Dynamical friction in a relativistic plasma

O. J. Pike* and S. J. Rose

Blackett Laboratory, Imperial College, London SW7 2AZ, United Kingdom

(Received 4 February 2013; published 29 May 2014)

The work of Spitzer on dynamical friction in a plasma [L. Spitzer, Jr., *Physics of Fully Ionized Gases*, 2nd ed. (Wiley, New York, 1962), Chap. 5] is extended to relativistic systems. We derive the force of dynamical friction, diffusion tensor, and test particle relaxation rates for a Maxwellian background in the same form as Trubnikov [B. A. Trubnikov, in *Reviews of Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1965), Vol. 1, p. 105], enabling high-temperature laboratory and astrophysical plasmas to be modeled in a consistent manner.

DOI: 10.1103/PhysRevE.89.053107 PACS number(s): 52.25.Dg, 52.20.−j, 52.27.Ny

I. INTRODUCTION

Relativistic plasmas are of relevance to both fusion energy research and high-energy astrophysics. Coulomb collisions influence behavior in many of these systems, such as transport in inertial fusion targets [1], the slowing of fast electrons formed in high-intensity laser-plasma interactions [2] (critical to the fast ignition fusion scheme [3]), current drive in tokamaks [4,5], the thermalization of astrophysical plasmas [6,7], and, potentially, gamma-ray burst emission [8].

Accurately modeling any of these processes, typically achieved using the Boltzmann equation, requires a relativistic treatment. Landau [9] first showed that for Coulomb interactions the Boltzmann collision integral may be written in the Fokker-Planck form. Rosenbluth *et al.* [10] and later Trubnikov [11] reformulated this in terms of the derivatives of two potentials, a phrasing much more amenable to numerical solution. These results were extended to relativistic plasmas some time ago: the Fokker-Planck-Landau collision operator by Beliaev and Budker [12] and the differential formulation by Braams and Karney [13].

However, the direct relativistic analog to the semianalytical results of Spitzer [14] (later reformulated by Trubnikov [15]) remains missing. These describe the motion of a test particle traveling through a thermal background of field particles and enable kinetic processes to be modeled without directly solving the Boltzmann equation. Adopting the more convenient notation of Trubnikov, the relaxation rates of a test particle (labeled a) traveling with a velocity $\mathbf{v}$ through a background of field particles (labeled b) are given by [15]

$$\frac{d\mathbf{v}}{dt} = -\frac{\Gamma^{a/b}}{v^3}\left(1+\frac{m_a}{m_b}\right)\mu(x)\mathbf{v}, \tag{1}$$

$$\frac{d}{dt}(\mathbf{v}-\bar{\mathbf{v}})^2_\parallel = \frac{\Gamma^{a/b}}{vx}\mu(x), \tag{2}$$

$$\frac{d}{dt}(\mathbf{v}-\bar{\mathbf{v}})^2_\perp = \frac{\Gamma^{a/b}}{vx}[2x(\mu(x)+\mu'(x))-\mu(x)], \tag{3}$$

$$\frac{d\epsilon}{dt} = \frac{\Gamma^{a/b}m_a}{v}\left(\mu'(x)-\frac{m_a}{m_b}\mu(x)\right), \tag{4}$$

with

$$\frac{d\epsilon}{dt} = \frac{m_a}{2}\left(2\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}+\frac{d}{dt}(\mathbf{v}-\bar{\mathbf{v}})^2_\parallel+\frac{d}{dt}(\mathbf{v}-\bar{\mathbf{v}})^2_\perp\right). \tag{5}$$

Equations (1)–(4) describe the rate of momentum loss, parallel and perpendicular momentum diffusion, and energy change of the test particle, respectively. Here, m_a and m_b are the species masses, $v=|\mathbf{v}|$ is the test particle speed, and $\epsilon = m_a v^2/2$ its kinetic energy. The function

$$\mu(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t}t^{1/2}dt \tag{6}$$

is the Maxwell integral, whose argument $x = v^2/2\Theta_b c^2$ (where $\Theta_b \equiv k_B T_b/m_b c^2$ is the reduced temperature of the background) and derivative $\mu'(x)\equiv d\mu/dx$. The coefficient $\Gamma^{a/b}$ is given by

$$\Gamma^{a/b} = \frac{n_b q_a^2 q_b^2 \ln\Lambda^{a/b}}{4\pi\epsilon_0^2 m_a^2},$$

where n_b is the density of the background, q_a and q_b are the species charges, $\ln\Lambda^{a/b}$ is the Coulomb logarithm, and ϵ_0 is the permittivity of free space.

In this work, we derive the Fokker-Planck coefficients for a relativistic Maxwellian plasma (Sec. II), which are subsequently used to obtain expressions for the relativistic test particle relaxation rates (Sec. III). These are presented in the same form as those of Trubnikov [Eqs. (1)–(4)] and can readily be seen to reduce to these results in the classical limit. Finally, we discuss the limits of applicability of this work in Sec. IV. We note that these results are the exact relativistic counterparts to those of Spitzer; those derived in previous works are either more complex or less general [16–20].

II. RELATIVISTIC FOKKER-PLANCK COEFFICIENTS FOR A MAXWELLIAN BACKGROUND

We begin by considering the collision operator of Braams and Karney [13] between species a and b, as expressed in Fokker-Planck form:

$$C^{a/b} = -\frac{\partial}{\partial\mathbf{u}}\cdot\left(\frac{\mathbf{F}^{a/b}}{m_a}f_a - \mathsf{D}^{a/b}\cdot\frac{\partial f_a}{\partial\mathbf{u}}\right), \tag{7}$$

where $f_a(\mathbf{r},\mathbf{u},t)$ is the distribution function of species a and $\mathbf{u}\equiv\mathbf{p}/m_a$ is the ratio of momentum to species mass. The force

*o.pike11@imperial.ac.uk

of dynamical friction $\mathbf{F}^{a/b}$ and diffusion tensor $\mathsf{D}^{a/b}$ are given by

$$\mathbf{F}^{a/b}(\mathbf{u}) = -\frac{4\pi\Gamma^{a/b}}{n_b}\frac{m_a^2}{m_b}\frac{1}{\gamma}\mathbf{K}\left(g_0 - \frac{2}{c^2}g_1\right), \tag{8}$$

$$\mathsf{D}^{a/b}(\mathbf{u}) = -\frac{4\pi\Gamma^{a/b}}{n_b}\left[\frac{1}{\gamma}\left(\mathsf{L} + \frac{\mathsf{I}}{c^2} + \frac{\mathbf{uu}}{c^4}\right)h_1 - \frac{4}{\gamma c^2}\left(\mathsf{L} - \frac{\mathsf{I}}{c^2} - \frac{\mathbf{uu}}{c^4}\right)h_2\right], \tag{9}$$

in which $g_{0,1}$ and $h_{1,2}$ denote four of five potentials (represented in general by the symbol χ), $\gamma = (1 + |\mathbf{u}|^2/c^2)^{1/2}$ is the Lorentz factor, I is the unit diagonal second-order tensor, and

$$\mathbf{K}\chi = \frac{1}{\gamma}\frac{\partial\chi}{\partial\mathbf{v}}, \quad \mathsf{L}\chi = \frac{1}{\gamma^2}\frac{\partial^2\chi}{\partial\mathbf{v}\partial\mathbf{v}} - \frac{\mathbf{v}}{c^2}\frac{\partial\chi}{\partial\mathbf{v}} - \frac{\partial\chi}{\partial\mathbf{v}}\frac{\mathbf{v}}{c^2},$$

where $\mathbf{v} = \mathbf{u}/\gamma$ and $\partial/\partial\mathbf{v} = \gamma(\mathsf{I} + \mathbf{uu}/c^2)\cdot\partial/\partial\mathbf{u}$. The potentials satisfy the equations

$$\begin{aligned} Lg_0 &= f_b, \quad [L + 1/c^2]h_0 = f_b, \\ Lg_1 &= g_0, \quad [L - 3/c^2]h_1 = h_0, \\ &\qquad\;\; [L - 3/c^2]h_2 = h_1, \end{aligned} \tag{10}$$

where h_0 is the fifth potential, f_b is the distribution function of species b, and

$$L\chi = \left(\mathsf{I} + \frac{\mathbf{uu}}{c^2}\right) : \frac{\partial^2\chi}{\partial\mathbf{u}\partial\mathbf{u}} + \frac{3\mathbf{u}}{c^2}\cdot\frac{\partial\chi}{\partial\mathbf{u}}.$$

It is instructive to switch to a spherical coordinate (u,θ,ϕ) system, such that the distribution functions f and potentials χ may be expressed as the complex amplitudes of an expansion in spherical harmonics, e.g.,

$$f(\mathbf{r},\mathbf{u},t) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_l^m(\mathbf{r},u,t)P_l^{|m|}(\cos\theta)e^{im\phi}, \tag{11}$$

where $u = |\mathbf{u}|$ and $f_l^{-m} = (f_l^m)^*$. Explicit forms for the potentials can then be found by solving Eqs. (10) through the construction of a Green's function. We consider the case of an isotropic background ($l,m = 0$), in which case all components of $\mathbf{F}^{a/b}$ and $\mathsf{D}^{a/b}$ vanish other than $F_{u,l=0}^{a/b}$, $D_{uu,l=0}^{a/b}$, and $D_{\theta\theta,l=0}^{a/b}$. After substitution of the potentials, these be can written as integrals over the background distribution f_b (for a detailed derivation, see Ref. [21]):

$$F_{u,0}^{a/b} = -\frac{4\pi\Gamma^{a/b}}{n_b}\frac{m_a^2}{m_b}\left(\int_0^u (\gamma^2 j'_{0[1]1} - 2j'_{0[2]11})\frac{1}{u^2}\frac{u'^2}{\gamma'}f_b(u')du' + \int_u^\infty 4\frac{u'}{u}j_{0[2]02}f_b(u')du'\right), \tag{12}$$

$$D_{uu,0}^{a/b} = \frac{4\pi\Gamma^{a/b}}{n_b}\left(\int_0^u (2\gamma^2c^2 j'_{0[2]02} - 8c^2 j'_{0[3]022}) \times \frac{\gamma}{u^3}\frac{u'^2}{\gamma'}f_b(u')du' + \int_u^\infty (2\gamma'^2c^2 j_{0[2]02} - 8c^2 j_{0[3]022})\frac{\gamma}{u^2}\frac{u'}{\gamma'}f_b(u')du'\right), \tag{13}$$

$$D_{\theta\theta,0}^{a/b} = \frac{2\pi\Gamma^{a/b}}{n_b}\left\{\int_0^u \left[j'_{0[1]2} - 2\left(\frac{c^2}{u^2} + \frac{1}{\gamma^2}\right)j'_{0[2]02} + \frac{8}{\gamma^2}\frac{c^2}{u^2}j'_{0[3]022}\right]\frac{\gamma}{u}\frac{u'^2}{\gamma'}f_b(u')du' + \int_u^\infty \left[\frac{\gamma'^2}{\gamma^2}j_{0[1]2} - 2\frac{u'^2}{u^2}\left(\frac{c^2}{u'^2} + \frac{1}{\gamma^2}\right)j_{0[2]02} + \frac{8}{\gamma^2}\frac{c^2}{u^2}j_{0[3]022}\right]\gamma\frac{u'}{\gamma'}f_b(u')du'\right\}, \tag{14}$$

where $j_{l[k]*}^{(\prime)} = j_{l[k]*}(u^{(\prime)}/c)$; these functions are cataloged for reference in Appendix A. In general, the integrals must be evaluated numerically and considerable care taken in the limit $u' \to 0$ due to large cancellations. However, when the background is a Maxwellian, $f_b = f_{bM}$, where

$$f_{bM}(u) = \frac{n_b e^{-\gamma/\Theta_b}}{4\pi c^3\Theta_b K_2(1/\Theta_b)} \tag{15}$$

(K_ν is the νth-order Bessel function of the second kind), the integrals may in their most part be evaluated analytically and those remaining cast in a much simpler form. After several pages of algebra, one arrives at

$$F_{u,0}^{a/b} = -\frac{\Gamma^{a/b}}{u^2}\frac{m_a^2}{m_b}\mu_1, \tag{16}$$

$$D_{uu,0}^{a/b} = \frac{\Gamma^{a/b}\gamma c^2}{u^3}\Theta_b\mu_1, \tag{17}$$

$$D_{\theta\theta,0}^{a/b} = \frac{\Gamma^{a/b}c^2}{2\gamma u^3}\left[\frac{u^2}{c^2}(\mu_0 + \gamma\Theta_b\mu'_1) - \Theta_b\mu_1\right], \tag{18}$$

where, in analogy with Trubnikov's formulation, we have introduced the functions

$$\mu_0(\gamma,\Theta_b) = \frac{\gamma^2 L_0 - \Theta_b L_1 + (\Theta_b - \gamma)ue^{-\gamma/\Theta_b}}{K_2(1/\Theta_b)c}, \tag{19}$$

$$\mu_1(\gamma,\Theta_b) = \frac{\gamma^2 L_1 - \Theta_b L_0 + (\Theta_b\gamma - 1)ue^{-\gamma/\Theta_b}}{K_2(1/\Theta_b)c}, \tag{20}$$

with

$$\mu'_1 \equiv \frac{d\mu_1}{d\gamma} = \frac{2\Theta_b\gamma L_1 + (1 + 2\Theta_b^2)ue^{-\gamma/\Theta_b}}{\Theta_b K_2(1/\Theta_b)c}.$$

The integrals $L_\nu = L_\nu(u,\Theta_b)$ are given by

$$L_0 = \int_0^u \frac{e^{-\gamma'/\Theta_b}}{\gamma'}du', \quad L_1 = \int_0^u e^{-\gamma'/\Theta_b}du'.$$

(See Appendix B for further details of this derivation.) The full forms of $\mathbf{F}^{a/b}$ and $\mathsf{D}^{a/b}$ may then be determined via the relations [22]

$$\mathbf{F}_{l=0}^{a/b} \equiv F_{u,0}^{a/b}\frac{\mathbf{u}}{u}, \tag{21}$$

$$\mathsf{D}_{l=0}^{a/b} \equiv \left(D_{uu,0}^{a/b} - D_{\theta\theta,0}^{a/b}\right)\frac{\mathbf{uu}}{u^2} + D_{\theta\theta,0}^{a/b}\mathsf{I}. \tag{22}$$

The classical results are retrieved in the limit $u \to v$ (where $v^2/c^2 \ll 1$) and $\Theta_b \ll 1$. Under these conditions, it is easily seen that $\mu_0,\mu_1 \to \mu(x)$ and $\Theta_b\mu_1' \to \mu'(x)$; Eqs. (16)–(22) then reduce to the results of Trubnikov [15]:

$$F_{v,0}^{a/b} = -\frac{\Gamma^{a/b}}{v^2}\frac{m_a^2}{m_b}\mu(x), \tag{23}$$

$$D_{vv,0}^{a/b} = \frac{\Gamma^{a/b}}{2vx}\mu(x), \tag{24}$$

$$D_{\theta\theta,0}^{a/b} = \frac{\Gamma^{a/b}}{4vx}[2x(\mu(x)+\mu'(x))-\mu(x)]. \tag{25}$$

In the classical theory, $\mu(x)$ represents the sum over the distribution up to the speed of the test particle; no momentum is exchanged with any field particle which has a greater momentum than the test particle. In the relativistic theory, the discontinuity disappears [12]; neither of the functions μ_0 and μ_1 represent a sum over the relativistic Maxwellian.

Braams and Karney obtained versions of these coefficients in the limit $(\gamma-1) \gg \Theta_b$ [13]. It is straightforward to verify these are consistent with Eqs. (16)–(18) by noting $L_\nu(u,\Theta_b) \to cK_\nu(1/\Theta_b)$ as $u \to \infty$ and using the recurrence relations of the Bessel functions [23].

III. RELATIVISTIC TEST PARTICLE RELAXATION RATES

Now that the forms of the force of dynamical friction and diffusion tensor are known, it is straightforward to compute the various relaxation rates. Following the approach of Trubnikov [15], we consider an ensemble of test particles (a) with an initial distribution $f_a(t,\mathbf{r},\mathbf{u})|_{t=0} = n_a\delta(\mathbf{u}-\mathbf{u}_0)$ traveling in an infinite uniform background of field particles (b). The first two moments of the distribution are defined by

$$\bar{u}_i \equiv \frac{1}{n_a}\int u_i f_a d^3u, \tag{26}$$

$$\overline{(u-\bar{u})_i(u-\bar{u})_j} \equiv \frac{1}{n_a}\int (u-\bar{u})_i(u-\bar{u})_j f_a d^3u, \tag{27}$$

where u_i is the ith component of momentum and the bar above a quantity represents an ensemble average. Assuming there are no external fields, the Boltzmann equation is given simply as $\partial f_a/\partial t = C^{a/b}$. The rate of momentum loss is then calculated by taking the time derivative of the first moment. This gives

$$\left.\frac{d\bar{u}_i}{dt}\right|_{t=0} = \left(\frac{F_i^{a/b}}{m_a} + \frac{\partial}{\partial u_k}D_{ik}^{a/b}\right)_{\mathbf{u}=\mathbf{u}_0}.$$

Omitting the zero subscripts (as is done for subsequent results) and substituting for $\mathbf{F}^{a/b}$ and $\mathsf{D}^{a/b}$, we arrive at

$$\frac{d\mathbf{u}}{dt} = -\frac{\Gamma^{a/b}}{u^3}\left(\frac{1}{\gamma}\mu_0 + \frac{m_a}{m_b}\mu_1\right)\mathbf{u} \tag{28}$$

[cf. Eq. (1)]. Note that, outside the classical limit, the mean force acting on the test particle $\mathbf{F}_a \equiv m_a d\mathbf{u}/dt$ is not proportional to the force of dynamical friction $\mathbf{F}^{a/b}$. This should not be of concern as the latter is no more than a mathematical construct; the former is the physical force acting on the particle. Similarly considering the time derivative of the second moment yields

$$\left.\frac{d}{dt}\overline{(u-\bar{u})_i(u-\bar{u})_j}\right|_{t=0} = \left(2D_{ij}^{a/b}\right)_{\mathbf{u}=\mathbf{u}_0},$$

from which the parallel and perpendicular momentum diffusion rates may straightforwardly be shown to be

$$\frac{d}{dt}(\mathbf{u}-\bar{\mathbf{u}})_\parallel^2 = \frac{2\Gamma^{a/b}\gamma c^2}{u^3}\Theta_b\mu_1, \tag{29}$$

$$\frac{d}{dt}(\mathbf{u}-\bar{\mathbf{u}})_\perp^2 = \frac{2\Gamma^{a/b}c^2}{\gamma u^3}\left[\frac{u^2}{c^2}(\mu_0+\gamma\Theta_b\mu_1') - \Theta_b\mu_1\right] \tag{30}$$

[cf. Eqs. (2) and (3)].

The rate of energy exchange between the two species is defined as [22]

$$\frac{dE_a}{dt} \equiv 4\pi m_a c^2\int_0^\infty (\gamma-1)C^{a/b}u^2du, \tag{31}$$

in which we take $f_a(u) = n_a\delta(u-u_0)/4\pi u^2$, where $u_0 = |\mathbf{u}_0|$; generalizing to the isotropic distribution is possible as the energy exchange does not depend on the direction of the particle beam. In this case,

$$C^{a/b} = -\frac{1}{u^2}\frac{\partial}{\partial u}\left(u^2\frac{F_{u,0}^{a/b}}{m_a}f_a - u^2D_{uu,0}^{a/b}\frac{\partial f_a}{\partial u}\right).$$

(All other components vanish.) Substituting for $\mathbf{F}^{a/b}$ and $\mathsf{D}^{a/b}$ and evaluating the integral yields for the energy change of the test particle

$$\frac{d\epsilon}{dt} = \frac{1}{n_a}\frac{dE_a}{dt} = \frac{\Gamma^{a/b}m_a}{\gamma u}\left(\Theta_b\mu_1' - \frac{m_a}{m_b}\mu_1\right) \tag{32}$$

[cf. Eq. (4)].

The relation between the four relaxation rates can be found by writing the rate of change of energy of the test particle as

$$\begin{aligned}\frac{d\epsilon}{dt} &= m_ac^2\frac{d}{dt}\overline{(\gamma-1)}\\ &= m_ac^2\lim_{\delta t\to 0}\overline{[\gamma(\mathbf{u}+\delta\mathbf{u})-\gamma(\mathbf{u})]}/\delta t,\end{aligned}$$

which, after calculating the second order Taylor expansion of $\gamma(\mathbf{u}+\delta\mathbf{u})$, may be rewritten as

$$\frac{d\epsilon}{dt} = \frac{m_a}{2\gamma}\left(2\mathbf{u}\cdot\frac{d\mathbf{u}}{dt} + \frac{1}{\gamma^2}\frac{d}{dt}(\mathbf{u}-\bar{\mathbf{u}})_\parallel^2 + \frac{d}{dt}(\mathbf{u}-\bar{\mathbf{u}})_\perp^2\right) \tag{33}$$

[cf. Eq. (5)]. This expression is, as required, consistent with the relaxation rates previously derived.

For completeness, we note that the rate of thermal equilibration between two Maxwellian populations may also be found using Eq. (31). In this case, we take $f_a = f_{aM}$ and $f_b = f_{bM}$, which yields

$$\begin{aligned}\frac{dE_a}{dt} &= \frac{\Gamma^{a/b}m_an_a}{m_bc^3}\frac{k_B(T_b-T_a)}{K_2(1/\Theta_a)K_2(1/\Theta_b)}\\ &\quad\times\left[\frac{2(\Theta_a+\Theta_b)^2+1}{\Theta_a+\Theta_b}K_1(z)+2K_0(z)\right],\end{aligned} \tag{34}$$

where $z = (\Theta_a+\Theta_b)/\Theta_a\Theta_b$. It may easily be seen that this expression satisfies energy conservation:

$dE_b/dt = -dE_a/dt$. This result has been obtained previously by Stepney [6].

IV. LIMITS OF VALIDITY

Finally, we discuss the limits of validity of this work. As we have assumed a stationary Maxwellian [Eq. (15)], these results are valid only in the rest frame of the background plasma (species b). Test particle momenta should first be transformed into this frame before using the relaxation rates. (The same restriction applies in the classical case.)

The relativistic Fokker-Planck collision operator is valid for energies $E \ll (\ln\Lambda/\alpha)^{1/2}mc^2$, where α is the fine structure constant [12]; otherwise, the process of bremsstrahlung becomes significant and collisions can no longer be considered to be elastic. As with the nonrelativistic theory, $\ln\Lambda \gg 1$ is required for the small-angle scattering approximation to be valid. We note that, under this condition, the Fokker-Planck collision operator is highly consistent with the full quantum mechanical expression [24] (calculated using, e.g., the Møller cross section for electron-electron scattering, Bhabha cross section for electron-positron scattering [25]); discrepancies between the two are smaller than or, at most, of the same order as the inaccuracies introduced from neglecting moments higher than the second in the expansion of the collision operator ($\sim 1/\ln\Lambda$).

V. CONCLUSIONS

In summary, we have presented in Eqs. (16)–(18) the dynamical friction and diffusion coefficients for a relativistic Maxwellian plasma. From these the rates of momentum loss, parallel and perpendicular momentum diffusion and energy change for a test particle have been obtained [Eqs. (28), (29), (30), and (32)]. These are presented in a simple form, such that may be used straightforwardly in the semianalytical modeling of high-temperature plasmas. They reduce to Trubnikov's well-known results (1)–(4) in the classical limit.

ACKNOWLEDGMENTS

The authors are grateful to E. G. Hill, M. Sherlock, and A. E. Turrell for helpful comments. This work was supported by the EPSRC and AWE, Aldermaston.

APPENDIX A: THE $j_{l[k]*}$ FUNCTIONS

We catalog the $j_{l[k]*}$ functions as given by Braams and Karney [21] for $l = 0,1$:

$$j_{0[1]0} = \sigma/z, \quad j_{0[1]1} = 1, \quad j_{0[1]2} = \gamma,$$
$$j_{0[2]02} = (z\gamma - \sigma)/4z, \quad j_{0[2]11} = (\gamma\sigma - z)/2z,$$
$$j_{0[2]22} = [-z\gamma + \sigma(1+2z^2)]/8z,$$
$$j_{0[3]022} = [-3z\gamma + \sigma(3+2z^2)]/32z,$$
$$j_{1[1]0} = (\gamma\sigma - z)/z^2, \quad j_{1[1]1} = (z\gamma - \sigma)/2z^2,$$
$$j_{1[1]2} = z/3, \quad j_{1[2]02} = [-3\gamma\sigma + 3z + z^3]/12z^2,$$
$$j_{1[2]11} = [-3z\gamma + \sigma(3+2z^2)]/8z^2,$$
$$j_{1[2]22} = [-\sigma\gamma(3-6z^2) + 3z - 5z^3]/72z^2,$$
$$j_{1[3]022} = [\sigma\gamma(15+6z^2) - 15z - 11z^3]/288z^2,$$

where $z = u/c$, $\gamma = (1+z^2)^{1/2}$, and $\sigma = \sinh^{-1} z = \cosh^{-1}\gamma$.

APPENDIX B: DERIVATION OF RELATIVISTIC FOKKER-PLANCK COEFFICIENTS FOR A MAXWELLIAN BACKGROUND

To derive the simplified form of the dynamical friction coefficient $F_{u,0}^{a/b}$, we begin by rewriting Eq. (12) with the explicit forms of the $j_{l[k]*}$ functions (see Appendix A) in the case of a Maxwellian background $f_b = f_{bM}$:

$$F_{u,0}^{a/b} = -\frac{\Gamma^{a/b}}{c^3\Theta_b K_2(1/\Theta_b)}\frac{m_a^2}{m_b} \times \left\{\int_0^u \left(\gamma^2 - \frac{\gamma'\sinh^{-1}(u'/c) - (u'/c)}{(u'/c)}\right) \times \frac{1}{u^2}\frac{u'^2}{\gamma'}e^{-\gamma'/\Theta_b}du' + \int_u^\infty \frac{u'}{u}\left(\frac{(u/c)\gamma - \sinh^{-1}(u/c)}{(u/c)}\right)e^{-\gamma'/\Theta_b}du'\right\}. \tag{B1}$$

Making the change of variable $\sinh\phi = (u'/c)$ for the former integral and $\gamma' = (1+u'^2/c^2)^{1/2}$ for the latter yields

$$F_{u,0}^{a/b} = -\frac{\Gamma^{a/b}}{\Theta_b K_2(1/\Theta_b)}\frac{m_a^2}{m_b}\frac{1}{u^2}\left\{\int_0^{\sinh^{-1}(u/c)}(\gamma^2\sinh\phi - (\phi\cosh\phi - \sinh\phi))\sinh\phi e^{-\cosh\phi/\Theta_b}d\phi + [(u/c)\gamma - \sinh^{-1}(u/c)]\int_\gamma^\infty \gamma' e^{-\gamma'/\Theta_b}d\gamma'\right\}. \tag{B2}$$

The latter integral may be straightforwardly integrated by parts. The former is more involved, but may also be evaluated analytically (at least in part) by noting

$$\frac{d}{d\phi}\left(\frac{\sinh\phi}{\phi}\right) = \frac{1}{\phi^2}(\phi\cosh\phi - \sinh\phi).$$

Following this program through, we find

$$F_{u,0}^{a/b} = -\frac{\Gamma^{a/b}}{\Theta_b K_2(1/\Theta_b)c}\frac{m_a^2}{m_b}\frac{1}{u^2}\left[-L_0\left(\frac{1}{2}+\Theta_b^2+\frac{\gamma^2}{2}\right) - L_1\Theta_b + L_2\left(\frac{1}{2}+\frac{\gamma^2}{2}\right) + \gamma u\Theta_b e^{-\gamma/\Theta_b}(\gamma+\Theta_b)\right], \tag{B3}$$

where

$$L_\nu(u,\Theta_b) = c\int_0^{\sinh^{-1}(u/c)} e^{-\cosh\phi/\Theta_b}\cosh(\nu\phi)d\phi$$

is the general form of the L_ν functions. Using their recurrence relation

$$L_{\nu+1} = 2\nu\Theta_b L_\nu - \frac{[(u/c+\gamma)^{2\nu}-1]}{(u/c+\gamma)^\nu} c\Theta_b e^{-\gamma/\Theta_b} + L_{\nu-1}, \quad \text{(B4)}$$

and simplifying gives, finally, Eq. (16). The simplified forms of the parallel and perpendicular diffusion coefficients may be obtained in a similar manner.

[1] J. D. Lindl, P. Amendt, R. L. Berger, S. G. Glendinning, S. H. Glenzer, S. W. Haan, R. L. Kauffman, O. L. Landen, and L. J. Suter, Phys. Plasmas **11**, 339 (2004).
[2] S. C. Wilks, W. L. Kruer, M. Tabak, and A. B. Langdon, Phys. Rev. Lett. **69**, 1383 (1992).
[3] M. Tabak, J. Hammer, M. E. Glinsky, W. L. Kruer, S. C. Wilks, J. Woodworth, E. M. Campbell, M. D. Perry, and R. J. Mason, Phys. Plasmas **1**, 1626 (1994).
[4] C. F. F. Karney and N. J. Fisch, Phys. Fluids **28**, 116 (1985).
[5] Y. J. Hu, Y. M. Hu, and Y. R. Lin-Liu, Phys. Plasmas **18**, 022504 (2011).
[6] S. Stepney, Mon. Not. R. Astron. Soc. **202**, 467 (1983).
[7] A. G. Aksenov, R. Ruffini, and G. V. Vereshchagin, Phys. Rev. Lett. **99**, 125003 (2007).
[8] A. M. Beloborodov, Mon. Not. R. Astron. Soc. **407**, 1033 (2010).
[9] L. D. Landau, Phys. Z. Sowjetunion **10**, 154 (1936).
[10] M. N. Rosenbluth, W. M. MacDonald, and D. L. Judd, Phys. Rev. **107**, 1 (1957).
[11] B. A. Trubnikov, Zh. Eksp. Teor. Fiz. **34**, 1341 (1958) [Sov. Phys.–JETP **7**, 926 (1958)].
[12] S. T. Beliaev and G. I. Budker, Dokl. Akad. Nauk SSSR **107**, 807 (1956) [Sov. Phys.–Dokl. **1**, 218 (1956)].
[13] B. J. Braams and C. F. F. Karney, Phys. Rev. Lett. **59**, 1817 (1987).
[14] L. Spitzer, Jr., *Physics of Fully Ionized Gases*, 2nd ed. (Wiley, New York, 1962), Chap. 5.
[15] B. A. Trubnikov, in *Reviews of Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1965), Vol. 1, p. 105.
[16] N. E. Frankel, K. C. Hines, and R. L. Dewar, Phys. Rev. A **20**, 2120 (1979).
[17] C. D. Dermer, Astrophys. J. **295**, 28 (1985).
[18] E. Haug, Astron. Astrophys. **191**, 181 (1988).
[19] P. de Gottal and J. Gariel, Phys. A (Amsterdam) **157**, 1059 (1989).
[20] J. Robiche and J. M. Rax, Phys. Rev. E **70**, 046405 (2004).
[21] B. J. Braams and C. F. F. Karney, Phys. Fluids B **1**, 1355 (1989).
[22] I. P. Shkarofsky, Phys. Plasmas **4**, 2464 (1997).
[23] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965), Chap. 9.
[24] H. Akama, J. Phys. Soc. Jpn. **28**, 478 (1970).
[25] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons*, 2nd ed. (Springer, Berlin, 1980), Chap. 12.