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Solitary state at the edge of synchrony in ensembles with attractive and repulsive interactions

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We discuss the desynchronization transition in networks of globally coupled identical oscillators with attractive and repulsive interactions. We show that, if attractive and repulsive groups act in antiphase or close to that, a *solitary state* emerges with a single repulsive oscillator split up from the others fully synchronized. With further increase of the repulsing strength, the synchronized cluster becomes *fuzzy* and the dynamics is given by a variety of stationary states with zero common forcing. Intriguingly, solitary states represent the natural link between coherence and incoherence. The phenomenon is described analytically for phase oscillators with sine coupling and demonstrated numerically for more general amplitude models.

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Mean field approximation, or global coupling, is widely used in the description of oscillator networks with high degree of connectivity. In the case of weak interactions, the theoretical analysis of the dynamics is typically performed with the help of phase approximation [1,2], most frequently with the use of the analytically solvable Kuramoto-Sakaguchi model [1,3–5]. A topic of recent interest is the investigation of interactions of several globally coupled ensembles [6,7], in particular, with attracting (positive) and repulsive (negative) couplings [8,9]. These studies are partially motivated by the problems of neuroscience where many highly connected groups of neurons interact via excitatory and inhibitory connections [10].

One of the intriguing effects in ensembles of globally coupled identical oscillators is clustering (see, e.g., [11,12], and references therein). It appears that randomly chosen initial states in the course of evolution can eventually become identical, and the final configuration consists of clusters of identically equal states. In this Rapid Communication, we discuss the formation of clusters and desynchronization transition in finite-size ensembles of identical oscillators with attractive and repulsive coupling and demonstrate the following scenario: when the repulsion starts to prevail over the attraction, a solitary oscillator leaves the synchronous cluster creating a so-called solitary state. With further increase of the repulsion, the solitary state loses its stability. More and more oscillators leave the synchronous group, which becomes a *fuzzy cluster*. Our aim is to describe this scenario for the M-group Kuramoto-Sakaguchi model

$$\dot{\theta}_i^\sigma = \omega + \sum_{\sigma'=1}^{M} \frac{K_{\sigma\sigma'}}{N} \sum_{j=1}^{N_{\sigma'}} \sin\left(\theta_j^{\sigma'} - \theta_i^\sigma + \alpha_{\sigma\sigma'}\right),$$

where θ_i^σ is the phase of the ith oscillator in group σ, N_σ is the number of oscillators in the group, and $N = N_1 + \cdots + N_M$. Elements of the $M \times M$ matrices $K_{\sigma\sigma'}$ and $\alpha_{\sigma\sigma'}$ represent the coupling strength and the phase shift of each oscillator in group σ' acting on each oscillator in group σ. By transformation to a corotating coordinate frame we can put $\omega = 0$.

We introduce the effect starting with the two-group model ($M = 2$), assuming $K_{\sigma\sigma'} = K_{\sigma'}$, $\alpha_{\sigma\sigma'} = \alpha_{\sigma'}$ for all $\sigma,\sigma' = 1,2$, i.e., that the coupling strengths and the phase shifts are determined by the acting group only:

$$\dot{\theta}_i^\sigma = \sum_{\sigma'=1}^{2} \frac{K_{\sigma'}}{N} \sum_{j=1}^{N_{\sigma'}} \sin\left(\theta_j^{\sigma'} - \theta_i^\sigma + \alpha_{\sigma'}\right). \quad (1)$$

Furthermore, we suppose $K_1 > 0$, $K_2 < 0$ and $-\pi/2 < \alpha_1, \alpha_2 < \pi/2$ such that the first group acts attractively on all oscillators in the network and the second group acts repulsively (cf. [13]). Such coupling configuration is a prototype of neuronal networks with excitatory ($K_1 > 0$) and inhibitory ($K_2 < 0$) neurons [10].

By renormalizing the time, $t \to K_1 t$, we write the coupling coefficients as $K_1 = 1$, $K_2 = -(1 + \varepsilon)$, where the new coupling parameter

$$\varepsilon = -(1 + K_2/K_1) \quad (2)$$

quantifies the *excess of the repulsion over the attraction*. Introducing the complex order parameter for both groups $Z_\sigma = N_\sigma^{-1} \sum_{j=1}^{N_\sigma} e^{i\theta_j^\sigma} = \rho_\sigma e^{i\Theta_\sigma}$ and relabeling the phases as $\theta_j = \theta_i^\sigma$, where $j = 1, \ldots, N$, $j = (\sigma - 1)N_1 + i$, we bring the system to the form $\dot{\theta}_j = h \sin(\Phi - \theta_j)$ with the common forcing

$$H = he^{i\Phi} = \frac{N_1}{N} e^{i\alpha} Z_1 - \frac{N_2}{N}(1 + \varepsilon) e^{i\beta} Z_2, \quad (3)$$

where, for convenience, we rename $\alpha = \alpha_1$ and $\beta = \alpha_2$.

We emphasize that, although the oscillators of two groups contribute differently to H, they evolve under the common forcing and therefore the whole population is effectively three dimensional: It can be described by three Watanabe-Strogatz (WS) equations [4,7,14]. This feature distinguishes our model from the six-dimensional "conformists and contrarians" model by Hong and Strogatz [9]. The WS equations describe the system via collective variables κ, Ψ, Γ, where $0 \leqslant \kappa \leqslant 1$ and Ψ, Γ are angles; these variables describe the amplitude and the phase of the collective mode (which roughly correspond to amplitude and phase of the complex Kuramoto order parameter), and the phase shift of individual oscillators, respectively. The WS equations also contain N constants of motion (angles) χ_j, determined from initial conditions; χ_j obey three additional constraints. The original phases θ_j are restored by the transformation $e^{i\theta_j} = e^{i\Gamma}(\kappa + e^{i(\chi_j - \Psi)})(\kappa e^{i(\chi_j-\Psi)} + 1)^{-1}$. If the system evolves to a state with $\kappa = 1$, then all initially different phases become identical (one-cluster state). Exceptional is the case when $\kappa = 1$ and $\chi_j - \Psi = \pi$ for some $j = n$ [15]; then θ_n may differ from all other phases. Such *solitary states*, when all the phases but one are identical, are of our main interest here. Below we show that stable solitary states naturally appear in our model in the course of the desynchronization transition. Notice that other clustered states, except for fully synchronous and solitary ones, are not allowed by WS theory. Instead, the model exhibits a variety of neutrally stable *fuzzy clusters*, where some number of oscillators are split up from the others "almost" synchronized.

To describe the desynchronization transition we first check that the fully synchronous state $\theta_j \equiv \varphi$ of model (1) is stable for

$$\varepsilon < \varepsilon_{\rm cr} = \frac{N_1 \cos\alpha}{N_2 \cos\beta} - 1. \tag{4}$$

For $\varepsilon > \varepsilon_{\rm cr}$, we look for a solitary state $\theta_1 = \cdots = \theta_{N-1} \equiv \varphi$, $\theta_N \equiv \psi$, i.e., when one repulsive unit splits up from all others [16]. Dynamics of this state are given by two equations which can be easily obtained by direct substitution of φ and ψ into Eq. (1):

$$\begin{aligned} \dot\varphi &= -\frac{1+\varepsilon}{N}[\sin(\eta+\beta) + (N_2-1)\sin\beta] + \frac{N_1}{N}\sin\alpha, \\ \dot\psi &= \frac{1+\varepsilon}{N}[(N_2-1)\sin(\eta-\beta) - \sin\beta] - \frac{N_1}{N}\sin(\eta-\alpha), \end{aligned} \tag{5}$$

where we denote $\eta = \psi - \varphi$. After straightforward manipulations, this system can be reduced to a scalar equation for the phase difference η:

$$\dot\eta = A[\sin(\eta - \eta^*) + \sin\eta^*]. \tag{6}$$

Here $A \geqslant 0$, and η^* is expressed, using $p = N_1/N$, as

$$\eta^* = \arctan\frac{(1-p-2N^{-1})(1+\varepsilon)\sin\beta - p\sin\alpha}{(1-p)(1+\varepsilon)\cos\beta - p\cos\alpha}. \tag{7}$$

Equation (6) has two equilibria $\eta_{\rm syn} = 0$ and $\eta_{\rm sol} = 2\eta^* + \pi$ which describe, respectively, the full synchrony ($\psi = \varphi$) and the solitary state ($\psi = \varphi + \eta_{\rm sol}$) in the original model (1). It can be easily checked [17] that these states exchange their stability exactly at $\varepsilon_{\rm cr}$. Hence, the solitary state is stable for all $\varepsilon > \varepsilon_{\rm cr}$ within the two-cluster manifold (φ,ψ). To complete the stability analysis of the solitary state we have to examine in the phase space the directions, transversal to the manifold (φ,ψ), i.e., to quantify the stability of the main synchronized cluster of $N-1$ elements. For this goal we write the Jacobian for the system (1) at solitary state $\psi = \varphi + \eta_{\rm sol}$. Due to the matrix symmetry we find that the Jacobian has $N-2$ equal eigenvalues [18] which are, actually, transversal Lyapunov exponents (LE) of the solitary state, all equal:

$$\begin{aligned} \lambda_\perp = &-p\cos\alpha + \left(1 - p - \frac{1}{N}\right)(1+\varepsilon)\cos\beta \\ &- \frac{1+\varepsilon}{N}\cos(2\eta^* + \beta). \end{aligned} \tag{8}$$

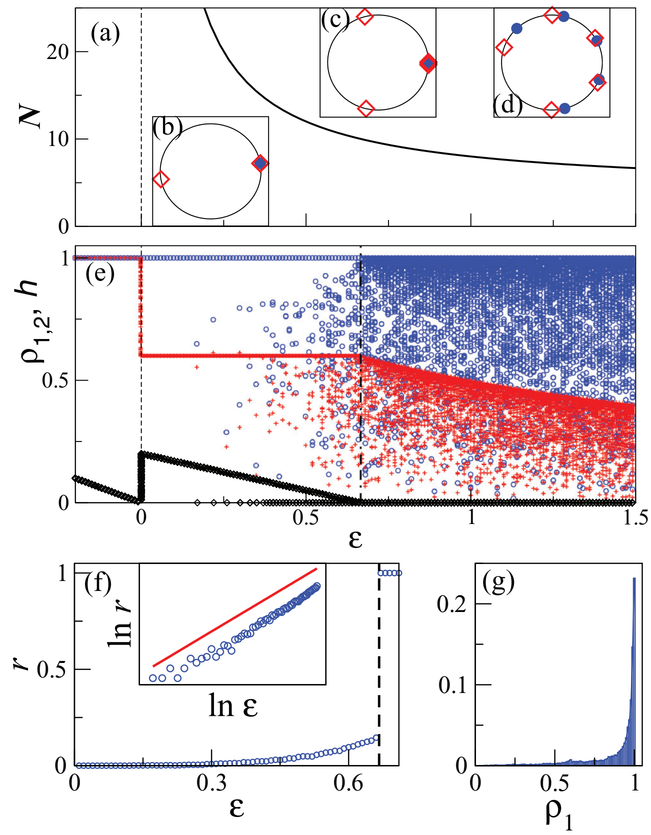

FIG. 1. (Color online) Desynchronization transition for $N_{1,2} = 5, \alpha = \beta = 0$. (a) Bifurcation diagram. Dashed vertical line at $\varepsilon_{\rm cr} = 0$ shows the border between the full synchrony and the solitary state, and the solid hyperbolic curve shows where the latter loses its stability, i.e., $\varepsilon_{\rm sol}^+$. The inset (b) presents an example of the solitary state, here for $\varepsilon = 0.25$: There exists a cluster of nine elements and one oscillator is exactly in antiphase to it, hence, $\rho_1 = 1$, $\rho_2 = 0.6$ [cf. panel (e)]. Insets (c) and (d) present two examples for $\varepsilon = 1 > \varepsilon_{\rm sol}^+$; (c) is a state with a fuzzy cluster, where all phases but two are very close though not identical ("mercedes state"), while (d) is a distributed state without fuzzy clusters. (e) Order parameters $\rho_{1,2}$ and the amplitude of the common forcing h (blue circles, red pluses, and black diamonds, respectively) as functions of ε, for 100 runs; in each run the initial phases are chosen from the uniform random distribution in $0,2\pi$. For full synchrony and for the solitary state the dependence $h(\varepsilon)$ perfectly agrees with Eq. (3) which yields for these states $h = -\varepsilon/2$ and $h = 0.2 - 0.3\varepsilon$, respectively. (f) The ratio r of nonsolitary states (blue circles), i.e., of states with $h = 0$, obtained in 10^4 runs with random initial conditions; it is well approximated by $r \sim \varepsilon^{10/3}$ (the slope of the solid red line in the inset is 10/3). (g) Histogram of ρ_1 for $\varepsilon = 1$, obtained from 10^4 runs with random initial conditions, demonstrates that the fuzzy clusters with $\rho_1 \lesssim 1$ are dominating. Thus, the desynchronized states $h = 0$ typically correspond to coherence of subpopulations.

For interpretation of the results we concentrate first on the simplest nontrivial case $N_1 = N_2 = N/2$ and $\alpha = \beta = 0$. Then $\eta_{sol} = \pi$, i.e., the solitary oscillator stays strictly in antiphase to all others, and Eqs. (4) and (8) yield the stability domain of this state:

$$0 < \varepsilon < \varepsilon_{sol}^{+} = 4(N-4)^{-1} \tag{9}$$

[see Fig. 1(a)] . It follows that the solitary state exists for arbitrary large network size N and that the ε width of the stability domain remains finite for any N, decreasing as $1/N$ as $N \to \infty$. Our numerical studies reveal, however, that the basin of attraction for the solitary state can be not of full measure. Indeed, solutions with $h = 0$ [see Eq. (3)], which are also stationary, coexist with the solitary state in the stability domain (9). As is illustrated in Fig. 1(e), immediately after the transition at ε_{cr} solitary states appear with probability one; soon after, the $h = 0$ states arise with nonzero probability which grows $\sim\varepsilon^{10/3}$ as ε approaches ε_{sol}^{+}. For $\varepsilon > \varepsilon_{sol}^{+}$ solitary state does not exist anymore. All stationary states fulfill the condition $h = 0$; then Eq. (3) yields $\rho_1 = \rho_2(1+\varepsilon)$. It turns out, that most likely are the states when the attractive units form a fuzzy cluster with $\rho_1 \lesssim 1$, and, respectively, $\rho_2 \approx (1+\varepsilon)^{-1}$. This is illustrated in Figs. 1(e)–1(g), where we present the results for numerical analysis with random initial conditions [19].

Now, we consider the case $\alpha \neq 0$, $\beta \neq 0$ (see Fig. 2). The solitary state is not stationary anymore [as follows from Eq. (5), all units rotate with a constant velocity], and its stability domain is obtained from Eqs. (7) and (8) as

$$N < \frac{\cos\beta + \cos(2\eta^* + \beta)}{(1-p)\cos\beta - p(1+\varepsilon)^{-1}\cos\alpha}. \tag{10}$$

If $\alpha = \beta \neq 0$, the solitary stability region coincides with those for the $\alpha = \beta = 0$ case. If $\alpha \neq \beta$, the region pulls down and shrinks rapidly as the difference between α and β increases. The solitary phenomenon becomes essentially low dimensional, i.e., it does not arise for large N [Fig. 2(a)]. Withal, the solitary angle $\eta_{sol} = 2\eta^* + \pi$ is not equal to π anymore, contrary to the case $\alpha = \beta = 0$. As ε crosses ε_{cr}, η_{sol} splits up from 0 and monotonically increases or decreases with ε, eventually approaching the value $(1 - 2/N)\tan\beta$ as $\varepsilon \to \infty$ [see Fig. 2(b)]. The increase or decrease of η_{sol} is determined by the sign of the difference $\tan\alpha - (1 - 4/N)\tan\beta$; if the difference equals 0, $\eta_{sol} = 2(N-2)/(N-4)\tan\alpha + \pi$ for all $\varepsilon > \varepsilon_{cr}$.

We conclude that the desynchronization transitions at $\varepsilon = \varepsilon_{cr}$ immediately yield the solitary state only if $\alpha = \beta$. Otherwise, there is always a gap between the full synchronization and the solitary behavior. Solitary states stabilize later at some $\varepsilon_{sol}^{-} > \varepsilon_{cr}$, and they lose their stability at $\varepsilon_{sol}^{+} > \varepsilon_{sol}^{-}$. The bifurcation values ε_{sol}^{-} and ε_{sol}^{+} depend on N,α,β,p, and can be obtained from Eq. (10). For $\alpha \neq \beta$ and N large enough, the solitary behavior does not appear and the transition can occur via the mechanism of *coherency exchange*, illustrated in Fig. 3 for $N_{1,2} = 5$, $\alpha = 2\pi/3$, and $\beta = 0$. As follows from Eq. (3), the condition $h = 0$ implies that for $\varepsilon = -1$ and arbitrary N, the order parameter $\rho_1 = 0$, i.e., the first,

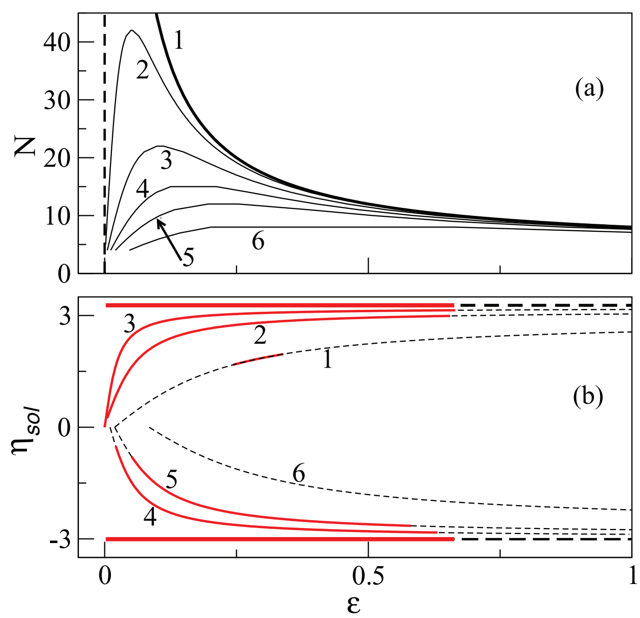

FIG. 2. (Color online) (a) Domains of solitary state for $N_1 = N_2$, $\alpha = 0.05$, and for different β. The dashed vertical line marks ε_{cr}, where the full synchrony becomes unstable. The bold line 1 marks the right border ε_{sol}^{+} of the domain for $\alpha = \beta = 0.05$ [notice that it coincides with the curve $4(N-4)^{-1}$ shown in Fig. 1(a) for the case $\alpha = \beta = 0$]. Curves 2–6 show the domains for $\beta = 0.1$, 0.15, 0.2, 0.25, and 0.35, respectively; these domains shrink with the increase of $|\alpha - \beta|$. Interestingly, for $\alpha \neq \beta$ the full synchrony and the solitary states are separated by an interval of incoherent dynamics. (b) Solitary angle η_{sol} as a function of the coupling parameter ε, for $N_1 = N_2 = 5$, $\alpha = 0.05$, and $\beta = -0.2$, 0, 0.05, 0.15, and 0.2 (curves 1–5), dashed lines; the intervals where the solitary state is stable are shown by bold red lines. Bold horizontal dashed lines show $\lim_{\varepsilon\to\infty}\eta_{sol}$; here it is ≈ 3.275. For $\beta = 0.0832$, η_{sol} equals this value for all $\varepsilon > \varepsilon_{cr}$.

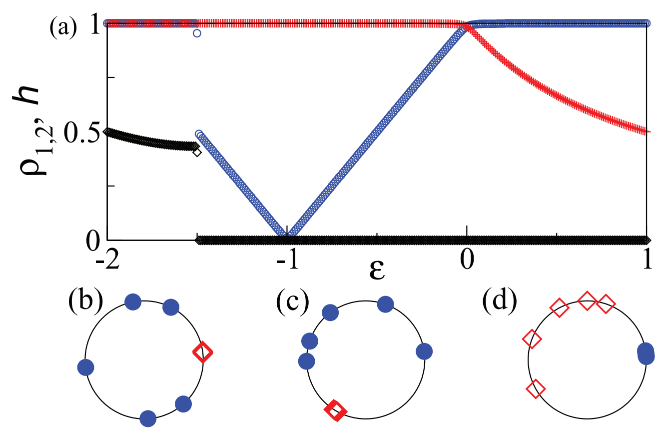

FIG. 3. (Color online) (a) An example of coherence exchange for $N_{1,2} = 5, \alpha = 2\pi/3, \beta = 0$. Here $\rho_{1,2}$ and h are shown by blue circles, red pluses, and black diamonds, respectively. For $\varepsilon < \varepsilon_{cr} = -1.5$ all oscillators are synchronized. For $\varepsilon > \varepsilon_{cr}$ the repulsive units form a fuzzy cluster, $\rho_2 \lesssim 1$, while the attractive oscillators first desynchronize, so that $\rho \approx 0$ for $\varepsilon = -1$ and then synchronize again. When ε becomes positive, the attractive oscillators form a fuzzy cluster, while the repulsive desynchronize. Initial phases for both groups are placed on two arcs of arbitrary length, shifted by π. Panels (b)–(d) are snapshots for $\varepsilon = -1$, $\varepsilon = -0.5$, and $\varepsilon = 0.5$.

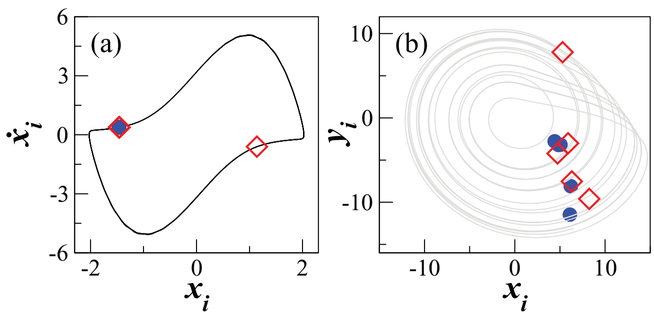

FIG. 4. (Color online) Solitary state in the ensemble of van der Pol oscillators (a) and of chaotic Rössler oscillators (b). Blue circles and red diamonds show the elements of the attractive and the repulsive group, respectively. Gray line shows the limit cycle of a single oscillator (a) and projection of the strange attractor of an autonomous Rössler system (b). In (a) all oscillators except one are in the cluster at $x_j \approx -1$; the solitary element is at $x_j \approx 1$, i.e., nearly in antiphase. In (b) all oscillators except for one repulsive element group with approximately the same phase (fuzzy cluster), as is typical for chaotic oscillators, where phase locking is generally not perfect.

attractive group necessarily desynchronizes. The straight lines for $-1.5 < \varepsilon < 0$ and the hyperbolic curve for $\varepsilon > 0$ are also explained by Eq. (3).

To test if the solitary states appear in more general networks, we analyzed a three-group Kuramoto model

$$\dot{\theta}_i^\sigma = \sum_{\sigma'=1}^{3} \frac{K_{\sigma'}}{N} \sum_{j=1}^{N/3} \sin\left[\theta_j^{\sigma'} - \theta_i^\sigma - (\sigma'-1)\frac{2\pi}{3}\right], \quad (11)$$

where $\sigma,\sigma' = 1,2,3$, $K_{1,2} = 1$, and $K_3 = 1+\varepsilon$. Here the first group is attractive, and two others are repulsive, quantified by phase shifts $\mp 2\pi/3$. If $\varepsilon > 0$, the repulsive action of the third group is stronger. Then, a solitary state is born at $\varepsilon_{\rm cr} = 0$, and its stability region has the same shape as in the two-group $\alpha = \beta$ case: $0 < \varepsilon < 6(N-6)^{-1}$. As expected, the solitary oscillator belongs to the third group; it splits up from all others remaining fully synchronized by the angle $\eta_{\rm sol} = \arctan \frac{3\sqrt{3}(1+\varepsilon)}{N\varepsilon} + \arcsin \frac{\sqrt{3}(6(1+\varepsilon)-N\varepsilon}{2\sqrt{27(1+\varepsilon)^2+N^2\varepsilon^2}} + \pi$. The analysis of M-group models, $M \geqslant 4$, remains a subject of future studies; the preliminary analysis for $M = 4$ and $\alpha_\sigma = (\sigma'-1)\frac{\pi}{2}$, $\sigma' = 1,\ldots,4$ does not reveal the solitary states. To illustrate that the demonstrated effect is not restricted to sine-coupled phase oscillators, we performed a numerical analysis of two more realistic models. First, we simulated globally coupled van der Pol oscillators (cf. [20]), for the case $N_{1,2} = 20$:

$$\ddot{x}_j - 3(1-x_j^2)\dot{x}_j + x_j = K_1(X_1 - \dot{x}_j) - K_2(X_2 - \dot{x}_j),$$

where $X_1 = N_1^{-1}\sum_{k=1}^{N_1}\dot{x}_k$, $X_2 = N_2^{-1}\sum_{k=N_1+1}^{N}\dot{x}_k$. Figure 4(a) exhibits a solitary state for $K_1 = 0.1$, $K_2 = 0.105$. Next, we consider attractively repulsively coupled Rössler oscillators, $N_{1,2} = 5$,

$$\dot{x}_j = -y_j - z_j + K_1(X_1 - x_j) - K_2(X_2 - x_j),$$
$$\dot{y}_j = x + 0.15y + K_1(Y_1 - y_j) - K_2(Y_2 - y_j),$$
$$\dot{z}_j = 0.4 + z_j(x_j - 8.5),$$

where $X_1 = N_1^{-1}\sum_{k=1}^{N_1} x_k$, $X_2 = N_2^{-1}\sum_{k=N_1+1}^{N} x_k$, $Y_1 = N_1^{-1}\sum_{k=1}^{N_1} y_k$, and $Y_2 = N_2^{-1}\sum_{k=N_1+1}^{N} y_k$. Since the systems are chaotic we can expect only some quantitative correspondence with our theory. Indeed, we observe a narrow domain with (approximately) solitary states [see Fig. 4(b)], where $K_{1,2} = 0.05$.

In conclusion, we have identified a scenario for the coherence-incoherence transition in networks of globally coupled identical oscillators with attractive and repulsive interactions. The transition occurs via solitary state at the edge of synchrony. The phenomenon arises when attraction and repulsion act in antiphase and diminishes and becomes low dimensional when this condition is not exact. The solitary state is asymptotically stable in the Lyapunov sense and hence, it is robust with respect to small inhomogeneities; however, under the perturbations, the perfectly synchronized main cluster of the solitary state becomes generally a fuzzy one.

In the desynchronized state the system is highly multistable; in particular, it exhibits fuzzy clustering. Finally, we have found solitary states for more realistic oscillatory networks with both periodic and chaotic local dynamics. This indicates a general, probably universal desynchronization mechanism in networks of very different nature, due to attractive and repulsive interactions.

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[16] Since all oscillators are identical we can without a loss of generality take that the solitary unit has index N.
[17] Stability and instability of the states $\eta_{\text{syn}} = 0$ and η_{sol} are determined by the sign of $\cos\theta = (1-p)(1+\varepsilon)\cos\beta - p\cos\alpha$.
[18] Indeed, substituting $\lambda = \lambda_\perp$ in the matrix $J - \lambda E$ immediately yields $N-1$ identical rows.
[19] An example of a fuzzy cluster for $\varepsilon > \varepsilon_{\text{cr}}^{+}$ is given in Fig. 1(c), cf. [8]. It resembles the "mercedes" state, where two oscillators from the repulsive group have phase shift $\pm\arccos(1 - \frac{\varepsilon N}{4(1+\varepsilon)})$ with respect to $N-2$ fully synchronous units. However, as we have proved analytically, the latter state is not attractive: It is stable inside its three-dimensional clustered manifold for $4/N - 4 < \varepsilon < 8/N - 8$, but is only neutrally stable transversally. The further increase of ε gives birth to analogous states with three and more oscillators split up from the main synchronized group. Similarly, they are neutrally stable.
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