Excitation and control of chirped nonlinear ion-acoustic waves

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Large-amplitude ion acoustic waves are excited and controlled by a chirped frequency driving perturbation. The process involves capturing into autoresonance (a continuous nonlinear synchronization) with the drive by passage through the linear resonance in the problem. The transition to autoresonance has a sharp threshold on the driving amplitude. The theory of this transition is developed beyond the Korteweg–de Vries limit by using the Whitham's averaged variational principle within the water bag model and compared with Vlasov-Poisson simulations.

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I. INTRODUCTION

Waves in continuous media can be excited by a variety of processes involving resonant wave interactions. The approach requires phase matching and the examples range from optical fiber or superconducting parametric amplifiers [1,2] to the formation of electrostatic waves in stimulated Raman (SRS) and Brillouin (SBS) scattering in laser-plasma interactions [3] and more. The nonlinearity, as well as variation of the parameters of the background medium, shifts the frequencies and the wave vectors of the excited waves and thus tends to destroy the phase matching in resonant interactions leading to saturation of the excitation process. Nevertheless, under certain conditions, the nonlinearity and the variation of system parameters may work in tandem to dynamically preserve the phase matching. This phenomenon is called autoresonance [4]. It was studied in many applications, such as particle accelerators [5], fluids [6], planetary dynamics [7], atomic systems [8], and optics [9]. In plasmas, the autoresonance idea was used for generation of large-amplitude plasma waves in beat-wave accelerators [10], excitation of the diocotron modes in pure electron plasmas [11], in experiments at CERN on the formation of cold antihydrogen atoms [12], and in the SRS and SBS theory [13,14].

The ion-acoustic waves are low-frequency longitudinal plasma density oscillations. They were predicted by Tonks and Langmuir [15] in 1929 on the basis of the fluid theory and observed experimentally by Revans in 1933 [16]. Since then, this important branch of plasma waves was studied in a variety of different contexts such as laser-plasma interactions [3] and dusty [17], ionospheric [18], and ultracold [19] plasmas. Stimulated Brillouin scattering is one of the most important resonant three-wave interaction processes in laser fusion plasmas involving ion-acoustic waves [3,20]. It describes the decay of the incident high-power laser radiation (the pump) in the plasma into the scattered electromagnetic wave and an ion-acoustic wave. The process is one of the causes of depleting and redirecting the incident laser flux. Despite its importance, the theoretical understanding of this phenomenon is still incomplete for plasmas characteristic of many present experiments, the reason being the complexity involving such factors as the nonlinearity [21], plasma nonuniformity and time dependence [22], and the effects of resonant particles [23]. All these factors affect the phase-matching condition between the waves, while the kinetic effects lead to Landau damping, resonant trapping of plasma particles, and consequent saturation. Can one preserve the phase matching and avoid saturation of the ion-acoustic wave by autoresonance in the system? This work is devoted to studying this question in the small ion temperature limit $T_i/T_e \ll 1$ limit. Previous related work used the long-wavelength fluid-type driven Korteweg–de Vries (KdV) model [24,25]. We will advance the analysis by allowing for arbitrary nonlinearity, longer wavelength [$k\lambda_D \sim O(1)$], and finite ion temperature.

II. AUTORESONANT ION-ACOUSTIC WAVES IN SIMULATIONS

This work is motivated by numerical simulations of the following one-dimensional Vlasov-Poisson system describing a driven ion-acoustic wave

$$f_t + uf_x - \varphi_x f_u = 0, \quad \varphi_{xx} = \exp(\varphi + \varphi_d) - \int f\,du. \quad (1)$$

Here f and φ are the ion distribution and the electric potential and $\varphi_d = \varepsilon\cos\theta_d$, where $\theta_d = kx - \int \omega_d dt$ is a small-amplitude wavelike driving potential, having a slowly varying frequency $\omega_d(t)$. All dependent and independent variables in (1) are dimensionless such that the position, time, and velocity are rescaled with respect to the Debye length $\lambda_D = u_e/\omega_p$, the inverse ion plasma frequency $(m_i/m_e)^{1/2}\omega_p^{-1}$, and the modified electron thermal velocity $(m_e/m_i)^{1/2}u_e$. The distribution function and the potentials are rescaled with respect to $(m_i/m_e)^{1/2}n_0/u_e$ and k_BT_e/e, respectively (here $k_BT_{e,i} = m_{e,i}u_{e,i}^2$). We assume that the electrons follow a Boltzmann distribution in the combined driven and driving potentials. We also assume spatial periodicity of period $2\pi/k$ associated with the driving wave and solve the time-evolution problem, subject to the simplest initial equilibrium: $\varphi(x,0) = 0$ and $f(u,x,0) = (2\pi\sigma^2)^{-1/2}\exp(-u^2/2\sigma^2)$, where $\sigma^2 = T_i/T_e$. Note that σ and the driving parameters fully define our rescaled, dimensionless problem. We applied our Vlasov

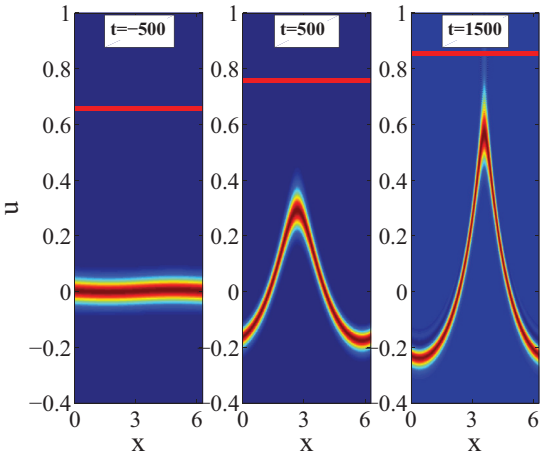

FIG. 1. (Color online) Autoresonant ion distribution in the driving wave frame at three successive times. The horizontal red lines show the location of the driving phase velocity.

code [26] for solving this problem numerically for a range of parameters and show the results of the simulations in the driving wave frame in Fig. 1 for $\sigma = 0.03$, i.e., $T_i \ll T_e$. We increase the driving frequency $\omega_d = \omega_0 + \alpha t$ and use the parameters $k = 1$, $\omega_0 = 0.66$, $\alpha = 0.0001$, and $\varepsilon = 0.0022$. Our simulations show that after the driving frequency passes the linear ion-acoustic frequency $\omega_a = k(1+k^2)^{-1/2}$, the system phase locks to the drive and one observes the formation of a growing-amplitude autoresonant deformation of the ion distribution in the figure. The associated density perturbation comprises a continuously-phase-locked, growing-amplitude ion-acoustic wave. Note that at all stages in this example, the driving phase velocity (its location is indicated by the straight red line in the figure) is well outside the ion distribution and thus the effect of resonant particles is negligible, until the final stage at $t_r = 1500$, when some resonant particles can be seen in the simulations. We found that the phase locking was lost beyond this stage and that t_r scales with the driving frequency chirp rate as α^{-1}. Figures 2(a) and 2(b) show the time evolution of the amplitude a of the first harmonic of the electric potential φ of the ion-acoustic wave and the phase mismatch Φ between the driven and driving waves as obtained in the simulations (solid lines) and theory (dotted lines) presented below. Importantly, we also found that the autoresonant excitation as seen in Fig. 1 took place only if the driving amplitude exceeded a threshold ε_{th} ($\varepsilon_{th} = 0.0017$ in our example). Below the threshold, the excitation saturates [see Figs. 2(c) and 2(d) for $\varepsilon = 0.0013$] and the phase locking discontinues.

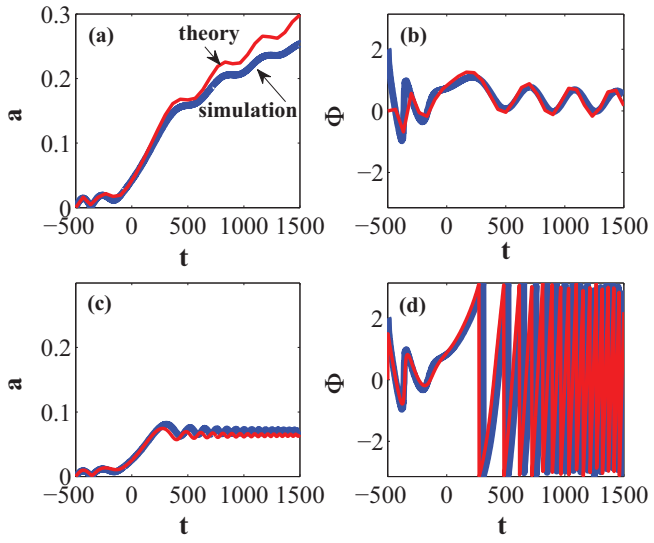

FIG. 2. (Color online) Evolution of the amplitude a and phase mismatch Φ (a) and (b) above and (c) and (d) below the threshold. The thin (red) lines show the theoretical results and the thick (blue) lines the Vlasov-Poisson simulation. Time $t = 0$ corresponds to $\omega_d = \omega_a$.

III. AVERAGED LAGRANGIAN APPROACH

Our theory of the autoresonant evolution of the ion-acoustic wave illustrated in Figs. 1 and 2 is based on the water bag model [27] of the ion distribution. We assume that the distribution is constant $f(u,x,t) = 1/2\Delta$ between two trajectories $u_{1,2}(x,t)$ in phase space and vanishes outside these trajectories (see Fig. 3). In this case, the problem can be described by the following set of the momentum and Poisson equations:

$$\begin{aligned} &u_{1t} + u_1 u_{1x} = -\varphi_x, \\ &u_{2t} + u_2 u_{2x} = -\varphi_x, \\ &\varphi_{xx} = \exp(\varphi + \varphi_d) - (u_1 - u_2)/2\Delta. \end{aligned} \tag{2}$$

If one defines $n(x,t) = (u_1 - u_2)/2\Delta$ and $u(x,t) = (u_1 + u_2)/2$, Eqs. (2) yield

$$\begin{aligned} &n_t + (un)_x = 0, \\ &u_t + uu_x = -\varphi_x - \Delta^2 n n_x, \\ &\varphi_{xx} = \exp(\varphi + \varphi_d) - n. \end{aligned} \tag{3}$$

Thus, our water bag model is isomorphic to the fluid limit of the driven ion-acoustic waves with Boltzmann electrons, the adiabatic ion pressure scaling $p \sim n^3$, and $\Delta^2 = 3\sigma^2$.

Next we observe that by defining the auxiliary potentials $\psi_{1,2}$ via $u_{1,2} = (\psi_{1,2})_x$, Eqs. (2) can be derived from the

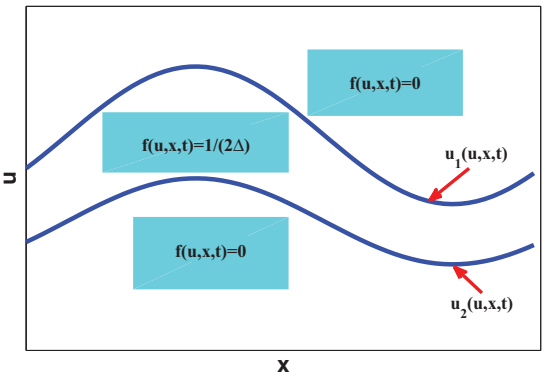

FIG. 3. (Color online) Water bag model. The ion distribution is confined between two limiting trajectories $u_{1,2}$.

variational principle with the following three-field Lagrangian

$$L = \frac{\varphi}{2\Delta}(\psi_{1x} - \psi_{2x}) - e^{\varphi+\varphi_d} - \frac{1}{2}\varphi_x^2 + \frac{1}{4\Delta}(\psi_{1x}\psi_{1t} - \psi_{2x}\psi_{2t}) + \frac{1}{12\Delta}(\psi_{1x}^3 - \psi_{2x}^3). \quad (4)$$

This Lagrangian can be used in the Whitham averaged variational principle [28] for studying the fluid limit of the driven-chirped ion-acoustic waves. The idea is to average (4) over the fast oscillations in the problem to get a new Lagrangian characterizing adiabatic modulations of the autoresonant wave parameters and thus describe the slow evolution of the system trapped in resonance with the driving wave. In studying the aforementioned autoresonance threshold phenomenon, we limit our theory to a weakly nonlinear evolution stage and consequently write the truncated harmonic decomposition of the three potentials [28]: $\psi_i \approx \xi_i + \frac{b_i}{k}\sin\theta + \frac{c_i}{2k}\sin(2\theta)$ and $\varphi = a_0 + a_1\sin\theta + a_2\sin(2\theta)$. Here the amplitudes a_i, b_i, and c_i are assumed to be slow functions of time, the wave phase θ and auxiliary phases ξ_i (necessary because ψ_i enter the Lagrangian via space-time derivatives only) are assumed to be fast [but $\theta_x = k$ and $(\xi_i)_x = \gamma_i$ are constants (given by initial conditions)], and $\theta_t = -\omega(t)$ and $(\xi_i)_t = -\alpha_i(t)$ are slow. Furthermore, we assume that the amplitudes of the zero and second harmonics scale quadratically with the amplitudes of the first harmonics. Then the substitution into (4) [with $e^{\varphi+\varphi_d}$ approximated as $e^{\varphi+\varphi_d} \approx 1 + \varphi + \varphi^2/2 + \varphi^3/6 + \varphi^4/24 + \varepsilon\varphi\cos(\theta - \Phi)$, where the phase mismatch $\Phi = \theta - \theta_d$ is assumed to be slow], averaging over θ between 0 and 2π, and truncating at fourth order in terms of the fundamental harmonic amplitudes yields the averaged slow Lagrangian

$$\Lambda = \Lambda_0(a_{0,1,2}, b_{1,2}, c_{1,2}; k, \omega, \gamma_{1,2}, \alpha_{1,2}) + \tfrac{1}{2}\varepsilon a_1\cos\Phi \quad (5)$$

in the problem. Note that k and $\gamma_{1,2}$ in our problem are given. Taking variations with respect to all slow amplitudes, a_1 yields six algebraic equations $\partial\Lambda_0/\partial A_m = 0$, where A_m represents the set $(a_{0,2}, b_{1,2}, c_{1,2})$. These equations allow us to eliminate all A_m from the problem and, after the substitution back into (5), we obtain a new slow Lagrangian

$$\Lambda' = \Lambda_0'(a_1; \theta_x, -\theta_t, (\xi_{1,2})_x, -(\xi_{1,2})_t) + \tfrac{1}{2}\varepsilon a_1\cos\Phi \quad (6)$$

involving the remaining amplitude a_1 and phases θ (via its derivatives and Φ) and $\xi_{1,2}$ (via their derivatives only). Taking the variations with respect to $\xi_{1,2}$ yields two algebraic equations $\partial\Lambda_0'/\partial\alpha_{1,2} = \text{const}$, which allow us to eliminate $\alpha_{1,2}$ and obtain the final Lagrangian of the form

$$\Lambda'' = \Lambda_0''(a_1; k, \omega) + \tfrac{1}{2}\varepsilon a_1\cos\Phi. \quad (7)$$

The evaluation of $\Lambda_0''(a_1; k, \omega)$ [to $O(a_1^4)$] in our problem involves a lengthy algebraic manipulation, which we performed by using *Mathematica* [29]. Here we present the final result in the limit $\Delta = 0$:

$$\Lambda'' = \tfrac{1}{2}B(k,\omega)a_1^2 - \tfrac{1}{4}C(k,\omega)a_1^4 + \tfrac{1}{2}\varepsilon a_1\cos\Phi, \quad (8)$$

where $B = \frac{1+k^2}{2\omega^2}(\omega^2 - \omega_a^2)$, $C = D/\{4\omega^6[k^2(4 - 16\omega^2) - 4\omega^2]\}$, and $D = 6k^4\omega^4 - 5k^6\omega^2 - 2\omega^8 + k^2\omega^6(1 - 4\omega^2) - 4k^8(1 + 5\omega^2)$. Next, we use (8) and take variations with respect to θ and a_1 to get

$$\frac{d}{dt}\left(\frac{\partial B}{\partial\omega}a_1^2\right) = -\frac{1}{2}\varepsilon a_1\sin\Phi, \quad (9)$$

$$B - Ca_1^2 + \frac{\varepsilon}{2a_1}\cos\Phi = 0. \quad (10)$$

At this stage, we write $\omega = \omega_a + \Delta\omega$, assume proximity to the linear resonance $\Delta\omega/\omega_a \ll 1$, and expand B and C in Eqs. (9) and (10) around ω_a to lowest significant order in $\Delta\omega$ to get

$$\frac{d}{dt}\left(\frac{\partial B(\omega_a)}{\partial\omega_a}a_1^2\right) = -\frac{1}{2}\varepsilon a_1\sin\Phi, \quad (11)$$

$$\Delta\omega = \left[C(\omega_a)a_1^2 - \frac{\varepsilon}{2a_1}\cos\Phi\right]\left[\frac{\partial B(\omega_a)}{\partial\omega_a}\right]^{-1}. \quad (12)$$

Then, after evaluating $\partial B(\omega_a)/\partial\omega_a = k^{-1}(1 + k^2)^{3/2}$ and $C(\omega_a) = (4 + 42k^2 + 93k^4 + 81k^6 + 24k^8)/48k^2$, we have

$$\frac{da_1}{dt} = -\frac{\varepsilon k}{4(1 + k^2)^{3/2}}a_1\sin\Phi \quad (13)$$

and (assuming passage through the linear resonance $\omega_d = \omega_a + \alpha t$)

$$\frac{d\Phi}{dt} = \Delta\omega - \alpha t = C'a_1^2 - \alpha t - \frac{\varepsilon k}{2(1 + k^2)^{3/2}a_1}\cos\Phi, \quad (14)$$

where $C' = C(\omega_a)k(1 + k^2)^{-3/2}$. Finally, we define $a = \alpha^{-1/2}C'^{1/2}a_1$, rescaled time $\tau = \alpha^{1/2}t$, and rescaled driving amplitude $\mu = C'^{1/2}\alpha^{-3/4}\varepsilon$. Note that Eqs. (13) and (14) can be combined into a single equation for $\Psi = a\exp(i\Phi)$:

$$i\Psi_\tau + (|\Psi|^2 - \tau)\Psi = \mu. \quad (15)$$

This one-parameter nonlinear Schrodinger-type equation is characteristic of passage through linear resonance in many

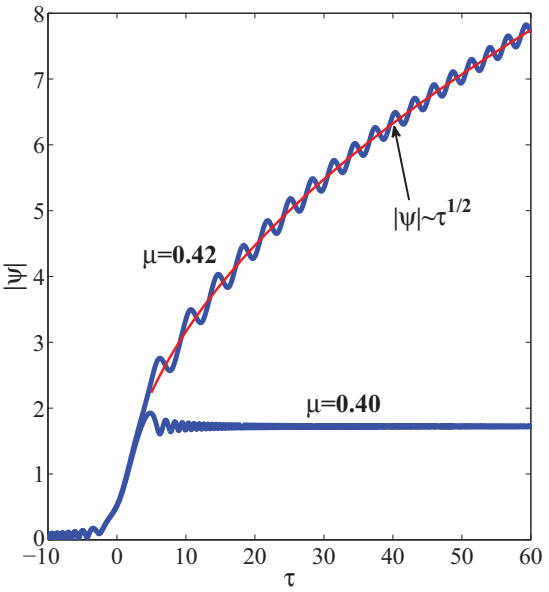

FIG. 4. (Color online) Solutions of Eq. (15) for $|\psi|$ just below ($\mu = 0.40$) and above ($\mu = 0.42$) the threshold. The thin (red) line shows the asymptotic autoresonant solution $|\psi| \sim \tau^{1/2}$. The linear resonance corresponds to $\tau = 0$.

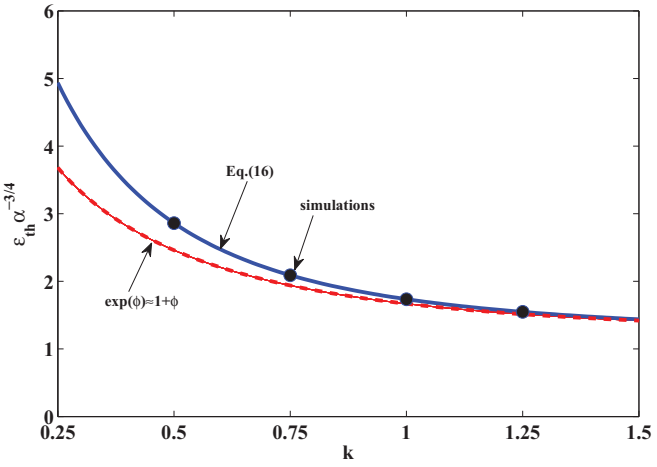

FIG. 5. (Color online) Threshold for the autoresonant transition vs k. The solid (blue) line shows Eq. (16), circles show simulations, and the dashed (red) line shows the linear approximation in the Poisson equation in Eqs. (2).

dynamical systems and predicts transition to autoresonance for $\mu > \mu_{th} = 0.41$ [4]. We illustrate this transition phenomenon in Fig. 4, showing the evolution of $|\psi|$ for μ just below and above μ_{th}. Returning to the original parameters in our driven ion-acoustic wave problem ($\varepsilon_{th} = 0.41\alpha^{3/4}C'^{-1/2}$), we obtain

$$\varepsilon_{th} = \frac{5.7\alpha^{3/4}(1+k^2)^{9/4}}{k^{1/2}(4+42k^2+93k^4+81k^6+24k^8)^{1/2}}. \quad (16)$$

The numerical solutions of Eqs. (13) and (14) shown in Fig. 2 ($\varepsilon_{th} = 0.0017$ in this example) are in an excellent agreement with Vlasov-Poisson simulations until the amplitude of the wave becomes large, in violation of our assumption of weak nonlinearity. In Fig. 5 we compare ε_{th} from Eq. (16) with Vlasov-Poisson simulations for different values of k. Finally, we found that if one uses a linear expansion $\exp(\varphi+\varphi_d) \approx 1+\varphi+\varphi_d$ (the usual assumption in deriving the KdV limit in the problem) instead of the fourth-order expansion used above, then $4+42k^2+93k^4+81k^6+24k^8$ in the denominator on the right-hand side in Eq. (16) should be replaced by $3(1+k^2)^3(3+8k^2)$ (see the corresponding dashed line in Fig. 5). This yields a significant difference in the threshold at smaller k.

IV. SUMMARY

We have studied autoresonant excitation of nonlinear ion-acoustic waves in the fluid approximation by passage through the linear resonance in the problem. The weakly nonlinear Whitham averaged variational principle was used in the theory of the autoresonant transition, yielding good agreement with Vlasov-Poisson simulations. Extension of the variational approach to include larger ion-acoustic wave amplitudes, resonant kinetics, and spatial nonuniformity effects seem to be important directions for future research.

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