gamma_{00}=\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{pmatrix},\quad \Gamma_{10}=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}.$$

Functions u and v are specified by

$$\begin{aligned}u_k^s(\alpha,\beta)&=\sum_{m=m_k+1}^{L}(-1)^{s(m+1)}f_m(p)f_m(p'),\\ \frac{v_k^s(\alpha,\beta)}{(-1)^{sn_k}}&=\left[f_{n_k}(q)f_{n_k-1}(q')+(-1)^s f_{n_k}(q')f_{n_k-1}(q)\right].\end{aligned} \tag{11}$$

It is also worth noting that we have a similar structure in the real space, where the total action Eq. (7) is expressed by $\mathcal{S}=\frac{1}{2}c_{mn}W_{mn,m'n'}c_{m'n'}+\sum_{\mathbf{r}_i}c_{m_in_i}h_i$, with W containing both the connectivity matrix M and the contribution of the line of defects V. A direct computation also leads to the factorization $\mathcal{Q}_n(\{\mathbf{r}_i\})=\mathrm{Pf}(W)\mathrm{Pf}(C)$, where $C_{ij}=(W^{-1})_{m_in_i,m_jn_j}$ is a $(n\times n)$ antisymmetric matrix.

Exact dimers enumeration algorithms [42] up to size of 10×10 has been widely used to compare with the theoretical prediction. For instance there are 636 072 different configurations of dimers with two monomers at coordinates $\mathbf{r}_1=(2,3)$ and $\mathbf{r}_2=(7,5)$ on a 8×8 lattice, in accordance with the computation of $\mathcal{Q}_2(\mathbf{r}_1,\mathbf{r}_2)$ taking $t_x=t_y=1$. As possible other application, we could obtain the full partition function of the monomer-dimer model by summing up over all the possible number of monomers and over all the possible positions. The result for the 8×8 lattice is 179,788,343,101,980,135 [25], compared with the 12,988,816 configurations without monomer. In Fig. 4, we have solved numerically for a size $L=96$ the modified correlation function $\mathcal{Q}_2(\{\mathbf{r}_1,\mathbf{r}_2\})\mathcal{Q}_0^{-1}=\mathrm{Pf}(M^{-1}W)\mathrm{Pf}(C)$, for two monomers at positions $\mathbf{r}_1=(m,L/2-k)$ and $\mathbf{r}_2=(m,L/2+k+1)$, $k=0\cdots L/2$, distant of $d=2k+1$. Due to finite-size effects, a curve for a given m is distinguished depending on the parity of k. In the large size limit, this difference is indiscernible. Figure 4 shows the crossover between a behavior in d^{-1} near the boundary $(m=96)$ to a bulk behavior [7] in

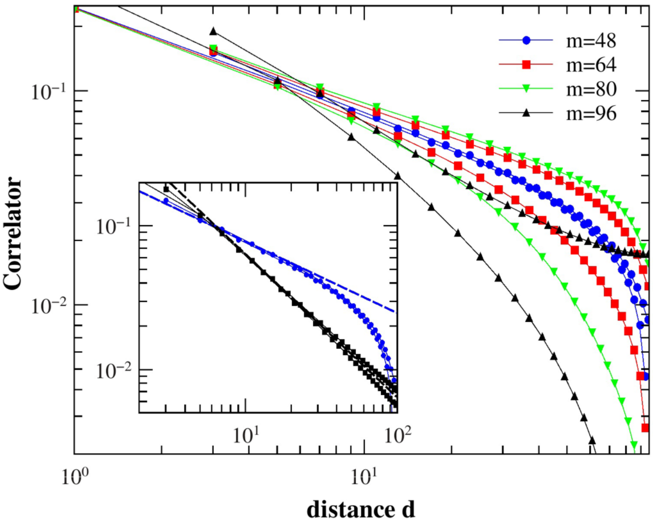

FIG. 4. (Color online) Correlation function $\mathcal{Q}_2(\{\mathbf{r}_1,\mathbf{r}_2\})\mathcal{Q}_0^{-1}$ for two monomers on a lattice of size $L=96$ as function of their distance $d=|\mathbf{r}_2-\mathbf{r}_1|$. They are positioned vertically, at locations $\mathbf{r}_1=(m,L/2-k)$ and $\mathbf{r}_2=(m,L/2+k+1)$, with $d=2k+1$. The curves represent different abscissa m successively from the right border $(m=L=96)$ to the center of the lattice $(m=48)$. Curves come by pair, with lower or higher correlations, depending if k is even or odd. Inset: Correlation function using Eq. (12) for two monomers on the boundary (black square symbols), at locations $n_1=L/2-k$ and $n_2=L/2+k+1$, as function of their distance $d=2k+1$. Lattice size is $L=1000$. Asymptotic limit $\frac{2}{\pi}d^{-1}$ (black dashed line) is shown for comparison. The bulk correlator (blue symbols $L=96$ and $m=48$) is also displayed, as well as its asymptotic limit $Bd^{-1/2}$ (blue dashed line). The value $B\simeq0.247$; see text.

$d^{-1/2}$ ($m = 48$). The amplitude B of the asymptotic two-point correlation function, which behaves like $Bd^{-1/2}$, has been determined explicitly in the thermodynamic limit [10], $B = 2^{-3/4}A^2 \approx 0.247$ with $A = 2^{1/12}e^{3\zeta'(-1)}$ and where $\zeta(s)$ is the Riemann zeta function. This value appears to be in good agreement with our numerical fit (see inset Fig. 4, dashed blue line). Interestingly, when the monomers are located exactly on the boundary ($m = L$), $V = 0$, and $W = M$, in this case $\mathcal{Q}_n(\{\mathbf{r}_i\}) = \mathcal{Q}_0 \mathrm{Pf}(C)$, and it is straightforward to compute exactly the elements of matrix C. In the discrete case one obtains

$$C_{ij} = \frac{4[(-1)^{n_i} - (-1)^{n_j}]}{(L+1)^2} \sum_{p,q=1}^{L/2} \frac{i^{1+n_i+n_j} t_y \cos\frac{\pi q}{L+1} \sin^2\frac{\pi p}{L+1}}{t_x^2 \cos^2\frac{\pi p}{L+1} + t_y^2 \cos^2\frac{\pi q}{L+1}} \times \sin\frac{\pi q n_i}{L+1} \sin\frac{\pi q n_j}{L+1}. \tag{12}$$

C_{ij} are zero if n_i and $n_j > n_i$ have the same parity. For example, fixing one monomer on the first site $n_1 = 1$ and taking $n_2 = 2k$, we have, for $t_x = t_y = 1$ in the asymptotic limit $L \to \infty$ and large k, the following expansion $C_{12} \simeq \frac{2}{\pi}k^{-1} - \frac{3}{2\pi}k^{-5}$. In the case $n_1 = L/2 - k$ and $n_2 = L/2 + k + 1$, as shown in inset of Fig. 4, $C_{12} \simeq \frac{2}{\pi}d^{-1} - \frac{2}{\pi}d^{-3}$ instead, with $d = 2k + 1$ and amplitude $2/\pi$. This result is in agreement with the work of Priezzhev and Ruelle [19] on the scaling limit of the correlation functions of boundary monomers in a system of closely packed dimers in terms of a 1D chiral free fermion theory [44].

In summary, we presented a practical fermionic solution of the 2D monomer-dimer model on the square lattice, which allows for expressing the correlation functions between monomers in terms of two pfaffians and gave an explicit formula for boundary correlations. This can also be used for studying more general n-point correlation functions, thermodynamical quantities, or transport phenomena of monomers. Other lattice types, such as hexagonal and other boundary conditions, can be considered as well.

ACKNOWLEDGMENTS

We are grateful to J. H. H. Perk for his knowledge in this domain and comments on the manuscript. This work was partly supported by the Collège Doctoral Leipzig-Nancy-Coventry-Lviv (Statistical Physics of Complex Systems) of UFA-DFH.

APPENDIX

In this section, we derive the continuum limit of the dimer action Eq. (5) and reformulate $\mathcal{S}_0$ in terms of two copies of Ising models. By an adequate change of variables [35] $c_{mn} \to i^{3/2+m^2} c_{mn}$, the action $\mathcal{S}_0$ can be written as a complex fermion field theory:

$$\mathcal{S}_0 = \sum_{m,n} \left[\frac{1}{2} t_x (c_{m+1n} c_{mn} - c_{m-1n} c_{mn}) + \frac{i}{2} t_y (c_{mn+1} c_{mn} - c_{mn-1} c_{mn}) \right].$$

We can introduce the formal derivative using series expansions $c_{m+1n} = c_{mn} + \partial_x c_{mn}$ and $c_{mn+1} = c_{mn} + \partial_y c_{mn}$, up to first order in lattice elementary step, so that the action can be recognized as a purely kinetic form with no mass contribution:

$$\mathcal{S}_0 = \sum_{m,n} [t_x \partial_x c_{mn} c_{mn} + i t_y \partial_y c_{mn} c_{mn}]. \tag{A1}$$

It is convenient to define the following fields:

$$\begin{aligned} c_-(m,n) &= c_{2m2n},\ c_+(m,n) = c_{2m2n+1}, \\ \bar{c}_-(m,n) &= c_{2m+12n+1},\ \bar{c}_+(m,n) = c_{2m+12n}, \end{aligned} \tag{A2}$$

and express the previous action in terms of these fields only:

$$\mathcal{S}_0 = -\sum_{m,n=1}^{L/2} \sum_{\sigma=\pm} [t_x (c_\sigma \partial_x \bar{c}_{-\sigma} + \bar{c}_\sigma \partial_x c_{-\sigma}) + i t_y (c_\sigma \partial_y c_{-\sigma} + \bar{c}_\sigma \partial_y \bar{c}_{-\sigma})]. \tag{A3}$$

Site variables (m,n) now designate the locations of reduced cells containing four sites and take values between 0 and $L/2$. Field vectors $(c_\sigma, \bar{c}_\sigma)$ are composed of two independent components and describe two coupled Ising models labeled by index $\sigma = \pm$. This action can be diagonalized with a linear transformation and new set of Grassmann variables:

$$\begin{aligned} \varphi_- &= \tfrac{1}{2}(c_- + c_+ + \bar{c}_- + \bar{c}_+), \\ \bar{\varphi}_- &= \tfrac{1}{2}(c_- + c_+ - \bar{c}_- - \bar{c}_+), \\ i\varphi_+ &= \tfrac{1}{2}(c_- - c_+ + \bar{c}_- - \bar{c}_+), \\ i\bar{\varphi}_+ &= \tfrac{1}{2}(c_- - c_+ - \bar{c}_- + \bar{c}_+). \end{aligned} \tag{A4}$$

We obtain finally a diagonalized form for $\mathcal{S}_0$, defining the complex derivative in two dimensions, $\partial = t_x \partial_x + i t_y \partial_y$ and $\bar{\partial} = t_x \partial_x - i t_y \partial_y$:

$$\mathcal{S}_0 = -\sum_{m,n=0}^{L/2} (\bar{\varphi}_+ \bar{\partial} \bar{\varphi}_+ - \varphi_+ \partial \varphi_+ - \bar{\varphi}_- \bar{\partial} \bar{\varphi}_- + \varphi_- \partial \varphi_-).$$

Following Plechko [43], it is useful to introduce Dirac matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and define spinor $\Psi_\sigma = \binom{\varphi_\sigma}{\bar{\varphi}_\sigma}$. It has to be noted that φ_σ and $\bar{\varphi}_\sigma$ are not conjugated but independent Grassmann variables. The action can then be put into a compact expression,

$$\mathcal{S}_0 = \sum_{m,n=0}^{L/2} \sum_{\sigma=\pm} {}^t\bar{\Psi}_\sigma (\sigma_1 \partial_1 + \sigma_2 \partial_2) \Psi_\sigma, \tag{A5}$$

where $\bar{\Psi}_\sigma = i\sigma_2 \Psi_\sigma$ and $\partial_1 = t_x \partial_x$, $\partial_2 = t_y \partial_y$. Here the resulting action is of Majorana form [43], equivalent to two independent Ising models at criticality, since no mass term is present.

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