Loop-erased random walk on a percolation cluster: Crossover from Euclidean to fractal geometry

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We study loop-erased random walk (LERW) on the percolation cluster, with occupation probability $p \geqslant p_c$, in two and three dimensions. We find that the fractal dimensions of LERW_p are close to normal LERW in a Euclidean lattice, for all $p > p_c$. However, our results reveal that LERW on critical incipient percolation clusters is fractal with $d_f = 1.217 \pm 0.002$ for $d = 2$ and 1.43 ± 0.02 for $d = 3$, independent of the coordination number of the lattice. These values are consistent with the known values for optimal path exponents in strongly disordered media. We investigate how the behavior of the LERW_p crosses over from *Euclidean* to *fractal* geometry by gradually decreasing the value of the parameter p from 1 to p_c. For finite systems, two crossover exponents and a scaling relation can be derived. This work opens up a theoretical window regarding the diffusion process on fractal and random landscapes.

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I. INTRODUCTION

The diffusion process in disordered media is *anomalous*, i.e., the mean square displacement (MSD) of the diffusing species has a nonlinear relationship with time, in contrast to diffusion on Euclidean lattices, where their MSD is proportional to time in all dimensions [1]. Such disordered media are typically simulated through percolation systems, and diffusion on percolation clusters has been studied in great detail [2,3]. One could restrict the diffusion of a simple random walk (RW) to the incipient infinite cluster; in this case finite-sized clusters are irrelevant. It is known that, above criticality $p > p_c$, diffusion is anomalous over short distances and normal over long distances [3]. As the occupation probability approaches the percolation threshold, diffusion becomes more anomalous over longer distances. Diffusion on critical incipient percolation clusters is anomalous on all length scales. On the other hand, one could erase the loops from the trajectory of the RW chronologically, and this operation results in a loop-erased random walk (LERW) [4]. This model is equivalent to the classical uniform spanning tree (UST) [5], the q-state Potts model in the limit $q \to 0$ [6], and the avalanche frontier in the Abelian sandpile model (ASM) [7]. It is known that the fractal dimension of LERW in $D = 2$ is $5/4$ and the upper critical dimension for LERW is $D = 4$, with $d_f = 2$ for $D \geqslant 4$. Although scaling and the universality class of LERW in an integer lattice are known, the universality class of this model in a fractal landscape and especially in critical percolation still need to be studied. In this paper we study the LERW on a percolation cluster, with an occupation probability above and equal to the critical value $p \geqslant p_c$. Our results show that, for all $p > p_c$, the scaling behavior of the obtained LERW_p curves is close to the exact results for LERW on Euclidean lattices [7]. However, our results reveal that the scaling behavior of this model near critical percolation is completely different, with a fractal dimension $d_f \approx 1.22$ in two dimensions (2D) and $d_f \approx 1.43$ in three dimensions (3D). Surprisingly, these values are close to a family of curves appearing in different contexts such as, e.g., polymers in strongly disordered media [8], watershed of random landscapes [9–11], ranked percolation [12], and optimal path cracks [13]. A crossover from *Euclidean* to *fractal* geometry can be observed by decreasing the value of the parameter p from 1 to p_c. To investigate how the behavior of the LERW_p crosses over from between these two universality classes, we have considered the mean total length of LERW_p as a homogeneous function on the lattice size and occupation probability. For the finite systems, two crossover exponents and a scaling relation can be derived. Our results for the crossover regime demonstrate that for a fixed lattice size L there are three distinct scaling regimes, as it has been reported in similar systems [13,14]. These regimes are separated by two crossovers. Finally, there is a scaling relation between the corresponding crossover exponents and the fractal dimension of LERW.

II. SIMULATION DETAILS

We start by constructing a porous landscape: We simulate a site percolation model on the square, triangular, honeycomb, and simple-cubic lattices of linear size L with free boundary conditions. We are especially interested in studying the diffusion properties in single percolation clusters with a fraction of occupied sites $p \geqslant p_c$; particularly, the single cluster containing the middle point of the lattice is generated in the same way as the algorithm [15]. To avoid an imprecise definition of the middle point for the lattice, we consider a lattice with an odd number of the side length L. If the obtained cluster is large enough to connect to at least one of the outer edges, we accept it; otherwise, we simply ignore it and produce another one. Once a large cluster is obtained, we start the diffusion process by setting a RW on the middle of the lattice and stop when it touches the outer edges for the first time. The LERW curve can be obtained by erasing the loops chronologically. In Fig. 1, a sample of LERW on a critical percolation cluster $p = p_c$ is shown. In planar LERW, simulations on critical percolation clusters were performed on a square lattice of size $L = 2^{n+4} + 1$, $n = 1,2,\ldots,8$. The number of samples generated for each lattice size ranges

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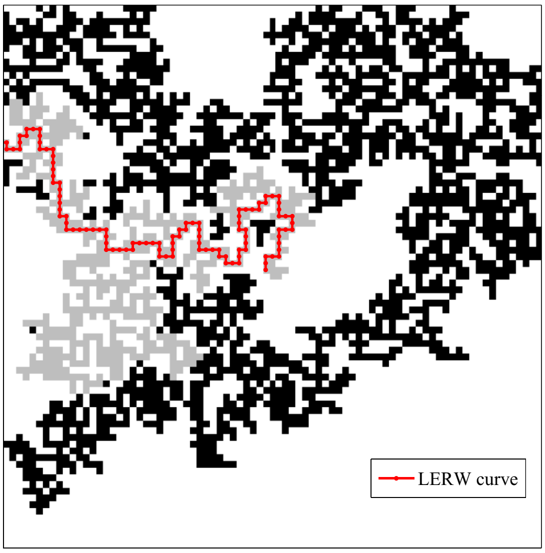

FIG. 1. (Color online) The generation process of a 2D LERW on a critical percolation cluster on a 81×81 lattice. A random walk starts at the middle of the lattice in a critical large cluster (shown in black color) and diffuses to the outer boundaries; the visited sites are shown in gray, and then the LERW curve (shown in red) can be obtained by erasing the loops chronologically.

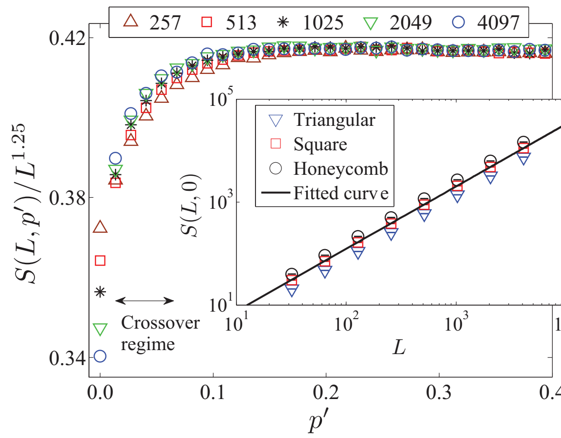

FIG. 2. (Color online) Dependence of the mean total length normalized with system size, $S(L,p')/L^{1.25}$, on p' for different lattice sizes. Above criticality, LERW$_p$ behaves as expected for Euclidean lattices. However, near the criticality, $S(L,0)$ grows slower than $L^{\frac{5}{4}}$. A crossover between two regimes can be observed for p in the neighborhood of p_c. Inset: Mean total length of LERW on a critical percolation cluster as a function of lattice size L for three different lattices (the statistical error bars are shown, but are quite shorter and appear as horizontal lines). The solid line represents the least-square fit of $\log[S(L,0)]$ to a function of the form $d_f \log(L) + a$, where the parameter a is an irrelevant constant that depends upon the lattice and the details of the definitions. We find $d_f = 1.217 \pm 0.002$ for the square lattice. The fit has a χ^2 statistic of 4.84 with six degrees of freedom for a p value of 0.57, so the fit passes the χ^2 test with a 95% confidence level.

from 4×10^6 for the smallest system sizes until about 10^4 for the largest system sizes. Moreover, to study this model in higher spatial dimensions, we simulate the same problem on a simple-cubic lattice. The actual numerical values of the site percolation threshold used for the square, triangular, honeycomb, and simple-cubic lattices are 0.592 746 02 [16], 0.5, 0.697 040 2 [16], and 0.311 607 7 [17], respectively.

III. THE FRACTAL DIMENSION

We estimate the fractal dimension for the obtained LERW$_p$ by computing the mean total length S for different lattice sizes L, comparing it with $S \sim L^{d_f}$. In the case of normal LERW ($p = 1$), the total length of the curves increases with system size as $S(L) \sim L^{\frac{5}{4}}$, consistent with the fractal dimension of Euclidean LERW. By decreasing the occupation probability p, the fractal dimension of these random curves remains unchanged. At percolation threshold, these curves are smoother than normal LERW, and the mean total length diverges with system size with a different exponent $S(L) \sim L^{d_f}$, with $d_f \approx 1.22$. Figure 2 shows the dependence of the $S(L)/L^{\frac{5}{4}}$ on p' for different system sizes, where p' is $p - p_c$. The overlap of the different curves confirms that the fractal dimension of the LERW$_p$ above p_c is $5/4$. A small deviation is observed due to finite-size effects. There is a crossover between two different regimes near the critical point $p \gtrsim p_c$, which can be observed in Fig. 2. The mean total length of LERW on a critical percolation cluster for different lattice sizes L is shown in the inset of Fig. 2. We obtain $d_f = 1.217 \pm 0.002$. In order to check the universality of this exponent and to show that it does not vary with the coordination number of the lattices, we perform the same simulations on the triangular and the honeycomb lattices. We found the same result (see the inset of Fig. 2), which provides strong evidence for the universality of this exponent.

IV. CROSSOVER SCALING FUNCTION

As shown in Fig. 2, the mean total length of LERW$_p$ increases with increasing occupation probability. For large systems, the mean total length of LERW$_p$ grows with p', such that $S(L,p') \sim p'^{\beta}$, where $\beta \approx 0.04$ is a different exponent, which we call the length-growth exponent. It is known that the anomalous diffusion in percolation clusters occurs only within the correlation length [3]. At high occupation probability, the correlation length is so small, therefore the mean length of LERW$_p$ can be described as a LERW in Euclidean geometry. When the occupation probability is reduced, the correlation length increases as $p'^{1/\nu}$ diverging at p_c, where ν is the correlation length exponent of the percolation model, with $\nu = 4/3$ in 2D, so the system becomes a self-similar random fractal leading to a different universality class. There is a crossover behavior, as depicted in Fig. 2, from Euclidean to fractal geometry. For the complete crossover scaling of the mean length, S can be considered as a homogeneous function on the relevant scaling fields, $S(bL, b^{y_p} p') = b^{-y_s} S(L,p')$, where b is a scaling parameter and y_s and y_p are relevant exponents for S and p scaling parameters, respectively. One

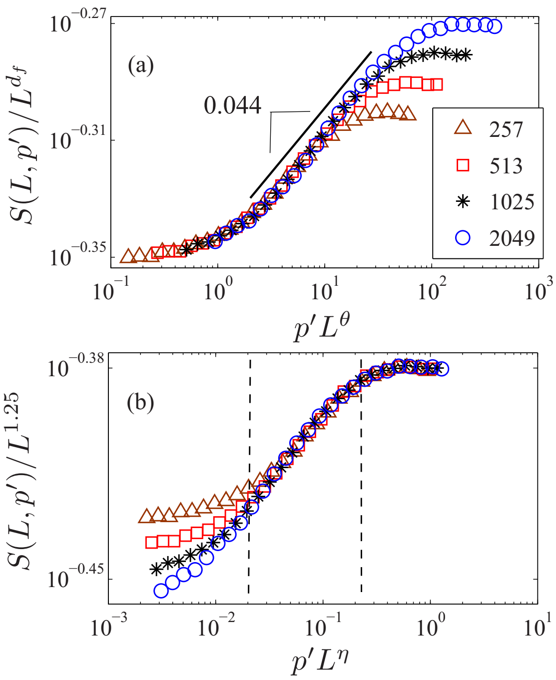

FIG. 3. (Color online) Crossover scaling and data collapse for LERW_p in different system sizes. (a) Rescaled mean total length S/L^{d_f}, where $d_f = 1.217$, vs $p'L^{\theta}$ for different system sizes. The scaling function given by Eq. (1) is applied, with $\theta = 0.90 \pm 0.05$. (b) Rescaled mean total length $S/L^{1.25}$ vs $p'L^{\eta}$, with $\eta = 0.15 \pm 0.02$, for different system sizes. For each finite lattice size L, there are three regimes separated by two crossovers. These crossovers scale with two crossover exponents, i.e., θ and η. By collapsing the data of different lattice sizes in the intermediate regime, a more precise estimate for β can be obtained, which is $\beta = 0.044 \pm 0.002$. All results have been averaged over 4×10^4 samples.

could restrict attention to the $p \to p_c$ regime, and then for a finite size of L, it is expected that S scales with d_f, so in this regime $y_s = d_f$. The next exponent can be found by trying to collapse the data (setting $b = L^{-1}$). The scaling ansatz for the mean total length is given by

$$S(L,p') = L^{d_f}\mathcal{G}[p'L^{\theta}], \tag{1}$$

where $\mathcal{G}[u]$ is a scaling function, such that $\mathcal{G}[u] \sim u^{\beta}$ for small values of u, and is nonzero at $u \to 0$. The exponent $\theta = y_p$ is the crossover exponent in the $p \to p_c$ regime. Figure 3(a) shows crossover scaling for different lattice sizes, close to the critical point. As it is shown, we have a good data collapse for small values of u with $\theta = 0.90 \pm 0.05$. For each finite lattice size L, there is a crossover point such that $p'_{\times_1}$ scales as $L^{-\theta}$, which, for $u \ll 1$, we have a saturation regime, and for $u \gg 1$ the results are consistent with u^{β} for all lattice sizes L. However, for large values of $p'L^{\theta}$, we do not observe a data collapse and the mean total length behaves as $L^{1.25-d_f}$. On the other hand, one could look at large values of p, and it is expected that the mean total length scales with a fractal dimension of Euclidean geometry, so $d_s = \frac{5}{4}$ in this regime. If we follow the same strategy as above, we could find another scaling function,

$$S(L,p') = L^{\frac{5}{4}}\mathcal{F}[p'L^{\eta}], \tag{2}$$

where the scaling function $\mathcal{F}[x]$ has a saturation regime for large values of x, and the exponent $\eta = y_p$ is the corresponding crossover exponent in this regime. In fact, we could find another crossover point, $p'_{\times_2}$ scaling with $L^{-\eta}$, for which the mean total length behaves as $\mathcal{F}[x] \sim x^{\beta}$ for $x \ll 1$, and is a constant value for $x \gg 1$. Figure 3(b) shows the scaling behaviors for different lattice sizes of L. As it is shown, we have a good data collapse with $\eta = 0.15 \pm 0.02$, and this clearly shows that the argument of $p'L^{\eta}$ in the crossover point should be independent of lattice size, so the crossing probability $p'_{\times_2}$ scales as $L^{-\eta}$ with system size. The overlap of the different curves confirms that the fractal dimension of the LERW_p curves above p_c is $\frac{5}{4}$. Three different regimes, as shown in Fig. 3, are clearly identified; for $p' < p'_{\times_1}$, the mean total length scales as $S \sim L^{d_f}$, for $p'_{\times_1} < p' < p'_{\times_2}$, S has a power law behavior as p'^{β}, and finally, for $p'_{\times_2} < p'$, it scales with a Euclidean exponent, i.e., $L^{1.25}$. Therefore, the following relation can be obtained,

$$\theta - \eta = \beta^{-1}\left(\tfrac{5}{4} - d_f\right), \tag{3}$$

which is in good agreement with our obtained numerical values for the exponents.

V. THREE-DIMENSIONAL LERW_p

Our interesting results for planar LERW_p motivate us to investigate it on higher dimensions, i.e., 3D. The scaling exponents of LERW on a Euclidean lattice in three dimensions are not rigorously known. However, the fractal dimension of this model, based on Monte Carlo simulations, has been reported to be ≈1.624 [18–20]. Since a single exponent would not be enough to describe the scaling behavior of the mean total length S, as discussed in Refs. [19,20], we cannot study $S(L,p')/L^{d_f}$ for all p' values as in Fig. 2. In fact, in three dimensions, due to stronger finite-size effects than two-dimensional LERW, corrections to scaling are needed to be considered. Here, we restrict our attention to only the problem of LERW on a critical percolation cluster. We compute the mean total length S for different lattice sizes L, and Fig. 4 shows the dependence of $S(L,0)$ on lattice size L. The best fit to our data collected for sizes $25 \leqslant L \leqslant 257$ yields the fractal dimension $d_f = 1.43 \pm 0.02$. To estimate d_f to four decimals from the data, corrections to scaling are necessary. Interestingly, this value is similar to the fractal dimensions of the optimal paths in strongly disordered media in 3D [21]. Our results, which are summarized in Table I, indicate that the LERW on a percolation cluster can be classified into distinct universality classes.

VI. SUMMARY AND DISCUSSION

In this paper, we have investigated the geometrical behavior of LERW_p on a percolation cluster, with an occupation probability above and equal to the critical value $p \geqslant p_c$. Our results show that the scaling behavior of planar LERW_p for $p > p_c$ is the same as the LERW on Euclidean lattices,

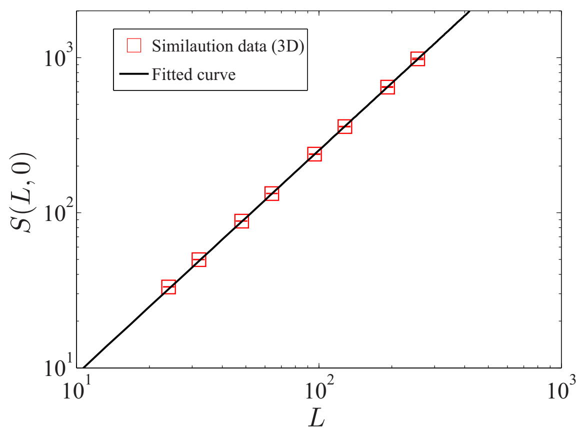

FIG. 4. (Color online) Mean total length of 3D LERW on a critical percolation cluster as a function of lattice size L. Simulations were performed on a simple-cubic lattice of size L ranging from 25 to 257. The number of samples generated for each lattice size ranges from 10^5 for the smallest system sizes down to about 2×10^3 for the largest system sizes. Error bars are smaller than the symbol size. The solid line represents the least-square fit of $\log[S(L,0)]$ to a function of the form $d_f \log(L) + a$, where $d_f = 1.43 \pm 0.02$. The fit has a χ^2 statistic of 5.7 with six degrees of freedom for a p value of 0.46, so the fit passes the χ^2 test.

which has been rigorously proven recently [26]. However, the LERW on critical percolation clusters scales with an anomalous exponent, and our results reveal that the fractal dimension of this model is $d_f \approx 1.22$ in 2D. Interestingly, this value is statistically identical to a family of curves appearing in different contexts such as, e.g., polymers in strongly disordered media [8], watershed of random landscapes [9,10,27], ranked percolation [12], minimum spanning tree (MST) [22], and optimal path cracks [13]. Also, our attempts to understand LERW_p in three dimensions are focused on the critical percolation cluster; the fractal dimension of this model on 3D percolation clusters is ≈1.43. These exponents in both two and three dimensions are similar to the fractal dimensions of the optimal paths in strongly disordered media [21]. This fact clearly indicates that optimal paths in strongly disordered media are related to the well-known LERW on a critical percolation cluster. Since LERW is equivalent to UST and the

TABLE I. Classification of LERW and some related models into two distinct universality classes (the watershed and ranked percolation in three dimensions are random surfaces).

Models	$d = 2$	$d = 3$
MST	1.22 ± 0.01 [22]	1.46 ± 0.02 [23]
Optimal paths	1.22 [24,25]	1.44 [24,25]
Invasion percolation	1.22 ± 0.01 [23]	1.42 ± 0.02 [23]
Watershed	1.217 ± 0.0015 [11]	
Ranked percolation	1.215 ± 0.003 [12]	
LERW_p at $p = p_c$	1.217 ± 0.002	1.43 ± 0.02
Euclidean LERW (UST, ASM, and q-state Potts in $q \to 0$)	$\frac{5}{4}$ [6,7]	1.62400 ± 0.00005 [18]

q-state Potts model (in the limit of $q \to 0$), it is interesting to study these models in a diluted lattice generating by a sequence of random deletions. Because of the negative specific heat exponent of the pure system, the Harris criterion [28] claims that the universality of this model should remain unchanged. However, as it was reported for spanning trees on a critical percolation cluster [29,30], it is in a different universality class from the UST model. Although here we restrict our attention to two and three dimensions, the findings for spanning trees on a critical percolation cluster [29] indicate that the upper critical dimension of this model is likely to be the same as the percolation model. The LERW on a percolation cluster is related to a more general concept of random walks on a fractal landscape; as another example, it was shown recently that the fractal dimension of LERW on a Sierpinski gasket is ≈1.194 [31,32]. Finally, our results, summarized in Table I, demonstrate that this model can be classified into two distinct universality classes, LERW on *Euclidean* and *fractal* geometries. Near the percolation threshold p_c, there is a crossover regime, shown in Fig. 2, between these two universality classes. To achieve a better understanding of this regime, we have considered the mean total length of LERW_p as a homogeneous function on the lattice size and occupation probability. Our findings for the crossover regime, shown in Fig. 3, clearly demonstrate that for a fixed lattice size L, three distinct scaling regimes have to be distinguished: (a) a fractal regime for $p' < L^{-\theta}$; (b) a Euclidean geometry regime for $p' > L^{-\eta}$; and (c) a transition regime from the fractal to the Euclidean behaviors for $L^{-\theta} < p' < L^{-\eta}$. These regimes are separated by two crossovers. Finally, there is a scaling relation between the corresponding crossover exponents and the fractal dimension of LERW. In general, the existence of three different scaling behaviors in a system often leads to three distinct regimes, as it has been reported in both related [13,14] and unrelated systems (for example, see Refs. [33,34]). Although there are some similar reports in which the crossover behavior has been investigated by just one crossover exponent [12,35–37], the existence of three regimes clearly can be observed in the related figures. Moreover, the proposed scaling ansatz for the systems is not valid in the entire crossover region. The connection between LERW_p and other important models such optimal paths in strongly disordered media and the watershed model allows one to look at such random paths with a different eye and to build bridges between the connectivity in disordered media and other research areas in mathematics, percolation, and quantum field theory. This work opens up several challenges. Besides the need for more precise numerical simulations in higher dimensions to study the fractal properties and the crossover exponents, it would be interesting to formulate a field theory scheme in a fractal landscape. Since the continuum limit of the planar LERW on the Euclidean lattice can be descried with the Schramm-Loewner evolution [38], another interesting possibility is to find a conformal field theory for LERW on critical percolation clusters [39].

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