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Rapid geometrical chaotization in slow-fast Hamiltonian systems

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In this Rapid Communication we demonstrate effects of a new mechanism of adiabaticity destruction in Hamiltonian systems with a separatrix in the phase space. In contrast to the slow diffusive-like destruction typical for many systems, this new mechanism is responsible for very fast chaotization in a large phase volume. To investigate this mechanism we consider a Hamiltonian system with two degrees of freedom and with a separatrix in the phase plane of fast variables. The fast chaotization is due to an asymmetry of the separatrix and corresponding geometrical jumps of an adiabatic invariant. This system describes the motion of charged particles in a inhomogeneous electromagnetic field with a specific configuration. We show that geometrical jumps of the adiabatic invariant result in a very fast chaotization of particle motion.

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Theory of adiabatic invariants is an important element of modern theoretical physics. Progress in this theory is often related to the investigation of charged particles motions in electromagnetic fields. This line of research was started with the discovery of the conservation of magnetic moment for charged particles motion in a strong magnetic field [1]. Further research on confinement of particles in magnetic traps resulted in development of the theory of eternal adiabaticity [2] as well as the theory of violation of adiabaticity [3]. For many systems, the destruction of adiabatic invariance corresponds to crossings of separatrices in a phase space by phase trajectories. Although, the evolution of adiabatic invariants in systems with separatrix crossings has been mentioned already by Ehrenfest [4], this problem was thoroughly studied for the first time for the charged particle trappings by waves [5]. Further, the asymptotic expressions for jumps of adiabatic invariants due to separatrix crossings were derived for a wide class of Hamiltonian systems [6,7]. In systems with a certain symmetry, accumulation of these jumps results in a slow diffusive-like destruction of adiabatic invariance.

The presence of an adiabatic invariant in a Hamiltonian system is associated with a certain periodicity of motion in the phase space. Namely, dynamics is represented as a composition of a fast periodic motion and a slow evolution of this motion. The parameter $\kappa \ll 1$ defines the ratio of time scales of these two motions. Averaging over the periodic motion excludes its phase from the Hamiltonian. The canonically conjugate to this phase variable is an invariant of the averaged system. It is an adiabatic invariant of the exact system [8]. General theory [6,7] predicts that the adiabatic invariant experiences so-called jumps when the phase point crosses a narrow neighborhood of the separatrix. Each jump consists of two parts: 1) the dynamical jump resulting from the singularity of the period of motion in the vicinity of the separatrix; 2) the geometrical jump resulting from the difference of areas surrounded by a periodic trajectory before and after separatrix crossing. Although, certain theories predict that geometrical jumps are more important for the destruction of the adiabatic invariant than dynamical jumps [9,10], only dynamical jumps were considered before for physical systems with multiple crossings of the separatrix. The amplitude of the dynamical jump is $\sim\kappa \ln\kappa$ [6,7]. A precise value of the dynamical jump strongly depends on the particular location of the crossing at the separatrix and could be considered as a quasi-random value [6,7]. Thus, due to dynamical jumps the adiabatic invariant changes only slightly for $\kappa \ll 1$ and might be considered as a random quantity for a single crossings. In this case, chaotization of motion has a diffusive-like character. In this Rapid Communication we demonstrate that in systems with asymmetric phase portraits the asymmetry results in the appearance of a new effect—the geometrical chaotization, which could become much faster than the typical chaotization of motion due to separatrix crossings investigated before. The main role in this mechanism is played by geometrical jumps of the adiabatic invariant.

The important and characteristic example of the destruction of the adiabaticity due to separatrix crossings can be found in the system describing the dynamics of charged particles in the current sheet [11]. Particle dynamics in the magnetic field configuration with the current sheets is described by a nonautonomous Hamiltonian system with two degrees of freedom (two pairs of conjugate variables), whose Hamiltonian is

$$H = \tfrac{1}{2}p_z^2 + \tfrac{1}{2}(p_x - sz)^2 + \tfrac{1}{2}\left(\kappa x - \tfrac{1}{2}z^2 - \varepsilon t\right)^2. \quad (1)$$

Here we use normalized variables (z,p_z), (x,p_x) and parameters $\kappa \ll 1$, $\varepsilon \leqslant \kappa$, and $0 \leqslant s < 1$ (see the relations of dimensionless parameters κ, s, and ε to the corresponding physical effects at the end of this Rapid Communication).

For $s = 0$ one gets the Hamiltonian system described in details in [12]. In this case, one can change variables $\kappa x \to \kappa x - \varepsilon t$ and $p_x \to p_x - u_d$, where $u_d = \varepsilon/\kappa$. The new Hamiltonian is

$$\bar{H} = \tfrac{1}{2}p_z^2 + \tfrac{1}{2}p_x^2 + \tfrac{1}{2}\left(\kappa x - \tfrac{1}{2}z^2\right)^2 \quad (2)$$

and $\bar{H} = H + (1/2)u_d^2$. Hamiltonian (2) is symmetric relative to the plane $z = 0$, i.e., transformation $z \to -z$ does not change the Hamiltonian.

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For $s \neq 0$ we have the following Hamiltonian obtained from Eq. (1) after the same change of variables $\kappa x \to \kappa x - \varepsilon t$, $p_x \to p_x - u_d$:

$$\bar{H} = \tfrac{1}{2}p_z^2 + \tfrac{1}{2}(p_x - sz)^2 + \tfrac{1}{2}\left(\kappa x - \tfrac{1}{2}z^2\right)^2 - szu_d. \quad (3)$$

Hamiltonian (3) cannot be transformed to Eq. (2) due to the term $\sim szu_d$. This term makes the system asymmetric relative the plane $z = 0$ and plays an important role in dynamics. For the particular case with $\varepsilon = 0$ (and $u_d = 0$) we obtain the Hamiltonian described in details in [13]:

$$\bar{H} = \tfrac{1}{2}p_z^2 + \tfrac{1}{2}(p_x - sz)^2 + \tfrac{1}{2}\left(\kappa x - \tfrac{1}{2}z^2\right)^2. \quad (4)$$

This Hamiltonian is invariant relative to the transformation $z \to -z$, $p_x \to -p_x$. Below we show that the effect of chaotization due to geometrical jumps is present only in the system with $\varepsilon \neq 0$ and $s \neq 0$, while even one additional symmetry for systems with $s = 0$, $\varepsilon \neq 0$ [12] and $s \neq 0$, $\varepsilon = 0$ [13] eliminates this effect.

Hamiltonians (2), (3), and (4) do not explicitly depend on time. Therefore, phase points move in a four-dimensional space $(z, p_z, \kappa x, p_x)$ at the three-dimensional surface $\bar{H}(z, p_z, \kappa x, p_x) = h = \text{const}$. Variables $(\kappa x, p_x)$ change with time substantially slower than variables (z, p_z), because of $\kappa \ll 1$ smallness. Thus, we have a slow-fast Hamiltonian system. For the frozen slow variables $(\kappa x, p_x)$ any phase trajectory is represented as some curve in the plane (z, p_z). For the simplest case of $s = 0$ [system (2)] the phase portraits of this plane are shown in Fig. 1(a). Phase domains filled by trajectories of two types (shown in left and right panels) are demarcated by the separatrix (shown by the red curve). A slow change of $(\kappa x, p_x)$ results in the evolution of the trajectory in the (z, p_z) plane. Eventually, the trajectory crosses the separatrix and simultaneously the projection of the phase point onto the $(\kappa x, p_x)$ plane crosses the so-called uncertainty curve in this plane [shown by the red curve in Fig. 1(b) for system with $s = 0$]. The uncertainty curve demarcates domains which correspond to different types of motion with $\bar{H}(z, p_z, \kappa x, p_x) = h$ in the (z, p_z) plane. Each point in the grey domain in Fig. 1(b) corresponds to two possible trajectories in the (z, p_z) plane. Thus, there are two copies of this domain.

In the plane (z, p_z) trajectories are closed. Thus, one can introduce the adiabatic invariant $I_z = (1/2\pi)\oint p_z dz$ [8]. This is an approximate integral of motion in the slow-fast system. The equation $I_z(\kappa x, p_x, h) = \text{const}$ determines trajectories in the plane $(\kappa x, p_x)$ for a fixed value of h [see Fig. 1(b)].

Crossing of the separatrix in the (z, p_z) plane results in the dynamical jump $\Delta I_z^{\text{dyn}} \sim \kappa \ln \kappa$ [$\Delta I_z^{\text{dyn}} \sim \kappa$ for symmetric Hamiltonian (2)] and the geometrical jump ΔI_z^{geom} [14]. Thus, trajectories deviate from the adiabatic approximation $I_z(\kappa x, p_x, h) = \text{const}$. In the adiabatic approximation a time interval $\sim 1/\kappa$ corresponds to two crossings of the separatrix by the trajectory: one with the decrease of κx and another one with the increase of κx [see Fig. 1(b)]. The sum of two dynamical jumps $\sum \Delta I_z^{\text{dyn}}$ is a random value with an amplitude $\sim \kappa$ [6,7]. For Hamiltonian (2) each point of the grey domain in the $(\kappa x, p_x)$ plane corresponds to two identical trajectories in the (z, p_z) plane. Thus, the first geometrical jump results in a doubling of the area surrounded by the trajectory, while the second geometrical jump exactly compensates this effect and the sum of two geometrical jumps is exactly equal to zero. In Fig. 2(a) an example of trajectory is presented for Hamiltonian (2). A compensation of two successive geometrical jumps is well seen, while dynamical jumps are responsible for small change of the trajectory in comparison with the adiabatic approximation $I_z(\kappa x, p_x, h) = \text{const}$.

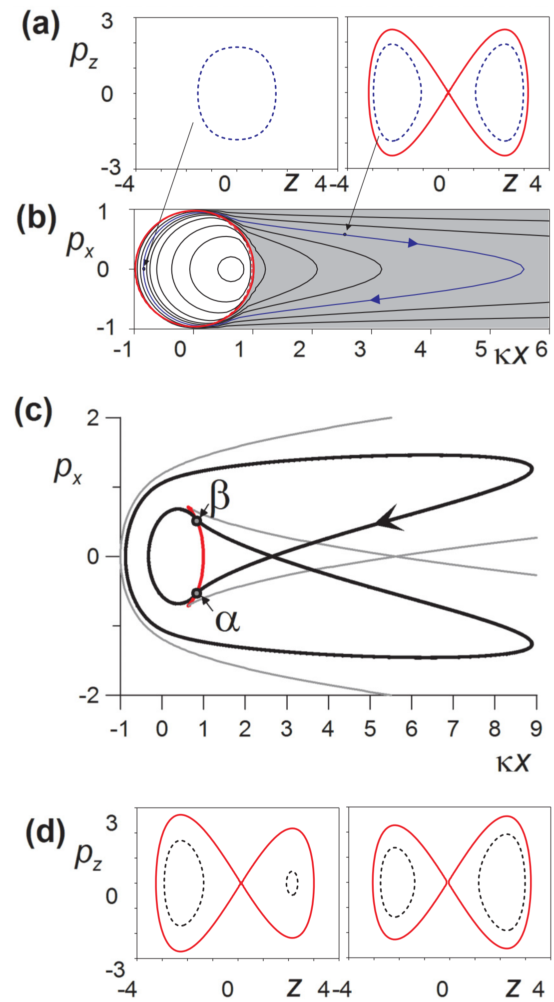

FIG. 1. (Color online) (a) shows two types of phase portrait of system (2) in the plane (z, p_z). (b) shows the phase portrait of system (2) in the $(\kappa x, p_x)$ plane. Each trajectory has the equation $I_z(\kappa x, p_x, h) = \text{const.}$ with a certain value of I_z. (c) shows an example of particle trajectory in the $(\kappa x, p_x)$ plane in system (4) with $s = 0.2$. (d) shows two examples of phase portrait of system (3) with $s = 0.2$ in the plane (z, p_z).

For system (2) the averaged value of ΔI_z^{dyn} for the long-term dynamics is equal to zero [11]. Thus, there is only a slow diffusion of I_z provided by random changes $\Delta I_z^{\text{dyn}} \sim \kappa$. The time interval between such changes is about κ^{-1} and, as a result, I_z could change substantially only at time interval $\sim \kappa^{-3}$. This relatively slow process is shown in Fig. 2(b) where values of the adiabatic invariant are calculated at points $p_x = 0, \kappa x < 0$ [one point for one period of oscillations in the $(\kappa x, p_x)$ plane].

The phase portrait in the plane (z, p_z) for Hamiltonian (4) is asymmetric relative to the plane $z = 0$. However, there is the additional symmetry: the transformation $z \to -z$ can be

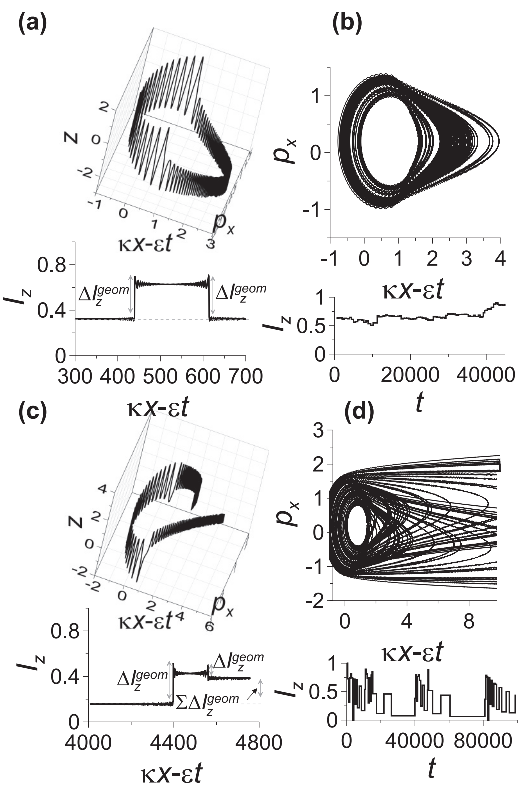

FIG. 2. (a) shows a fragment of particle trajectory in 3D space and corresponding evolution of I_z for system (2). (b) shows a particle trajectory for system (2) and I_z as a function of time (I_z is calculated at points $p_x = 0, \kappa x < 0$). (c) shows a fragment of particle trajectory in 3D space and corresponding evolution of I_z for system (3). (d) shows a particle trajectory for system (3) and I_z as functions of time (I_z is calculated at points $p_x = 0, \kappa x < 0$).

compensated by the transformation $p_x \to -p_x$. As a result, trajectories in the plane $(\kappa x, p_x)$ are mirror symmetric relative to the line $p_x = 0$ [see Fig. 1(c)]. In two points of the uncertainty curve crossing (α and β) phase portraits in the (z, p_z) plane are asymmetric relative to the plane $z = 0$, but these two portraits can be transformed one to another by the change $z \to -z$. Thus, geometrical jumps for Hamiltonian (4) results only in the splitting of adiabatic trajectories: instead of one trajectory in the $(\kappa x, p_x)$ plane we obtain some set of trajectories (each of these trajectories corresponds to some value of I_z), but the number of these trajectories is finite and well prescribed (see details in [13]).

Hamiltonian (3) corresponds to the asymmetric phase portrait in the plane (z, p_z) [see Fig. 1(d)]. Moreover, the term $\sim szu_d$ results in the asymmetry of trajectories in the plane $(\kappa x, p_x)$ relative to the line $p_x = 0$. Thus, parts of the surface $\bar{H}(z, p_z, \kappa x, p_x) = h$, that correspond to two regions inside separatrix loops in the (z, p_z) plane, are projected onto different domains in the $(\kappa x, p_x)$ plane. Geometrically, to represent these projections, one should take two copies of the grey domain in the $(\kappa x, p_x)$ plane [shown in Fig. 1(b)], shift them with respect to each other still keeping them glued at the uncertainty curve, and additionally deform (this is due to $\varepsilon \neq 0$). In each of these domains we have its own family of adiabatic trajectories $I_z(\kappa x, p_x, h) = \text{const}$. At the intersection of these domains, we have two families of adiabatic trajectories not coinciding with each other. As a result, trajectories in the $(\kappa x, p_x)$ plane cross the uncertainty curve with the decrease and the increase of the κx value in points corresponding to different phase portraits in the (z, p_z) plane. The sum of two successive geometrical jumps is already not equal to zero, $\sum I_z^{\text{geom}} \neq 0$. Moreover, the asymmetry of the phase portrait in the $(\kappa x, p_x)$ plane prevents from the compensation of geometrical jumps for long-term dynamics.

Consider a phase point that crosses a separatrix twice, first leaving one of the domains in the (z, p_z) plane, and then coming back to this domain. For such a phase point, the value of the adiabatic invariant after the return to the initial domain is substantially different from its initial value due to geometrical jumps [see Fig. 2(c)]. The comparison of Figs. 2(a) and 2(c) demonstrates the main effect of the nonzero sum of geometrical jumps—the adiabatic invariant I_z changes significantly already after only two uncertainty curve crossings. This change $\sum I_z^{\text{geom}}$ does not depend on κ and is determined only by the geometry of the phase portrait in the (z, p_z) plane (the controlling parameters are ε and s). Therefore, already several crossings of the separatrix result in a substantial modification of I_z and the corresponding trajectory fills the large domain in the $(\kappa x, p_x)$ plane, Fig. 2(d).

To demonstrate the principal role of geometrical jumps in fast chaotization, we integrated numerically 10^5 trajectories with initially the same value of I_z for three systems: 1) with $s = 0, \varepsilon = 0$; 2) with $s \neq 0, \varepsilon = 0$; and 3) with $s \neq 0, \varepsilon \neq 0$. For the system 1) with the symmetric phase portrait dynamical jumps ($\sim\kappa$) support the diffusion of I_z around the initial value $I_{z,\text{init}}$. Final I_z distribution has a strong maximum for $I_z/I_{z,\text{init}} = 1$ [see Fig. 3(a)]. For the system 2) the geometrical jumps create three maxima of I_z distribution (the additional symmetry allows only three possible I_z values [13]), while dynamical jumps organize some spreading around these three I_z values [see Fig. 3(b)]. For the system 3) with the asymmetric phase portrait the term $\sim szu_d$ in Hamiltonian (3) makes geometrical jumps uncompensated. Thus, I_z rapidly changes and the final distribution is substantially more smeared than the previous two distributions [see Fig. 3(c)].

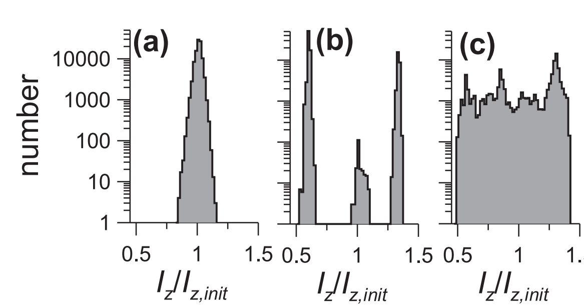

FIG. 3. Distribution of I_z values for the ensemble of 10^5 trajectories: (a) $s = 0, \varepsilon = 0$, (b) $s = 0.15, \varepsilon = 0$, (c) $s = 0.15, \varepsilon = 0.003$. For all calculations $\kappa = 0.01$. Time interval of numerical integration of trajectories is chosen so that each trajectory crosses the separatrix ten times.

As it was mentioned above, system (1) describes the motion of charged particles in a current sheet. Current sheets are widespread plasma structures observed in the laboratory [15], near-Earth plasma environment [16], stellar [17], and pulsar [18] magnetospheres. The most investigated configuration has a symmetric phase portrait ($s = 0$) [11]. Nonzero values of s and ε mean that the current sheet includes a shear (guide) component of the magnetic field and a finite electric field. Such topology of the magnetic field configuration is typical for current sheets located in the so-called outflow regions of the magnetic reconnection [19] and is extensively studied in numerical modeling [20] and laboratory experiments [21]. Thickness of the reconnected current sheet is comparable with ion gyroradius (or even smaller) and corresponding parameter $\kappa \sim 0.01$–0.3 [11]. Parameter s determines the amplitude of the guide field and for typical magnetotail [13] and astrophysical [22] reconnection $s \in [0,0.3]$. Amplitude of the reconnection electric field ε defines the reconnection rate and for realistic conditions in space plasmas [16] one has $\varepsilon \sim 0.1$–0.01. In this Rapid Communication we show how a combination of asymmetry $s \neq 0$ and the reconnection electric field $\varepsilon \neq 0$ quickly destroys the adiabaticity of ion motion. So fast chaotization can play an important role for ion heating [23], for the increase of collisionless conductivity [24], and for the stability of self-consistent plasma structures [19,21].

To conclude, we have shown that multiple crossings of the separatrix in systems with asymmetric phase portrait result in geometric chaotization. This mechanism of chaotization of the large phase volume for a short time interval is demonstrated in our Rapid Communication. Besides the reconnected current sheets, this mechanism should be important for planetary radiation belts [25]. Moreover, many problems in classical and celestial mechanics [10,26], fluid dynamics [27], statistical mechanics [28], quantum mechanics [29] are described by slow-fast Hamiltonian systems with separatrices in their phase space. In all these systems the proposed mechanism can play a significant role.

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