PHYSICAL REVIEW E **89**, 061003(R) (2014)

Selection of the Taylor-Saffman bubble does not require surface tension

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(Received 15 February 2013; revised manuscript received 18 June 2013; published 20 June 2014)

A new general class of exact solutions is presented for the time evolution of a bubble of arbitrary initial shape in a Hele-Shaw cell when surface tension effects are neglected. These solutions are obtained by conformal mapping the viscous flow domain to an annulus in an auxiliary complex plane. It is then demonstrated that the only stable fixed point (attractor) of the nonsingular bubble dynamics corresponds precisely to the selected pattern. This thus shows that, contrary to the established theory, bubble selection in a Hele-Shaw cell does not require surface tension. The solutions reported here significantly extend previous results for a simply connected geometry (finger) to a doubly connected one (bubble). We conjecture that the same selection rule without surface tension holds for Hele-Shaw flows of arbitrary connectivity.

DOI: 10.1103/PhysRevE.89.061003 PACS number(s): 47.20.Gv, 02.30.Ik, 47.20.Ma, 47.54.Bd

Introduction. It is remarkable that numerous processes of pattern formation, ranging from dendritic and fractal growth to viscous fingering and bacterial colony growth, have (after some idealization) the same compact mathematical formulation [1,2]. Subsequent development of this formulation, called *Laplacian growth*, significantly widened the list of connections by including one-dimensional (1D) turbulence and generation of complex shapes [3], quantum gravity, integrable systems, random matrices, and conformal theories [4]. The problem of great importance in some processes mentioned above was selection of the observed pattern from continuously many solutions.

Background. It has been widely accepted that surface tension is necessary for selecting a single pattern (moving two times faster than the background flow) from a continuum of solutions in interface dynamics after it was first conjectured by Saffman and Taylor for viscous fingers in a Hele-Shaw cell [5]. But verifying this selection scenario was not possible until much later because of significant mathematical difficulties related to including surface tension. After the seminal work of Kruskal and Segur [6], these difficulties were finally resolved, and in a series of works [7] it was shown that surface tension indeed selects the observed pattern; see [8] for details.

Recent development and new challenges. More recently, however, it was demonstrated in [9], by using time-dependent exact solutions without surface tension, that contrary to the previously mentioned works, selection of the Saffman-Taylor finger is determined entirely by the zero surface tension dynamics. This result makes us reconsider the role of surface tension in a more general problem: selection of multi-connected patterns. The significance of the multi-connected case is that simple conformal mappings used in [9] are no longer applicable, and even a *steady-state* case requires advanced mathematics that is less traditional for physical applications [10,11]. No wonder, then, that searching for *time-dependent* exact solutions in multi-connected geometry, the crucial step of studying selection, presents a mathematical challenge.

Outline of main results. In the present work we solve the selection problem for a doubly connected geometry and significantly extend the results obtained in [9] by addressing the dynamics of a bubble in a Hele-Shaw cell instead of a finger. This extension allows us to conjecture that surface tension (when small enough) is not required for pattern selection in Laplacian growth in domains of arbitrary connectivity.

The pattern selection problem for an inviscid bubble dragged by a viscous flow in a Hele-Shaw cell was posed by Taylor and Saffman [12] and was later addressed experimentally [13] and theoretically [14]. The problem was to select, from the continuum of steady-state solutions obtained for zero surface tension, the unique bubble [12] with velocity twice the background flow velocity. While it has long been known that the inclusion of surface tension leads to velocity selection [14], we demonstrate here that the selection mechanism does not require surface tension because the selected pattern is the only stable fixed point (attractor) of the nonsingular bubble dynamics *without surface tension*.

Problem formulation and plan of the paper. A top view of a Hele-Shaw channel with lateral walls at $y = \pm\pi$ in our scaled units and with the bubble moving to the right is shown in Fig. 1(a). The fluid (oil) velocity obeys the two-dimensional (2D) Darcy law, $\mathbf{v} = -\nabla p$, where p is scaled pressure. Far from the bubble the oil flows along the Ox axis with uniform velocity, $V = 1$; thus $p = -x$, when $|x| \to \infty$. Owing to incompressibility, $\nabla \cdot \mathbf{v} = 0$, p is harmonic, $\nabla^2 p = 0$, in the viscous domain $D(t)$, where t denotes time. It is thus convenient to introduce a complex potential, $W = \Phi + i\Psi$, where $\Phi = -p$ and Ψ is the stream function. In view of the uniform far-field velocity, one has $W \approx z$ for $|x| \to \infty$, so $\Psi = \pm\pi$ at $y = \pm\pi$ since the lateral walls are streamlines. Because pressure is constant (taken to be zero) in the inviscid bubble and continuous across the oil-bubble interface $\Gamma(t)$ if surface tension is neglected, then $\Phi = 0$ at $\Gamma(t)$. The fluid domain in the W plane is shown in Fig. 1(b), where the vertical slit maps to the interface $\Gamma(t)$ and the rest of the horizontal

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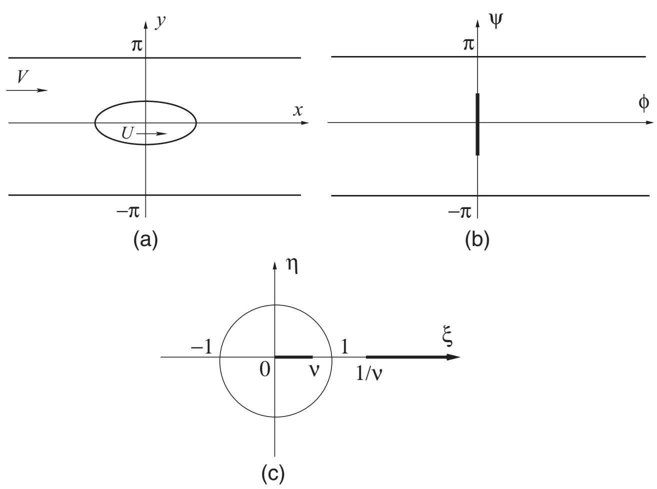

FIG. 1. (a) The fluid domain $D(t)$ for a moving bubble in a Hele-Shaw channel and the corresponding domains (b) in the complex potential plane and (c) in the auxiliary ζ plane.

strip, $-\pi < \Psi < \pi$, maps to the domain $D(t)$ in the physical plane [Fig. 1(a)]. The kinematic identity, $V_n = v_n$, stating the equality of the normal velocities of the interface, V_n, and of the fluid, v_n, completes the formulation of this free-boundary problem of finding $\Gamma(t)$ given $\Gamma(0)$.

This long-standing nonlinear unstable problem [15] was shown to have an integrable structure and to possess deep connections with other branches of mathematical physics [4]. While numerous time-dependent solutions were obtained (listed in [16]), almost all of them described simply connected domains.

Here we present exact solutions for an evolving bubble in a Hele-Shaw cell when the fluid domain $D(t)$ is doubly connected [see Fig. 1(b)]. We then show that these solutions explain not only *how* the moving bubble eventually reaches a stationary shape when $t \to \infty$ but also *why* the selected bubble moves precisely twice as fast as the background flow when surface tension effects become negligible, in agreement with [13].

In what follows, we first address the dynamics of a bubble with a shape symmetric with respect to the channel center line. The solutions we obtain for this simpler case illustrate our main result, namely, that all nonsingular solutions describing bubble shapes converge to the only stable fixed point of this infinitely dimensional dynamical system, which is precisely the selected member of the continuous family [12]. After that, we extend our solutions to asymmetric bubbles and arrive at the conclusion that $U = 2$ is still the selected value.

The conformal map and the equation of motion. For a symmetric bubble we introduce a conformal map $z = f(t,\zeta)$ from the exterior of the unit circle with a cut in the complex plane, $\zeta = \xi + i\eta$ [see Fig. 1(c)], onto the fluid domain $D(t)$ in the z plane, $z = x + iy$. The unit circle, $|\zeta| = 1$, maps onto the interface $\Gamma(t)$, and the cut sides, $\zeta = \xi \pm i0$, where $1 < 1/\nu(t) < \xi < \infty$, map onto the channel walls, $y = \pm\pi$, so that $\zeta = 1/\nu$ and $\xi = +\infty$ are mapped to $x = +\infty$ and $x = -\infty$, respectively. Thus, the polar angle $\phi = \arg(\zeta)$ parametrizes $\Gamma(t)$: $z = f(t,e^{i\phi})$. It is easy to see that the complex potential, $W = \Phi + i\Psi$, satisfying all aforementioned boundary conditions, is

$$W(t,\zeta) = \log\frac{1-\nu/\zeta}{1-\nu\zeta}. \tag{1}$$

Let us express the normal velocity of the interface V_n as

$$V_n = V_1 l_2 - V_1 l_1 = \mathrm{Im}(\bar{V} l) = \mathrm{Im}(\bar{z}_t z_s),$$

where $l = l_1 + i l_2 = dz/ds$ is the unit tangent vector along the interface, $\Gamma(t) = z(t,s)$, parametrized by an arclength s, and the subscripts are partial derivatives. The normal fluid velocity we rewrite as

$$v_n = \partial_n \Phi = \partial_s \Psi$$

(the Cauchy-Riemann condition). Equating the two last formulas, as required by the kinematic identity, and reparametrizing $s \to \phi$, we obtain $\mathrm{Im}(\bar{z}_t z_\phi) = \Psi_\phi$. Calculating Ψ_ϕ from (1) for $\zeta = e^{i\phi}$, we obtain the equation for the moving interface, $z(t,\phi) = f(t,e^{i\phi})$:

$$\mathrm{Im}(\bar{z}_t z_\phi) = \mathrm{Re}\frac{2\nu}{e^{i\phi} - \nu}. \tag{2}$$

For stationary solutions, $z(t,\phi) = Ut + Z(\phi)$, where the velocity U is a constant, Eq. (2) is simplified to

$$\mathrm{Im}(Z_\phi) = \frac{2\nu}{U}\mathrm{Re}\frac{1}{e^{i\phi} - \nu}. \tag{3}$$

The solution of (3) is the sum of two logarithms:

$$Z = -\log(1 - \nu e^{i\phi}) + \alpha \log(1 - \nu e^{-i\phi}), \tag{4}$$

where the coefficient of the first term is chosen to satisfy $W = z$ in (1) when $z \to \infty$. Expression (4) is precisely the one-parameter family of stationary bubbles obtained in [12]. Substituting (4) into (3), we obtain

$$U = \frac{2}{1+\alpha}. \tag{5}$$

We will show below that all solutions with $\alpha \neq 0$ are unstable and, if perturbed, move to the solution with $\alpha = 0$, which corresponds to the selected value, $U = 2$ [12,13].

The finite-parametric solutions. Being integrable, Eq. (2) possesses a rich list of exact solutions, many of which blow up in finite time. Leaving those aside as physically nonrealizable, we present here a new class of finite-parametric *nonsingular* solutions (analogous of those obtained earlier [17] for finger dynamics), which remain finite for all times:

$$z = \tau(t) - \log[1 - \nu(t)e^{i\phi}] + \sum_{k=0}^{N} \alpha_k \log[1 - a_k(t)e^{-i\phi}], \tag{6}$$

where $\alpha_0 = \alpha$ and τ are both real, $a_0 = \nu$, $|a_k| < 1$ for all times, and α_k are constants. These parameters must be chosen so that critical points of the conformal map always stay inside the unit circle to prevent blow ups [18]. Also, the symmetry of the bubble requires that each term with complex α_k, a_k implies the term with $\bar{\alpha}_k, \bar{a}_k$ in the sum. It is easy to verify that (6) is indeed a solution of (2), where the time dependence of τ, ν, and a_k is given by the following $N + 2$ equations:

$$\beta_k = \tau + \log \bar{a}_k/(\bar{a}_k - \nu) + \sum_{l=0}^{N} \alpha_l \log(1 - a_l \bar{a}_k), \tag{7a}$$

$$2t + 2t_0 = (1+\alpha)\tau + (\alpha^2 - 1)\log(1-\nu^2) + \alpha \sum_{k=1}^{N} \alpha_k \log(1-\nu a_k) + \sum_{k=1}^{N} \alpha_k \log(1 - a_k/\nu) \quad (7b)$$

$$A = -\log(1-\nu^2) + \sum_{k=0}^{N}\sum_{l=0}^{N} \alpha_k \bar{\alpha}_l \log(1 - a_k \bar{a}_l), \quad (7c)$$

with $k = 1, \ldots, N$ in (7a). Here the β_k's, the initial time t_0, and the bubble area A are the constants of motion.

The attractor. It follows from (7b) and (7c) that $\tau(t) \to Ut$ when $t \to +\infty$. Since β_k is a constant, a real part of at least one logarithm in (7a) should go to $-\infty$ in large times so as to compensate a divergent positive $\tau(t)$. This is possible only if all $a_k(t) \to 0$ for $k > 0$, as $t \to \infty$. Thus, we conclude that the origin, $\zeta = 0$, attracts all $a_k(t)$ for $k \geqslant 1$. Thus, for $t \to \infty$ the only nonvanishing parameter among the a_k's is a_0, which we have identified with ν, so $a_0(t) = \nu(t)$ for all times. But in this case, solution (6) asymptotically approaches the family of stationary bubbles (4) discussed above, namely,

$$z = \frac{2t}{1+\alpha} - \log(1 - \nu\, e^{i\phi}) + \alpha \log(1 - \nu e^{-i\phi}).$$

To test the stability of the trajectory $a_0(t) = \nu(t)$, we deviate $a_0(0)$ slightly from $\nu(0)$. Namely, we replace the $\alpha \log(1 - \nu e^{-i\phi})$ in (6) by $\frac{\alpha}{2}\log(1 - a_0 e^{-i\phi}) + \frac{\alpha}{2}\log(1 - \bar{a}_0 e^{-i\phi})$ following the symmetry mentioned earlier. Thus, we obtain the following dynamics for a_0, ν, and τ:

$$\beta_0 = \tau + \log \bar{a}_0/(\bar{a}_0 - \nu) + (\alpha/2)\log\left[(1-|a_0|^2)(1-\bar{a}_0^2)\right],$$
$$2(t+t_0) = \tau - \log(1-\nu^2) + \alpha \log|1 - a_0/\nu|,$$
$$A = -\log(1-\nu^2) + (\alpha^2/2)\log\left[(1-|a_0|^2)\left|1 - a_0^2\right|\right].$$

These equations clearly show that $a_0 \to 0$, $\tau \to 2t$, and $\nu \to \sqrt{1 - e^{-A}}$ when $t \to \infty$, so the interface becomes

$$z = 2t - \log(1 - \nu e^{i\phi}). \quad (8)$$

This is precisely the selected pattern with $U = 2$. Thus, $\zeta = \nu$ repels nearby singularities, which move toward zero. Therefore, the selected bubble (8) represents the only attractor, $\zeta = 0$, of the nonsingular subset of the finite-dimensional dynamical systems (6).

The infinite-parametric solutions. Extending (6) to the case of an infinite number of parameters, we obtain

$$z = \tau(t) + \log \frac{1}{1 - \nu(t) e^{i\phi}} + \sum_{k=0}^{N} \int_0^{a_k(t)} \frac{\rho_k(t,w)}{w - e^{i\phi}}\, dw, \quad (9)$$

where $a_0 = \nu$ and $|a_k| < 1$ for all k. Solution (9) coincides with (6), if the functions $\rho_k(t,w)$ are constants, and contains all previously known solutions for simply connected and symmetric doubly connected cases [19]. The constants of motion for (9), analogous to β_k in (7), are $B_k = f[1/\bar{a}_k(t)]$, which yield [20]

$$B_k = \tau(t) + \log \frac{\bar{a}_k(t)}{1 - \nu(t)\bar{a}_k(t)} + \sum_{l=0}^{N} \int_0^{a_l(t)} \frac{\rho_l(t,w)}{w - 1/\bar{a}_k(t)}\, d\zeta$$

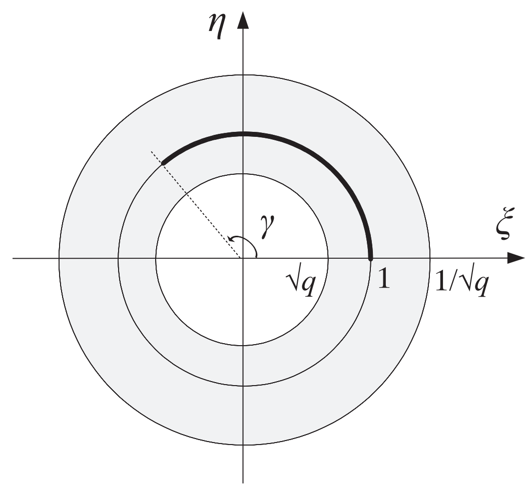

FIG. 2. Flow domain (shaded region) in the ζ plane for an asymmetric bubble; see text.

for $k = 1, \ldots, N$. Both a proof that $\tau \sim t$, as $t \to \infty$, and procedure of testing the points, ν and 0, for stability are the same as above with the same conclusion, that all $a_k \to 0$ for $t \to \infty$, implying that an arbitrary shape, expressed by (9), moves toward the selected bubble (8) with $U = 2$.

Nonsymmetric case. The complex potential for a nonsymmetric (with respect to the channel center line) shape requires infinite reflected images, so we conformally map the annulus, $0 < \sqrt{q} < |\zeta| < 1$, in the ζ plane (see Fig. 2) onto the fluid domain $D(t)$ in the z plane, so that the inner circle, $|\zeta| = \sqrt{q}$, is mapped onto the interface $\Gamma(t)$. The unit circle, $|\zeta| = 1$, is mapped onto the channel walls, $y = \pi$ and $y = 0$. Under inversion with respect to the unit circle (see Fig. 2), we obtain the preimage of the domain $\bar{D}(t)$, which is the complex conjugate of $D(t)$. We map

$$\{\sqrt{q} < |\zeta| < 1/\sqrt{q}\} \to \{D(t) \cup \bar{D}(t)\},$$

where the annulus $\sqrt{q} < |\zeta| < 1/\sqrt{q}$ is cut along the part of the unit circle where $0 < \arg\zeta < \gamma$, so that the inner (outer) cut side is mapped onto the upper wall (its mirror image), where $y = \Psi = \pm\pi$, while the complimentary arc along the unit circle, $\gamma < \arg\zeta < 2\pi$, is mapped onto the south wall, where $y = \Psi = 0$. We fix the map by sending the points $\zeta = 1$ to $x = +\infty$ and $\zeta = e^{i\gamma}$ to $x = -\infty$.

The complex potential for the nonsymmetric case is

$$W(\zeta) = i\gamma/2 + \log \frac{\Theta(e^{-i\gamma}\zeta)\,\Theta(q\zeta)}{\Theta(qe^{-i\gamma}\zeta)\,\Theta(\zeta)},$$

where

$$\Theta(\zeta) = (1-\zeta)\prod_{m=1}^{+\infty}(1 - q^{2m}\zeta)(1 - q^{2m}/\zeta) = \frac{\vartheta_4(\log(\sqrt{\zeta/q}), q)}{\prod_{m=1}^{+\infty}(1 - q^{2m})}$$

and

$$\vartheta_4(w,q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \exp(2nw)$$

is the Jacobi theta function [21]. It is easy to verify that $W(\zeta)$ satisfies the boundary conditions stated above.

Finite-parametric solutions for the interface in this case have the form

$$z(t,\phi) = \tau(t) + i\gamma(t)/2 + \log\frac{\Theta(e^{i(\phi-\gamma(t))})}{\Theta(e^{i\phi})} + \sum_{k=1}^{N}[\alpha_k \log\Theta(a_k(t)e^{-i\phi}) + \bar{\alpha}_k \log\Theta(\bar{a}_k(t)e^{i\phi})]. \quad (10)$$

Here all $|a_k| < 1$, and $\sum_{k=1}^{N}\alpha_k = 0$. Since y is a multiple of π when $|\zeta| = 1$, τ is purely real. Inserting (10) into (2) and integrating the resulting equations of motion, we obtain $N+2$ complex constants of motion: $\beta_k = f(t,q/\bar{a}_k)$ for $k = 1,\ldots,N$, $\beta_+ = f(t,q)$, and $\beta_- = f(t,qe^{i\gamma})$, where

$$\beta_k = \tau + i\gamma/2 + \log[\Theta(q\,e^{i\gamma}\bar{a}_k)/\Theta(q\bar{a}_k)] + \sum_{l=1}^{N}[\alpha_l \log\Theta(a_l\bar{a}_k/q) + \bar{\alpha}_l\,\log\,\Theta(q\bar{a}_l/\bar{a}_k)], \quad (11a)$$

$$\beta_\pm = \tau - 2t + i\gamma/2 \pm \log[\Theta(q\,e^{i\gamma})/\Theta(q)] + \sum_{l=1}^{N}[\alpha_l \log\Theta(e^{-i\gamma_\pm}\,a_l/q) + \bar{\alpha}_l \log\Theta(qe^{i\gamma_\pm}\bar{a}_l)], \quad (11b)$$

where $\gamma_+ = 1$ and $\gamma_- = \gamma$. The constants β_+ and β_- are not independent since $\mathrm{Im}\,\beta_+ = \mathrm{Im}\ \beta_- = \gamma/2 + \mathrm{Im}\sum_{l=1}^{N}\alpha_l \log a_l$. Formulas (11) constitute the full dynamics of a_k, γ, q, and τ. The bubble area A, while fixed in time, is not an independent constant of motion; it is neatly expressed through other constants as

$$A/\pi = \beta_+ - \beta_- + 2\mathrm{Re}\sum_{k=1}^{N}\bar{\alpha}_k\beta_k.$$

The attractor. After eliminating τ from (11a), by subtracting β_+ (or β_-) from β_k, we see that the term $2t$ in the resulting equation must be canceled by a divergent logarithmic term. This implies that all $a_k \neq q$ move to the point $qe^{i\gamma}$ when $t \to \infty$. The points q and $qe^{i\gamma}$ are the repeller and the attractor, respectively, for the dynamical system (11). If one of the a_k's, say, a_1, was initially at the repeller, $a_1 = q$, then β_1 diverges, and $U = 2/(1+\alpha_1) \neq 2$ when $t \to \infty$. After relocating a_1 out of q [by setting $a_1(0) \neq q(0)$], the asymptotic velocity of the bubble reaches the same selected value, $U = 2$, as in the symmetric case.

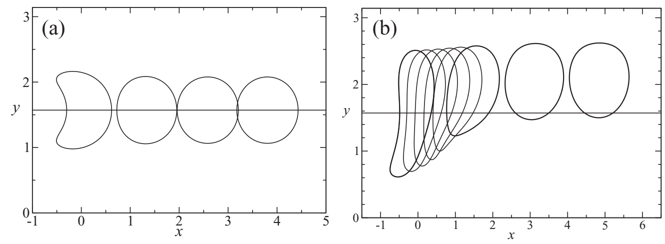

FIG. 3. Examples of bubble evolution: (a) symmetric solution and (b) asymmetric solution.

For $\gamma = \pi$, solution (10) describes symmetric bubbles and thus recovers (6), albeit in a different formulation. A symmetric bubble evolution is shown in Fig. 3(a), where the asymptotic shape corresponds to the Taylor-Saffman bubble [12] with $U = 2$, as described by (8). In Fig. 3(b) we show an asymmetric solution whose asymptotic shape coincides with the asymmetric bubble obtained in [10,14] for $U = 2$ [22]. Without presenting here the analysis for a nonsymmetric bubble with infinitely many parameters, let us mention that in this case the selected velocity is also $U = 2$ when $t \to \infty$.

Discussion. The results presented here and in [9] unambiguously indicate that the stability of the selected pattern, with respect to the rest of the family, is built in the Laplacian growth *without surface tension* [23]. In this context, surface tension is just one of infinitely many perturbations (perhaps the most relevant) which kick the system toward the attractor while also regularizing high curvatures. The selected pattern, so obtained, although linearly unstable in the absence of surface tension, is *stable asymptotically*: if perturbed, it eventually recovers its original shape.

Since the selection mechanism in both simply connected and doubly connected geometries is due to the attractor for all nonsingular solutions with zero surface tension, we conjecture that the same holds for Laplacian growth in domains of arbitrary connectivity.

Acknowledgments. One of us (M.M.-W.) thanks MPIPKS (Dresden) and UPFE (Recife) for hospitality while this work was done.

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[18] One should also stay away from initial data leading to the loss of univalence of the interface. It is easy to accommodate, and we do not discuss the details here.

[19] Apparently one can approximate any nonsingular solution of (2) by (9), although we have not proved this.

[20] The rest of the formulas related to solution (9) will be presented elsewhere.

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[22] Since all a_k move toward the same point, $a_k \to qe^{i\gamma}$, for $t \to \infty$, the sum in (11) vanishes because $\sum_{k=1}^{N} \alpha_k = 0$, so $\gamma(t) \to \mathrm{Im}\beta_+ = \mathrm{Im}\beta_-$, which is a constant of motion provided by the initial conditions. Thus it remains to be seen how to centralize a bubble in our framework (if possible at all) so that $\gamma \to \pi$ as $t \to \infty$.

[23] With properly chosen parameters, the obtained solutions, expressed by (6), (9), and (10), can faithfully describe dynamics with nonzero surface tension.