Hydrodynamic Burnett equations for inelastic Maxwell models of granular gases

Nagi Khalil, Vicente Garzó, and Andrés Santos

Departamento de Física, Universidad de Extremadura, E-06071 Badajoz, Spain
(Received 31 January 2014; published 5 May 2014)

The hydrodynamic Burnett equations and the associated transport coefficients are exactly evaluated for generalized inelastic Maxwell models. In those models, the one-particle distribution function obeys the inelastic Boltzmann equation, with a velocity-independent collision rate proportional to the γ power of the temperature. The pressure tensor and the heat flux are obtained to second order in the spatial gradients of the hydrodynamic fields with explicit expressions for all the Burnett transport coefficients as functions of γ, the coefficient of normal restitution, and the dimensionality of the system. Some transport coefficients that are related in a simple way in the elastic limit become decoupled in the inelastic case. As a byproduct, existing results in the literature for three-dimensional elastic systems are recovered, and a generalization to any dimension of the system is given. The structure of the present results is used to estimate the Burnett coefficients for inelastic hard spheres.

I. INTRODUCTION

Kinetic theory provides a fundamental and systematic way of deriving closed hydrodynamic equations for dilute molecular gases by means of the Chapman-Enskog (CE) method [1]. The essential ingredients of the method are a Boltzmann-like kinetic equation for the distribution function, an identification of the hydrodynamics fields, and an expansion in powers of gradients of those hydrodynamic fields [1,2]. The generality of the above scheme allows for the use of the CE method in the study of different systems, the reliability of the resulting description being dependent, on the one hand, on the validity of the kinetic equation used and the choice of the hydrodynamic variables and, on the other hand, on the fulfillment of the hypothesis of weak spatial gradients.

For granular fluids, which can be briefly defined as systems composed by macroscopic particles with short-ranged inelastic interactions (collisions), a closed hydrodynamic description based on the CE method has been derived for different models. Two of them are relevant here, the inelastic hard-sphere model (IHSM) and the inelastic Maxwell model (IMM) [3–7]. The minimal version of the IHSM corresponds to a collection of smooth hard spheres or disks that undergo inelastic collisions, with a velocity-independent coefficient of normal restitution α [8,9]. More sophisticated models, close to the IHSM, consider particle rotations with coefficients of normal and tangential restitution [10–16], velocity-dependent coefficients of restitution [9,17,18], polydispersity [19], presence of an interstitial fluid [20–23], etc. Some conclusions of the research carried out along the last few years in the minimal version of the IHSM, and also in some others, are that the inelastic Boltzmann equation is able to describe dilute (and moderately dense) systems (the fundamental hydrodynamic variables being the same as that of the ordinary elastic case, i.e., density, velocity, and temperature) and the Navier–Stokes (NS) hydrodynamic equations provided by the CE method are applicable for a generality of accessible situations with small spatial gradients. Therefore, the current attempts to extend the NS hydrodynamic description for dilute granular gases [24,25] focus on several fronts: denser regimes [26–28], even taking into account velocity correlations [29], inclusion of non-Newtonian states

like the uniform shear flow [30–34], the Fourier state [35,36], and high gradients [37].

The latter limitation (i.e., the weakness of the spatial gradients) of the usual hydrodynamic description is addressed in this work. More specifically, we apply the CE method to the next order to the NS one, namely the Burnett order, where the irreversible momentum and heat fluxes are obtained to second order in the hydrodynamic gradients. The importance of going beyond the NS order in granular gases, due to the inherent coupling between inelasticity and spatial gradients, was pointed out by pioneering works a few years ago [37,38]. On the other hand, the derivation of the Burnett equations in the framework of the Boltzmann equation for the IHSM is an extremely difficult task that requires the use of approximations to get high-degree collisional moments. In fact, to the best of our knowledge, the existing Burnett hydrodynamics description of the IHSM [37] makes use of the elastic forms of the Burnett transport coefficients [1].

A way of circumventing the above difficulty, while keeping the structure of the nonlinear Boltzmann equation, consists of using the IMM, where calculations can be made exactly for any degree of dissipation. In this model, the collision rate of the inelastic Boltzmann equation is assumed to be independent of the relative velocity of the colliding particles, just as in the case of elastic collisions [39,40]. Furthermore, in order to capture in an effective way the velocity dependence of the original IHSM collision rate, one usually assumes that the IMM collision rate is proportional to T^γ with $\gamma = \frac{1}{2}$, where T is the local granular temperature. In this paper, we take γ as a generalized exponent, so that different values of γ can be used to mimic different interactions. For instance, in the case of elastic collisions, a repulsive potential of the form $\phi(r) \sim r^{-s}$ corresponds to $\gamma = 1/2 - (d-1)/s$, where d is the dimensionality of the system [41], so that $\gamma = 0$ defines the standard Maxwell model [$s = 2(d-1)$], while $\gamma = \frac{1}{2}$ mimics hard spheres ($s \to \infty$).

The derivation of the Burnett equations for the IMM can be essentially done thanks to the exact knowledge of the collisional moments up to fourth degree for arbitrary values of the coefficient of restitution and the dimensionality of the system [42]. The price paid for obtaining exact results

is to have a less realistic description than with the IHSM. Nevertheless, it has been shown that the transport properties obtained from the IMM compare quite well with those of the IHSM [43–47]. Moreover, experiments for magnetic grains can be well described by means of the IMM [48]. In addition, the structure of the Burnett constitutive equations of the IMM are expected to be the same as that of IHSM. Apart from that, the results of the present work have their own interest since they constitute a natural extension of the Burnett hydrodynamic description of Maxwell molecules [1] to granular gases. As we will see, some Burnett transport coefficients having simple relationships in the elastic limit decouple in the inelastic case.

While the knowledge of the Burnett constitutive equations can be useful for the description of non-Newtonian granular flows, some caution is required because, as reported for ordinary gases in Bobylev's pioneering work [49], the Burnett hydrodynamic equations are *artificially* unstable. On the other hand, several methods of regularization of the Burnett equations have been proposed to overcome the above difficulty [38,50–54]. In principle, those methods could be applied to the inelastic case in order to disentangle Bobylev's instability from the clustering instability that can be present in granular gases [55,56].

This work is organized as follows. In Sec. II, the general CE method is applied to the inelastic Boltzmann equation. The IMM is introduced in Sec. III and the existing results in the literature for the zeroth (Euler) and first (NS) orders in gradients are generalized to arbitrary γ. In Sec. IV, the Burnett transport coefficient of the pressure tensor and the heat flux are calculated. They are explicitly given in terms of the coefficient of normal restitution, the dimensionality of the system, and the parameter γ. The most technical details are relegated to Appendixes B and C. The results are widely discussed in Sec. V, where the α dependence of the Burnett transport coefficients is presented and explicit expressions of the above coefficients in the elastic limit ($\alpha = 1$) are given and compared with those in the literature. In addition, based on the formal structure of the results for the IMM, estimates of the Burnett coefficients for the IHSM are displayed. Finally, the paper is closed in Sec. VI with some concluding remarks.

II. FROM KINETIC TO HYDRODYNAMIC DESCRIPTIONS

In this section, the CE method is described for a d-dimensional system composed by inelastic particles of mass m and coefficient of normal restitution α ($0 \leqslant \alpha \leqslant 1$).

First, the Boltzmann equation for a force-free d-dimensional granular gas is considered,

$$(\partial_t + \mathbf{v} \cdot \nabla) f(\mathbf{r},\mathbf{v},t) = J[\mathbf{v}|f,f], \quad (2.1)$$

where $f(\mathbf{r},\mathbf{v},t)$ is the distribution function of a particle at position $\mathbf{r}$, with velocity $\mathbf{v}$ at time t. The explicit form of the collision operator $J[\mathbf{v}|f,f]$ is so far not needed, except that it must reflect the collision rules relating the precollisional velocities $\{\mathbf{v}_1', \mathbf{v}_2'\}$ to the postcollisional velocities $\{\mathbf{v}_1, \mathbf{v}_2\}$:

$$\mathbf{v}_1' = \mathbf{v}_1 - \tfrac{1}{2}(1 + \alpha^{-1})(\mathbf{g} \cdot \widehat{\boldsymbol{\sigma}})\widehat{\boldsymbol{\sigma}}, \quad (2.2a)$$

$$\mathbf{v}_2' = \mathbf{v}_2 + \tfrac{1}{2}(1 + \alpha^{-1})(\mathbf{g} \cdot \widehat{\boldsymbol{\sigma}})\widehat{\boldsymbol{\sigma}}, \quad (2.2b)$$

where $\mathbf{g} = \mathbf{v}_1 - \mathbf{v}_2$ is the relative velocity and $\widehat{\boldsymbol{\sigma}}$ is a unit vector directed along the line of centers of the two colliding particles.

Secondly, as usual, the hydrodynamic fields are chosen to be the number density,

$$n(\mathbf{r},t) = \int d\mathbf{v} f(\mathbf{r},\mathbf{v},t), \quad (2.3)$$

the flow velocity,

$$\mathbf{u} = \frac{1}{n} \int d\mathbf{v} \, \mathbf{v} f(\mathbf{v},\mathbf{r},t), \quad (2.4)$$

and the granular temperature,

$$T(\mathbf{r},t) = \frac{m}{nd} \int d\mathbf{v} \, V^2 f(\mathbf{v},\mathbf{r},t), \quad (2.5)$$

where $\mathbf{V} = \mathbf{v} - \mathbf{u}$ is the peculiar velocity. By taking moments in the Boltzmann equation with respect to $1, \mathbf{v}$, and v^2, the balance equations are obtained:

$$\partial_t n + \nabla \cdot (n\mathbf{u}) = 0, \quad (2.6)$$

$$\partial_t u_i + u_j \nabla_j u_i + \frac{1}{\rho} \nabla_j P_{ij} = 0, \quad (2.7)$$

$$\partial_t T + \mathbf{u} \cdot \nabla T + \frac{2}{nd} (\mathsf{P} : \nabla \mathbf{u} + \nabla \cdot \mathbf{q}) = -\zeta T. \quad (2.8)$$

In Eq. (2.7), $\rho = mn$ is the mass density. The pressure tensor P, the heat flux $\mathbf{q}$, and the cooling rate ζ are defined in terms of the distribution function as

$$\mathsf{P}(\mathbf{r},t) = m \int d\mathbf{v} \, \mathbf{V}\mathbf{V} f(\mathbf{r},\mathbf{v},t), \quad (2.9)$$

$$\mathbf{q}(\mathbf{r},t) = \frac{m}{2} \int d\mathbf{v} \, V^2 \mathbf{V} f(\mathbf{r},\mathbf{v},t), \quad (2.10)$$

$$\zeta(\mathbf{r},t) = -\frac{m}{nTd} \int d\mathbf{v} \, v^2 J[\mathbf{v}|f,f]. \quad (2.11)$$

As the collision operator J conserves the number of particles and linear momentum, in the expression of the cooling rate (2.11), v^2 can be replaced by V^2.

Finally, the CE method is applied. This method provides a *normal* solution to the Boltzmann equation, i.e., a solution where all space and time dependence occurs through the hydrodynamic fields,

$$f(\mathbf{r},\mathbf{v},t) = f[\mathbf{v}|n(\mathbf{r},t),\mathbf{u}(\mathbf{r},t),T(\mathbf{r},t)], \quad (2.12)$$

and, as a consequence, a closed hydrodynamic description is obtained. The functional dependence on the hydrodynamic fields in Eq. (2.12) is made local in space by an expansion in spatial gradients as

$$f(\mathbf{r},\mathbf{v},t) = f^{(0)}(\mathbf{r},\mathbf{v},t) + \epsilon f^{(1)}(\mathbf{r},\mathbf{v},t) + \epsilon^2 f^{(2)}(\mathbf{r},\mathbf{v},t) + \cdots, \quad (2.13)$$

where the superscript denotes the order of the gradient and ϵ is a nonuniformity parameter. In this way, the perturbative orders denoted by ϵ are associated with the gradients, and hence the hydrodynamic fields are of zeroth order. The mean

difference of the CE method with respect to other perturbative schemes is the association of different time scales to different orders in gradients [2]. Therefore, the time derivative is also expanded as

$$\partial_t = \partial_t^{(0)} + \epsilon \partial_t^{(1)} + \epsilon^2 \partial_t^{(2)} + \cdots. \quad (2.14)$$

Once the ingredients of the method have been put together, the distribution function (and also a closed set of hydrodynamic equations) are obtained at the desired order in the gradients. In particular, the pressure tensor, the heat flux, and the cooling rate can be written as

$$\mathsf{P} = \mathsf{P}^{(0)} + \epsilon \mathsf{P}^{(1)} + \epsilon^2 \mathsf{P}^{(2)} + \cdots, \quad (2.15)$$

$$\mathbf{q} = \mathbf{q}^{(0)} + \epsilon \mathbf{q}^{(1)} + \epsilon^2 \mathbf{q}^{(2)} + \cdots, \quad (2.16)$$

$$\zeta = \zeta^{(0)} + \epsilon \zeta^{(1)} + \epsilon^2 \zeta^{(2)} + \cdots, \quad (2.17)$$

where the different powers of ϵ correspond to retaining the orders of the expansion (2.13) of the distribution function in the definitions (2.9)–(2.11). When the zeroth and first-order terms in Eqs. (2.15)–(2.17) are inserted into the balance equations (2.6)–(2.8), the Euler and NS hydrodynamic equations are obtained, respectively. The second-order terms yield the Burnett hydrodynamic equations. As said in Sec. I, the main objective of this paper is to derive the Burnett constitutive equations for the IMM with explicit expressions for all the involved transport coefficients.

III. INELASTIC MAXWELL MODELS. EULER AND NAVIER–STOKES ORDERS

The IMM collisional operator is [7]

$$J[\mathbf{v}_1|f,f] = \frac{(d+2)\nu_0}{2n\Omega_d} \int d\mathbf{v}_2 \int d\widehat{\boldsymbol{\sigma}} [\alpha^{-1} f(\mathbf{r},\mathbf{v}_1',t)$$
$$\times f(\mathbf{r},\mathbf{v}_2',t) - f(\mathbf{r},\mathbf{v}_1,t) f(\mathbf{r},\mathbf{v}_2,t)], \quad (3.1)$$

where $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the total solid angle in d dimensions and ν_0 is an effective collision frequency that is taken here to be proportional to the density and the γ power of the temperature,

$$\nu_0 \propto nT^\gamma. \quad (3.2)$$

The factor $(d+2)/2$ appearing on the right-hand side of Eq. (3.1) has been introduced to guarantee that the NS shear viscosity in the elastic limit ($\alpha = 1$) is simply $\eta_0 = p/\nu_0$, where $p = nT$ is the hydrostatic pressure. The class of models with general γ mimic other inelastic models with a collision rate proportional to a power of the relative velocity [57–59].

The specific IMM form (3.1) allows one to exactly express any collisional moment of degree k in terms of the moments of f of degree equal to or smaller than k. In particular, the cooling rate ζ is [43]

$$\zeta = \frac{d+2}{4d}(1-\alpha^2)\nu_0. \quad (3.3)$$

As a consequence, ζ does not depend on the hydrodynamic gradients and hence Eq. (2.17) implies

$$\zeta = \zeta^{(0)}, \quad (3.4)$$

$$\zeta^{(i)} = 0, \quad i \geqslant 1. \quad (3.5)$$

In the case of the IHSM, $\zeta^{(1)} = 0$ in the dilute limit [24,26] but $\zeta^{(2)} \neq 0$, although its influence on the energy balance equation is relatively very small.

A. Euler order

To zeroth order, Eq. (2.1) becomes

$$\partial_t^{(0)} f^{(0)} = J[\mathbf{v}|f^{(0)},f^{(0)}]. \quad (3.6)$$

As the time dependence of $f^{(0)}$ occurs through the hydrodynamic fields, the time derivative can be written as

$$\partial_t^{(0)} f^{(0)} = \frac{\partial f^{(0)}}{\partial n} \partial_t^{(0)} n + \frac{\partial f^{(0)}}{\partial u_i} \partial_t^{(0)} u_i + \frac{\partial f^{(0)}}{\partial T} \partial_t^{(0)} T. \quad (3.7)$$

The balance equations (2.6)–(2.8) to zeroth order read $\partial_t^{(0)} n = \partial_t^{(0)} u_i = 0$ and

$$\partial_t^{(0)} T = -\zeta T, \quad (3.8)$$

where in Eq. (3.8) we have taken into account Eq. (3.4). Using Eq. (3.8), Eq. (3.6) becomes

$$-T\zeta \partial_T f^{(0)} = J[\mathbf{v}|f^{(0)},f^{(0)}]. \quad (3.9)$$

As Eq. (3.9) is also verified by the distribution function of the homogeneous cooling state (HCS), the distribution function of zeroth order $f^{(0)}$ is the local version of the latter with the replacement $\mathbf{v} \to \mathbf{V}$. Since $f^{(0)}$ is an isotropic function (with respect to $\mathbf{V}$), then

$$P_{ij}^{(0)} = p\delta_{ij} = nT\delta_{ij}, \quad (3.10)$$

$$\mathbf{q}^{(0)} = 0. \quad (3.11)$$

Although the solution to Eq. (3.9) is not known, its velocity moments can be in principle obtained in a recursive way [5,6]. In particular, the fourth-degree cumulant is [43]

$$c \equiv \frac{d}{d+2} \frac{\langle V^4 \rangle}{\langle V^2 \rangle^2} - 1 = \frac{6(1-\alpha)^2}{4d-7+3\alpha(2-\alpha)}, \quad (3.12)$$

where $\langle A(\mathbf{V}) \rangle = n^{-1} \int d\mathbf{v} A(\mathbf{V}) f^{(0)}$.

B. NS order

Once $f^{(0)}$ is characterized, it is possible to consider the first order. Now, the equation for $f^{(1)}$ reads

$$\partial_t^{(0)} f^{(1)} + \mathcal{L} f^{(1)} = -\left(\partial_t^{(1)} + \mathbf{v} \cdot \nabla\right) f^{(0)}, \quad (3.13)$$

where

$$\mathcal{L} f^{(1)} = -J^{(1)}[f,f] = -J[f^{(1)},f^{(0)}] - J[f^{(0)},f^{(1)}] \quad (3.14)$$

is the linearized (inelastic) Boltzmann collision operator acting on $f^{(1)}$.

The right-hand side of Eq. (3.13) can easily be evaluated taking into account that the balance equations to first order become

$$\partial_t^{(1)} n + \boldsymbol{\nabla} \cdot (n\mathbf{u}) = 0, \tag{3.15}$$

$$\partial_t^{(1)} u_i + u_j \nabla_j u_i + \rho^{-1} \nabla_i p = 0, \tag{3.16}$$

$$\partial_t^{(1)} T + \mathbf{u} \cdot \boldsymbol{\nabla} T + \frac{2}{d} T \boldsymbol{\nabla} \cdot \mathbf{u} = 0. \tag{3.17}$$

Equation (3.13) is a linear integral equation for $f^{(1)}$ with an inhomogeneous term given by the right-hand side. It is straightforward to check that the inhomogeneous term is orthogonal to $(1, \mathbf{v}, v^2)$, i.e., the subspace associated with the null eigenvalue of the linear operator acting on $f^{(1)}$ (solubility conditions) [60]. The general solution to Eq. (3.13) is of the form [24,43],

$$f^{(1)}(\mathbf{r}, \mathbf{v}, t) = \mathcal{A}_i(\mathbf{V})\nabla_i \ln n + \mathcal{B}_i(\mathbf{V})\nabla_i \ln T$$
$$+ \mathcal{C}_{ij}(\mathbf{V})\left(\nabla_i u_j + \nabla_j u_i - \frac{2}{d}\delta_{ij}\boldsymbol{\nabla} \cdot \mathbf{u}\right), \tag{3.18}$$

where $\mathcal{A}_i(\mathbf{V})$, $\mathcal{B}_i(\mathbf{V})$, and $\mathcal{C}_{ij}(\mathbf{V})$ obey a set of linear integral equations. The absence of an independent scalar term proportional to $\boldsymbol{\nabla} \cdot \mathbf{u}$ in Eq. (3.18) implies that any isotropic moment of $f^{(1)}$ must vanish. On the other hand, for dense gases the above property does not apply [26,61].

The NS constitutive equations for the pressure tensor and the heat flux have the form,

$$P_{ij}^{(1)} = -\eta\left(\nabla_i u_j + \nabla_j u_i - \frac{2}{d}\delta_{ij}\boldsymbol{\nabla} \cdot \mathbf{u}\right), \tag{3.19}$$

$$\mathbf{q}^{(1)} = -\mu \boldsymbol{\nabla} n - \kappa \boldsymbol{\nabla} T, \tag{3.20}$$

where, by dimensional analysis, the shear viscosity η, the thermal conductivity κ, and the coefficient μ have the following scaling properties:

$$\eta = \eta_0 \eta^*(\alpha), \quad \kappa = \kappa_0 \kappa^*(\alpha), \quad \mu = \frac{T\kappa_0}{n}\mu^*(\alpha). \tag{3.21}$$

Here, $\eta_0 = p/\nu_0$ and $\kappa_0 = [d(d+2)/2(d-1)]\eta_0/m$ are the shear viscosity and thermal conductivity coefficients in the elastic limit.

In the case of the IMM, the transport coefficients can be obtained without the need of determining the unknown functions $\mathcal{A}_i(\mathbf{V})$, $\mathcal{B}_i(\mathbf{V})$, and $\mathcal{C}_{ij}(\mathbf{V})$. The method consists of multiplying Eq. (3.13) by $V_i V_j - d^{-1}V^2\delta_{ij}$ and $V^2\mathbf{V}$, integrating over velocity, and applying Eqs. (3.15)–(3.17). The results are

$$\eta^* = \frac{1}{\nu_{0|2}^* - (1 - \gamma)\zeta^*}, \tag{3.22}$$

$$\kappa^* = \frac{d-1}{d}\frac{1 + 2c}{\nu_{2|1}^* - 2\zeta^*}, \tag{3.23}$$

$$\mu^* = \frac{\kappa^*}{1 + 2c}\frac{\zeta^* + \nu_{2|1}^* c}{\nu_{2|1}^* - (2 - \gamma)\zeta^*}. \tag{3.24}$$

Upon deriving these equations, use has been made of the exact expressions for the second- and third-degree collisional moments for IMM [see Eqs. (2.17) and (2.20) of Ref. [42]]. In Eqs. (3.22)–(3.24),

$$\nu_{0|2}^* = \frac{(1 + \alpha)(d + 1 - \alpha)}{2d}, \tag{3.25}$$

$$\nu_{2|1}^* = \frac{(1 + \alpha)[5d + 4 - \alpha(d + 8)]}{8d}, \tag{3.26}$$

and $\zeta^* = \zeta/\nu_0$. Equations (3.22)–(3.24) for $\gamma = \frac{1}{2}$ were first obtained in Ref. [43]. Here, they are generalized to arbitrary γ.

It is interesting to observe that the structure of Eqs. (3.22)–(3.24) for the NS transport coefficients (with $\gamma = \frac{1}{2}$) coincides with that of the IHSM, except that the α dependence of the cumulant c, the cooling rate ζ^*, and the collision frequencies $\nu_{0|2}^*$ and $\nu_{2|1}^*$ are different [24,62–65]. The IHSM expressions can be found in Appendix A.

IV. BURNETT ORDER

In this section, the Burnett constitutive equations for the pressure tensor and heat flux are derived and the corresponding transport coefficients are evaluated. The procedure is similar to the one followed at zeroth and first orders and makes use of the preceding results. As is usually done in the case of elastic collisions [1,60,66], we will choose $p = nT$ instead of n as a hydrodynamic variable in the Burnett order.

To second order in ϵ, the kinetic equation for f reads

$$\partial_t^{(0)} f^{(2)} + \left(\partial_t^{(1)} + \mathbf{v} \cdot \boldsymbol{\nabla}\right)f^{(1)} + \partial_t^{(2)} f^{(0)} = J^{(2)}[f, f], \tag{4.1}$$

where

$$J^{(2)}[f, f] = J[f^{(2)}, f^{(0)}] + J[f^{(0)}, f^{(2)}] + J[f^{(1)}, f^{(1)}]. \tag{4.2}$$

Equation (4.1) can be rewritten as

$$\partial_t^{(0)} f^{(2)} + \mathcal{L} f^{(2)} = -\left(\partial_t^{(1)} + \mathbf{v} \cdot \boldsymbol{\nabla}\right)f^{(1)}$$
$$- \partial_t^{(2)} f^{(0)} + J[f^{(1)}, f^{(1)}]. \tag{4.3}$$

As in the case of Eq. (3.13), the inhomogeneous term (right-hand side) of Eq. (4.3) is orthogonal to $(1, \mathbf{v}, v^2)$, so that the solubility conditions are satisfied. To evaluate $\partial_t^{(2)} f^{(0)}$, the balance equations to second order are needed,

$$\partial_t^{(2)} n = 0, \tag{4.4}$$

$$\partial_t^{(2)} u_i + \frac{1}{\rho}\nabla_j P_{ij}^{(1)} = 0, \tag{4.5}$$

$$\partial_t^{(2)} T + \frac{2}{nd}\left(P_{ij}^{(1)}\nabla_i \mathbf{u}_j + \boldsymbol{\nabla} \cdot \mathbf{q}^{(1)}\right) = 0. \tag{4.6}$$

The aim of this section is to determine the pressure tensor and heat flux to second order in the spatial gradients. This is accomplished by taking the corresponding moments in Eq. (4.1). Each quantity will be considered separately. Since the algebra involved is rather cumbersome, we give here the final results, the mathematical details being postponed to Appendixes B and C.

A. Pressure tensor

The Burnett constitutive equation for the pressure tensor $P_{ij}^{(2)}$ can be written as

$$
P_{ij}^{(2)} = a_1 \frac{\kappa_0}{v_0} \left(\nabla_i \nabla_j T - \frac{1}{d} \delta_{ij} \nabla^2 T \right) + a_2 \frac{T \kappa_0}{p v_0} \left(\nabla_i \nabla_j p - \frac{1}{d} \delta_{ij} \nabla^2 p \right) + a_3 \frac{\kappa_0}{T v_0} \left[(\nabla_i T)(\nabla_j T) - \frac{1}{d} \delta_{ij} (\boldsymbol{\nabla} T)^2 \right]
$$

$$
+ a_4 \frac{T \kappa_0}{p^2 v_0} \left[(\nabla_i p)(\nabla_j p) - \frac{1}{d} \delta_{ij} (\boldsymbol{\nabla} p)^2 \right] + a_5 \frac{\kappa_0}{p v_0} \left[(\nabla_i T)(\nabla_j p) + (\nabla_i p)(\nabla_j T) - \frac{2}{d} \delta_{ij} (\boldsymbol{\nabla} p) \cdot (\boldsymbol{\nabla} T) \right]
$$

$$
+ a_6 \frac{\eta_0}{v_0} D \left(D_{ij} - \frac{1}{d} \delta_{ij} D \right) + a_7 \frac{\eta_0}{v_0} \left[D_{ik} D_{kj} - \omega_{ik} \omega_{kj} - \frac{1}{d} \delta_{ij} (D_{\ell k} D_{k\ell} - \omega_{\ell k} \omega_{k\ell}) + \omega_{ik} D_{kj} - D_{ik} \omega_{kj} \right], \quad (4.7)
$$

where

$$
D \equiv \boldsymbol{\nabla} \cdot \mathbf{u}, \qquad (4.8)
$$

$$
D_{ij} \equiv \tfrac{1}{2} (\nabla_i u_j + \nabla_j u_i), \qquad (4.9)
$$

$$
\omega_{ij} \equiv \tfrac{1}{2} (\nabla_j u_i - \nabla_i u_j). \qquad (4.10)
$$

The terms in Eq. (4.7) fall into two classes [66]: those which are linear in second derivatives of T and p and those which are quadratic in the first derivatives of T, p, and $\mathbf{u}$. The coefficients a_1 and a_2 correspond to the first class, while the coefficients a_3–a_7 correspond to the second class.

The reduced Burnett coefficients a_i are dimensionless quantities that are consistently determined in Appendix B. While the coefficients a_1–a_5 (involving terms associated with pressure and temperature gradients) obey a set of coupled linear equations, the coefficients a_6 and a_7 are decoupled from the rest. They are given by

$$
a_6 = \frac{2}{d} \frac{d - 2(2 - \gamma)}{v_{0|2}^* - (1 - 2\gamma) \zeta^*} \eta^*, \qquad (4.11)
$$

$$
a_7 = \frac{2\eta^*}{v_{0|2}^* - (1 - 2\gamma) \zeta^*}. \qquad (4.12)
$$

As shown in Appendix D, the coefficients a_6 and a_7 agree with the results obtained in the zero strain rate limit of the viscometric functions defined in the non-Newtonian uniform shear and uniform longitudinal flows.

The two linear Burnett coefficients a_1 and a_2 obey a closed set of two equations whose solution is

$$
a_1 = \frac{4}{(d+2)\Delta} \left\{ [v_{0|2}^* - (3 - 2\gamma) \zeta^*](\kappa^* - \mu^*) \right.
$$

$$
\left. + (1 - \gamma) \zeta^* \left(\frac{d-1}{d} \eta^* - \mu^* \right) \right\}, \qquad (4.13)
$$

$$
a_2 = -\frac{4}{(d+2)\Delta} \left[\frac{d-1}{d} - \frac{\mu^*}{\eta^*} - \zeta^* (\kappa^* - \mu^*) \right], \quad (4.14)
$$

where

$$
\Delta \equiv [v_{0|2}^* - (2 - \gamma) \zeta^*][v_{0|2}^* - 2(1 - \gamma) \zeta^*]. \quad (4.15)
$$

The remaining three coefficients are given by

$$
\begin{pmatrix} a_3 \\ a_4 \\ a_5 \end{pmatrix} = \mathsf{L}^{-1} \cdot \mathsf{X}, \qquad (4.16)
$$

where L is the square matrix,

$$
\mathsf{L} = \begin{pmatrix} v_{0|2}^* & 0 & 2(1 - \gamma) \zeta^* \\ 0 & v_{0|2}^* - 2(2 - \gamma) \zeta^* & -2\zeta^* \\ -\zeta^* & (1 - \gamma) \zeta^* & v_{0|2}^* - (2 - \gamma) \zeta^* \end{pmatrix}, \qquad (4.17)
$$

and X is the column matrix,

$$
\mathsf{X} = \frac{4}{d+2} \begin{pmatrix} (1 - \gamma)(\kappa^* - \mu^*) \\ \frac{d-1}{d} \eta^* - \mu^* \\ \left(1 - \frac{\gamma}{2}\right) \mu^* - \frac{d-1}{2d} \eta^* \end{pmatrix}
$$

$$
- \begin{pmatrix} \gamma(1 - \gamma) \zeta^* & -(1 - \gamma)(2 - \gamma) \zeta^* \\ 0 & -2\zeta^* \\ -\gamma \zeta^* & 2(1 - \gamma) \zeta^* \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \qquad (4.18)
$$

B. Heat flux

The structure of the Burnett constitutive equation for the heat flux $\mathbf{q}^{(2)}$ is

$$
q_i^{(2)} = b_1 \frac{T \kappa_0}{v_0} \nabla^2 u_i + b_2 \frac{T \kappa_0}{v_0} \nabla_i D + b_3 \frac{\kappa_0}{v_0} D_{ij} \nabla_j T
$$

$$
+ b_4 \frac{\eta_0}{\rho v_0} D_{ij} \nabla_j p + b_5 \frac{\kappa_0}{v_0} \omega_{ij} \nabla_j T + b_6 \frac{\eta_0}{\rho v_0} \omega_{ij} \nabla_j p
$$

$$
+ b_7 \frac{\kappa_0}{v_0} D \nabla_i T + b_8 \frac{\eta_0}{\rho v_0} D \nabla_i p. \qquad (4.19)
$$

Analogously to the case of the pressure tensor, Eq. (4.19) contains linear Burnett terms (with coefficients b_1 and b_2) and nonlinear Burnett terms (with coefficients b_3–b_8).

The procedure to obtain the coefficients b_i is described in Appendix C. In the case of the linear Burnett coefficients b_1 and b_2, the results are

$$
b_1 = \frac{d-1}{d(d+2)} \frac{\psi - (d+2) \eta^*}{v_{2|1}^* - 2(1 - \gamma) \zeta^*}, \qquad (4.20)
$$

$$
b_2 = \frac{\frac{(d-1)(d-2)}{d^2(d+2)} [\psi - (d+2) \eta^*] - \frac{2}{d} \kappa^* - \mu^*}{v_{2|1}^* - 2(1 - \gamma) \zeta^*}, \qquad (4.21)
$$

where

$$
\psi \equiv \frac{(d+4)(1+c) + d\lambda^* \eta^*}{v_{2|2}^* - (2 - \gamma) \zeta^*}, \qquad (4.22)
$$

with

$$\nu_{2|2}^* = \frac{(1+\alpha)}{8d(d+4)}[7d^2 + 31d + 18 - \alpha(d^2 + 14d + 34)$$
$$+ 3\alpha^2(d+2) - 6\alpha^3], \qquad (4.23)$$

$$\lambda^* = \frac{(1+\alpha)^2}{8d^2}[d^2 + 5d - 2 - 3\alpha(d+4) + 6\alpha^2]. \quad (4.24)$$

The remaining coefficients obey pairs of linear equations whose solutions are

$$\begin{pmatrix} b_3 \\ b_4 \end{pmatrix} = \mathsf{M}^{-1} \cdot \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \qquad (4.25)$$

$$\begin{pmatrix} b_5 \\ b_6 \end{pmatrix} = \mathsf{M}^{-1} \cdot \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \qquad (4.26)$$

$$\begin{pmatrix} b_7 \\ b_8 \end{pmatrix} = \mathsf{M}^{-1} \cdot \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \qquad (4.27)$$

where

$$\mathsf{M} = \begin{pmatrix} \nu_{2|1}^* - (1-\gamma)\zeta^* & -\frac{2(d-1)}{d(d+2)}(\gamma-1)\zeta^* \\ -\frac{d(d+2)}{2(d-1)}\zeta^* & \nu_{2|1}^* - (3-2\gamma)\zeta^* \end{pmatrix}, \quad (4.28)$$

$$A_1 = \frac{2(d-1)}{d}\left[\frac{2-\gamma}{d+2}\psi - (1-\gamma)\eta^*\right] + \frac{4(\kappa^* - \mu^*)}{d+2}, \qquad (4.29)$$

$$A_2 = -2\left(\eta^* - \frac{d}{d-1}\mu^*\right), \qquad (4.30)$$

$$B_1 = 2(\kappa^* - \mu^*), \quad B_2 = \frac{d+2}{d-1}\mu^*, \qquad (4.31)$$

$$C_1 = -\frac{2(d-1)}{d^2}\left[\frac{2-\gamma}{d+2}\psi - (1-\gamma)\eta^*\right]$$
$$+ \frac{d^2 + 2\gamma(d+2) - 8}{d(d+2)}(\kappa^* - \mu^*), \qquad (4.32)$$

$$C_2 = \frac{2}{d}\eta^* + \frac{d^2 + 2\gamma(d+2) - 8}{2(d-1)}\mu^*. \qquad (4.33)$$

V. DISCUSSION

A. Structure of the Burnett coefficients

The main results of the paper, derived for the IMM, are summarized by Eqs. (4.7) and (4.19), complemented by the explicit expressions for the (reduced) Burnett transport coefficients $\{a_i\}$ and $\{b_i\}$. They are exactly given by Eqs. (4.11)–(4.16), (4.20), (4.21), and (4.25)–(4.27) for arbitrary values of the dimensionality d, the model parameter γ, and the coefficient of normal restitution α.

In principle, the coefficients of $D_{ik}D_{kj} - \frac{1}{d}\delta_{ij}D_{\ell k}D_{k\ell}$, $\frac{1}{d}\delta_{ij}\omega_{\ell k}\omega_{k\ell} - \omega_{ik}\omega_{kj}$, and $\omega_{ik}D_{kj} - D_{ik}\omega_{kj}$ in Eq. (4.7) do not need to be the same. However, our results show that the three coefficients degenerate into a single one (a_7) in the IMM.

The seven coefficients $\{a_i\}$ associated with the pressure tensor depend on α *only* through a dependence on the reduced

TABLE I. Relationship between the coefficients ϖ_i and a_i and between the coefficients θ_i and b_i.

$P_{ij}^{(2)}$ coefficients	$q_i^{(2)}$ coefficients
$\varpi_1 = a_6 + \frac{6}{d}a_7$	$\theta_1 = \frac{d(d+2)}{2(d-1)}\left(\frac{d-2}{d}b_1 - b_2 + \frac{b_3+b_5}{d} + b_7\right)$
$\varpi_2 = -\frac{d(d+2)}{2(d-1)}a_2$	$\theta_2 = \frac{d^2(d+2)}{4(d-1)}\left(\frac{d-2}{d}b_1 - b_2\right)$
$\varpi_2' = \frac{d(d+2)}{2(d-1)}a_4$	$\theta_2' = \frac{d(d+2)}{4(d-1)}b_5$
$\varpi_2'' = a_7$	$\theta_3 = b_4 + b_6$
$\varpi_3 = \frac{d(d+2)}{2(d-1)}a_1$	$\theta_4 = \frac{d(d+2)}{d-1}b_1$
$\varpi_4 = \frac{d(d+2)}{2(d-1)}(2a_5 + a_4)$	$\theta_5 = \frac{d(d+2)}{6(d-1)}(b_3 + b_5)$
$\varpi_5 = \frac{d(d+2)}{2(d-1)}a_3$	$\theta_6 = \frac{1}{d}(b_4 + b_6) + b_8$
$\varpi_6 = 4a_7$	$\theta_7 = -b_6$

cooling rate ζ^* and the three dimensionless NS coefficients η^*, κ^*, and μ^*. Note that the dependence on $\nu_{0|2}^*$ can be eliminated in favor of η^* and ζ^* via Eq. (3.22). Therefore, there must exist only three α-*independent* equations relating the seven coefficients $\{a_i\}$. One of those relations is, simply,

$$\frac{a_6}{a_7} = 1 - \frac{2(2-\gamma)}{d}. \qquad (5.1)$$

In the case of the eight coefficients $\{b_i\}$ associated with the heat flux, they depend on α through the same coefficients as before (ζ^*, η^*, κ^*, and μ^*) plus the coefficient ψ. Note that the dependence on $\nu_{2|1}^*$ can be eliminated in favor of κ^*, μ^*, and ζ^* via Eqs. (3.23) and (3.24). Therefore, there must exist again only three independent relations among the coefficients $\{b_i\}$. Finally, the 15 coefficients $\{a_i\}$ and $\{b_i\}$ depend on the five coefficients ζ^*, η^*, κ^*, μ^*, and ψ, so that the total number of constraints is 10. Since six of them involve either only the $\{a_i\}$ or the $\{b_i\}$, there are four conditions relating all the coefficients.

The Burnett constitutive equations (4.7) and (4.19) can be written in other equivalent forms [1,60,66]. In particular, in the form found in Chapman and Cowling's standard textbook [1], the pressure tensor is

$$P_{ij}^{(2)} = \varpi_1 \frac{\eta_0^2}{p}D\widetilde{D}_{ij} - \frac{\eta_0^2}{2p}\Delta_{ijk\ell}\left\{\varpi_2\rho^{-1}\nabla_k\nabla_\ell p\right.$$
$$+ \varpi_2'(\nabla_k\rho^{-1})(\nabla_\ell p) + \varpi_2''[(\nabla_k u_m)(\nabla_m u_\ell)$$
$$+ 2(\nabla_k u_m)\widetilde{D}_{m\ell}] - \varpi_3\frac{p}{\rho T}\nabla_k\nabla_\ell T$$
$$- \varpi_4\frac{1}{\rho T}(\nabla_k p)(\nabla_\ell T) - \varpi_5\frac{p}{\rho T^2}(\nabla_k T)(\nabla_\ell T)$$
$$\left. - \varpi_6\widetilde{D}_{km}\widetilde{D}_{m\ell}\right\}, \qquad (5.2)$$

where

$$\widetilde{D}_{ij} \equiv D_{ij} - \frac{D}{d}\delta_{ij}, \quad \Delta_{ijk\ell} \equiv \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk} - \frac{2}{d}\delta_{ij}\delta_{k\ell}. \qquad (5.3)$$

The first column of Table I shows the relations between the coefficients ϖ_i of Eq. (5.2) and the coefficients a_i of Eq. (4.7). Note that in Ref. [1], which is restricted to elastic gases, $\varpi_2 = \varpi_2' = \varpi_2''$. This degeneracy is broken in the inelastic case.

TABLE II. Burnett coefficients in the elastic limit ($\alpha = 1$).

Eqs. (4.7) and (4.19)	Eqs. (5.2) and (5.5)
$a_1 = \frac{4}{d+2}$	$\varpi_1 = \frac{2(d+2+2\gamma)}{d}$
$a_2 = -\frac{4(d-1)}{d(d+2)}$	$\varpi_2 = 2$
$a_3 = \frac{4(1-\gamma)}{d+2}$	$\varpi_2' = \varpi_2'' = 2$
$a_4 = \frac{4(d-1)}{d(d+2)}$	$\varpi_3 = \frac{2d}{d-1}$
$a_5 = -\frac{2(d-1)}{d(d+2)}$	$\varpi_4 = 0$
$a_6 = \frac{2(d-4+2\gamma)}{d}$	$\varpi_5 = \frac{2d(1-\gamma)}{d-1}$
$a_7 = 2$	$\varpi_6 = 8$
$b_1 = \frac{2}{d+2}$	$\theta_1 = \frac{d(d+2)(d+2+2\gamma)}{2(d-1)^2}$
$b_2 = -\frac{2(5d-2)}{d(d-1)(d+2)}$	$\theta_2 = \frac{d^2(d+2)}{2(d-1)^2}$
$b_3 = \frac{2[d^2+7d-6-2(d-1)\gamma]}{(d-1)(d+2)}$	$\theta_2' = \frac{d^2(d+2)}{2(d-1)^2}$
$b_4 = -\frac{2d}{d-1}$	$\theta_3 = -\frac{2d}{d-1}$
$b_5 = \frac{2d}{d-1}$	$\theta_4 = \frac{2d}{d-1}$
$b_6 = 0$	$\theta_5 = \frac{d[2d^2+9d-6-2(d-1)\gamma]}{3(d-1)^2}$
$b_7 = \frac{d^3-2d^2-18d+12+2(d^2+4d-2)\gamma}{d(d-1)(d+2)}$	$\theta_6 = 0$
$b_8 = \frac{2}{d-1}$	$\theta_7 = 0$

It is also interesting to note that, according to Eq. (5.1), the following relations hold for any α:

$$\varpi_6 = 4\varpi_2'' = \frac{4d}{d+2+2\gamma}\varpi_1. \qquad (5.4)$$

In the case of the heat flux, one can rewrite Eq. (4.19) as [1]

$$q_i^{(2)} = \frac{\eta_0^2}{\rho T}\left[\theta_1 D\nabla_i T - \frac{2}{d}\theta_2\nabla_i(DT) - 2\theta_2'(\nabla_i u_j)(\nabla_j T)\right.$$

$$+ \theta_3\frac{T}{p}\widetilde{D}_{ij}\nabla_j p + \theta_4 T\nabla_j\widetilde{D}_{ij} + 3\theta_5\widetilde{D}_{ij}\nabla_j T$$

$$\left. + \theta_6\frac{T}{p}D\nabla_i p + \theta_7\frac{T}{p}(\nabla_i u_j)(\nabla_j p)\right], \qquad (5.5)$$

where the relations between the coefficients θ_i defined in Eq. (5.5) and b_i defined in Eq. (4.19) are given in the second column of Table I.

B. Elastic limit

Before analyzing the α dependence of the Burnett coefficients, it is worthwhile considering the elastic limit ($\alpha = 1$). In that case, one has $c = \zeta^* = \mu^* = 0$, $\eta^* = \kappa^* = \nu_{0|2}^* = 1$, $\nu_{2|1}^* = (d-1)/d$, and $\psi = d + 4$. Inserting those values into the expressions of the coefficients a_i and b_i, one obtains the values displayed in the first column of Table II. The values of the alternative coefficients ϖ_i and θ_i are then obtained from the expressions in Table I, the results being shown in the second column of Table II.

Obviously, some of the Burnett coefficients depend explicitly on the parameter γ. In the elastic case, it is legitimate to relate that parameter with the power of a repulsive interaction potential $\phi(r) \sim r^{-s}$ as $s = 2(d-1)/(1-2\gamma)$. In that case, it is well known that $\eta_0 \propto T^{1-\gamma}$. The generalization to *any* (short-range) interaction potential can be simply achieved by the replacement $\gamma \to 1 - \partial \ln \eta_0/\partial \ln T$ in Table II. Particular-
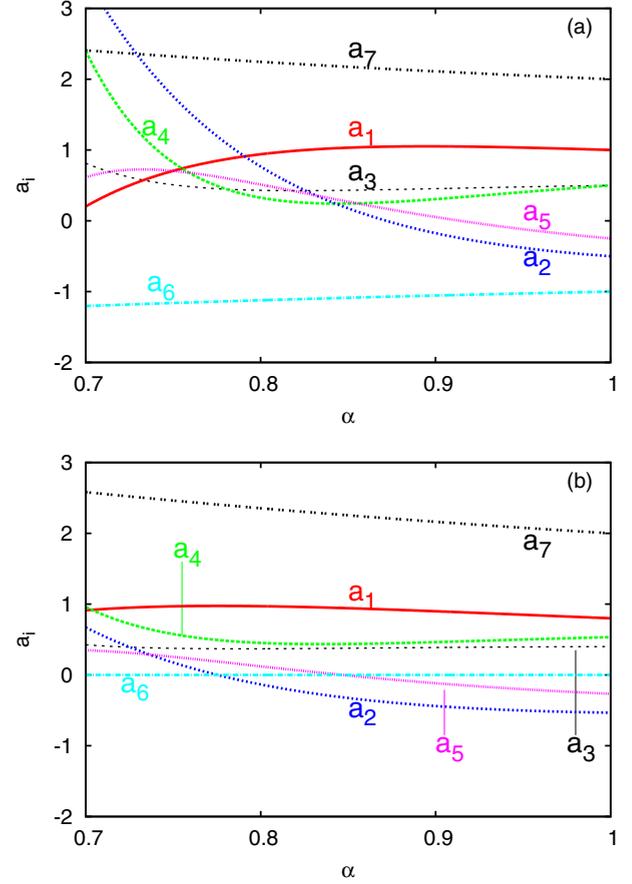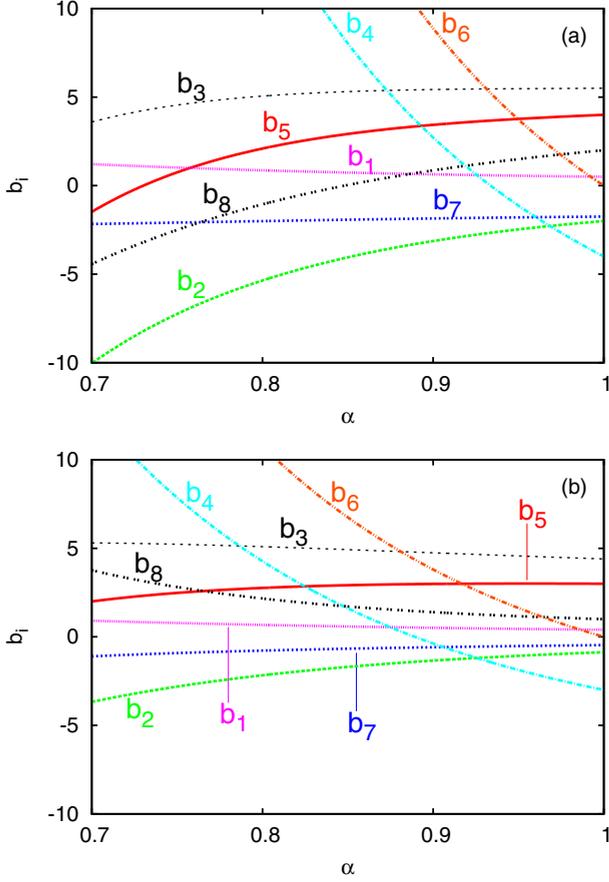

FIG. 1. (Color online) IMM Burnett transport coefficients $a_1, \ldots, a_7$ as functions of the coefficient of normal restitution in the case $\gamma = \frac{1}{2}$ for (a) $d = 2$ and (b) $d = 3$.

izing to three-dimensional systems ($d = 3$), one then recovers the expressions for the coefficients ϖ_i and θ_i given in Ref. [1] for an arbitrary potential in the first (Sonine) approximation. This is a stringent consistency test of the results derived in this paper. Moreover, Table II, with the replacement $\gamma \to 1 - \partial \ln \eta_0/\partial \ln T$, provides a generalization to any dimensionality of the Burnett coefficients given by Ref. [1] in the first approximation. To the best of our knowledge, this generalization had not been derived before. In particular, the results of Table II show that the exact universal relations [1,39,67] $\varpi_3 = \theta_4$, $\varpi_3 + \varpi_4 + \theta_3 = 0$, $(d/2)\varpi_1/\varpi_2 = (d/2)\theta_1/\theta_2 = (d+4)/2 - \partial \ln \eta_0/\partial \ln T$ hold for any dimensionality.

It is instructive to note that the simple relations (holding in the elastic case) $\varpi_2 = \varpi_2' = \varpi_2''$, $\varpi_4 = 0$, $\theta_2 = \theta_2'$, and $\theta_6 = \theta_7 = 0$ disappear if $\alpha < 1$. Except for $\varpi_2 = \varpi_2'$, this is due to the presence of the NS transport coefficient μ in the coefficients $c_{P,i}$ and $c_{q,i}$ appearing in Eqs. (B13) and (C12). On the other hand, while $c_{P,2} = -pc_{P,4}$, the relation $a_2 = -a_4$, and hence $\varpi_2 = \varpi_2'$, is broken down in the inelastic case due to the fact that $\partial_t^{(0)} T \neq 0$.

C. Inelastic case

The Burnett transport coefficients a_i associated with the pressure tensor are plotted in Fig. 1 as functions of α for two- and three-dimensional systems. In both cases, we have

FIG. 2. (Color online) IMM Burnett transport coefficients $b_1, \ldots, b_8$ as functions of the coefficient of normal restitution in the case $\gamma = \frac{1}{2}$ for (a) $d = 2$ and (b) $d = 3$.

FIG. 3. (Color online) (a) Regions in the (α, γ) plane where the IMM Burnett coefficients for a two-dimensional system diverge. The coefficient b_1 diverges in region A_I, while the coefficients a_1–a_5 and b_2–b_8 diverge in regions A_I and A_{II}. (b) Regions in the (α, γ) plane where the Burnett coefficients for a three-dimensional system diverge. The coefficient b_1 diverges in region A_I, while the coefficients b_2–b_8 diverge in regions A_I and A_{II}, the coefficients a_1 and a_2 diverge in regions A_I, A_{II}, and B_I, and the coefficients a_3–a_5 diverge in regions A_I, A_{II}, B_I, and B_{II}.

chosen $\gamma = \frac{1}{2}$, which, as said before, mimics the hard-sphere model. Similarly, Fig. 2 shows the α dependence of the Burnett transport coefficients b_i associated with the heat flux.

Within the range $0.7 \leqslant \alpha \leqslant 1$, we observe that, whereas some coefficients (a_3, a_6, b_1, b_3, and b_7) exhibit a weak dependence on dissipation, other coefficients (a_2, a_4, b_2, b_4, and b_6) are quite sensitive to α. The remaining coefficients (a_1, a_5, a_7, b_5, and b_8) present an intermediate behavior. It is especially interesting to note that the coefficient b_6, which vanishes in the elastic limit, grows very rapidly with increasing dissipation. In general, the impact of dissipation on the Burnett coefficients is more significant for $d = 2$ than for $d = 3$.

It is known that the heat flux NS coefficients κ^* and μ^* for the two- and three- dimensional IMM diverge, thus implying a breakdown of hydrodynamics, for very low values of α [7,43,68]. As seen from Eq. (3.23), the threshold for the divergence occurs when $\nu_{2|1}^* = 2\zeta^*$, i.e., at $\alpha = (4 - d)/3d$, what corresponds to $\alpha = \frac{1}{3}$ and $\frac{1}{9}$ for $d = 2$ and 3, respectively. While the Burnett coefficients a_6 and a_7 are always convergent, the remaining coefficients may diverge. This is discussed in Appendix E. The regions in the (α, γ) plane where the coefficients diverge are presented in Fig. 3.

In any case, from a practical point of view, all the NS and Burnett coefficients are well defined in the region of physical interest $\alpha \geqslant 0.5$.
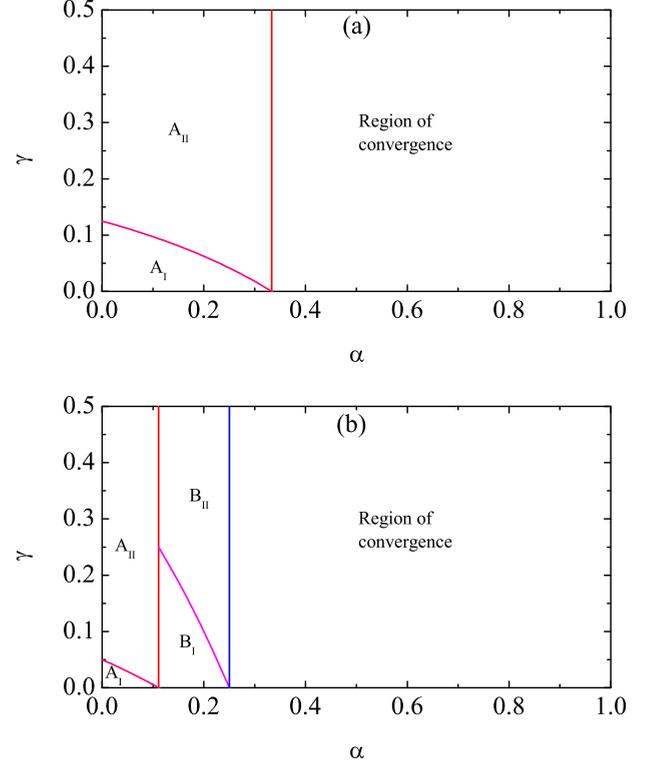
D. Estimates of IHSM coefficients

Although this paper is focused on the IMM, it is tempting to use the results derived here to *estimate* the Burnett transport coefficients for the more realistic case of the IHSM. It is reasonable to expect that the mathematical structures of the constitutive equations (4.7) and (4.19) are essentially preserved in the IHSM case.

As said before, the structure of the NS coefficients, Eqs. (3.22)–(3.24), is exactly the same for both inelastic models, the differences lying in the α dependence of the cumulant c, cooling rate ζ^*, and collision frequencies $\nu_{0|2}^*$ and $\nu_{2|1}^*$. Although the latter quantities are not exactly known for the IHSM, good estimates have been obtained from improved Sonine approximations [24,63–65]. Their expressions are given in Appendix A.

We recall that, according to Eqs. (4.11)–(4.18), the IMM Burnett coefficients a_1–a_7 associated with the pressure tensor depend on α *only* through the four quantities c, ζ^*, $\nu_{0|2}^*$, and $\nu_{2|1}^*$. This suggests that educated guesses for the corresponding IHSM Burnett coefficients can be obtained by inserting the corresponding IHSM values for c, ζ^*, $\nu_{0|2}^*$, and $\nu_{2|1}^*$ into Eqs. (4.11)–(4.18) with $\gamma = \frac{1}{2}$. The results are displayed in Fig. 4. Comparison with Fig. 1 shows qualitatively similar
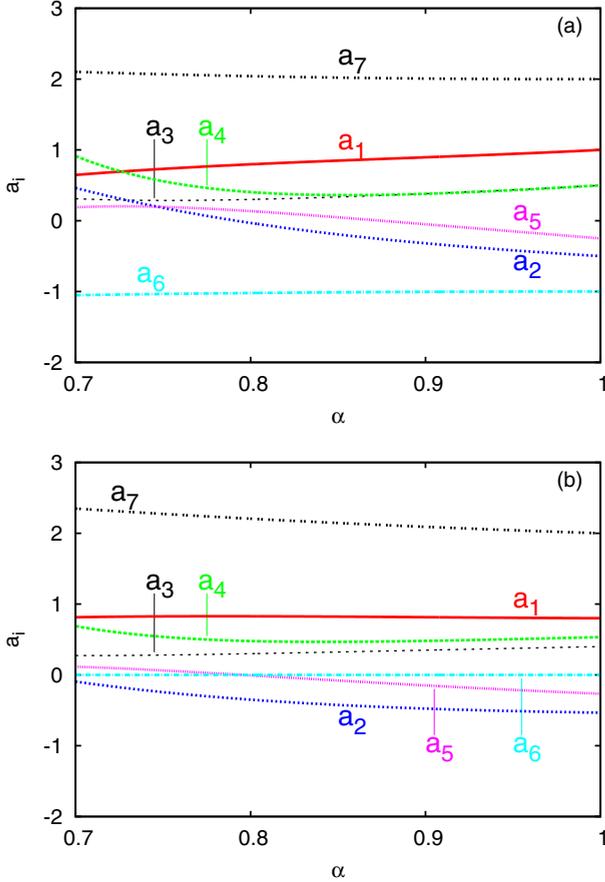
FIG. 4. (Color online) Estimates of the IHSM Burnett transport coefficients $a_1, \ldots, a_7$ as functions of the coefficient of normal restitution for (a) $d = 2$ and (b) $d = 3$.
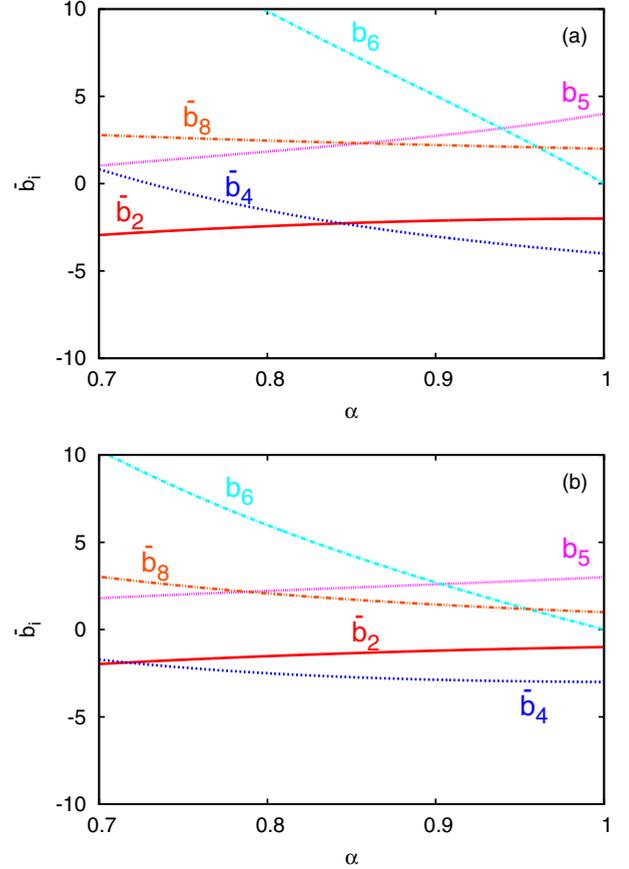

FIG. 5. (Color online) Estimates of the IHSM Burnett transport coefficients $\bar{b}_2, \bar{b}_4, b_5, b_6,$ and $\bar{b}_8$ as functions of the coefficient of normal restitution for (a) $d = 2$ and (b) $d = 3$.

behaviors, except that the influence of inelasticity is milder in the IHSM than in the IMM. This is essentially related to the different types of high-velocity tails of the HCS distribution. While the tail is algebraic in the case of the IMM [69,70], it has a stretched exponential form in the case of the IHSM [63]. As a matter of fact, we have checked that the only diverging coefficients (at $\alpha = 0.046$ and $\alpha = 0.015$ for $d = 2$ and $d = 3$, respectively) are a_3–a_5. This divergence takes place when $\nu_{0|2}^* = 2\zeta^*$, but it cannot be discarded that the divergence would disappear if more accurate expressions for $\nu_{0|2}^*$ and ζ^* were used in the region of extreme inelasticity.

In the case of the Burnett coefficients b_1–b_8 associated with the heat flux, Eqs. (4.20)–(4.33), b_5 and b_6 depend on α only through c, ζ^*, $\nu_{0|2}^*$, and $\nu_{2|1}^*$, but the remaining ones include an extra dependence through the quantity ψ, which is unknown in the IHSM. On the other hand, the combinations,

$$\bar{b}_2 \equiv b_2 - \frac{d-2}{d} b_1 = -\frac{\frac{2}{d}\kappa^* + \mu^*}{\nu_{2|1}^* - 2(1-\gamma)\zeta^*}, \qquad (5.6)$$

$$\bar{b}_4 \equiv b_4 - \frac{d(d+2)}{2(d-1)} \frac{\zeta^*}{\nu_{2|1}^* - (3-2\gamma)\zeta^*} b_3$$

$$= \frac{A_2}{\nu_{2|1}^* - (3-2\gamma)\zeta^*}, \qquad (5.7)$$

$$\bar{b}_8 \equiv b_8 - \frac{d(d+2)}{2(d-1)} \frac{\zeta^*}{\nu_{2|1}^* - (3-2\gamma)\zeta^*} b_7$$

$$= \frac{C_2}{\nu_{2|1}^* - (3-2\gamma)\zeta^*}, \qquad (5.8)$$

do not include ψ and thus can be estimated for the IHSM (with $\gamma = \frac{1}{2}$). The results for $\bar{b}_2$, $\bar{b}_4$, b_5, b_6, and $\bar{b}_8$ are plotted in Fig. 5. A comparison with a similar plot for the IMM (not shown) again exhibits qualitative similarities with a weaker dependence on inelasticity in the case of the IHSM.

VI. CONCLUSIONS

The main objective of this paper was to derive the constitutive equations for the pressure tensor and the heat flux of a granular gas by means of the CE method up to second (Burnett) order in the hydrodynamic gradients, with explicit expressions for the corresponding transport coefficients. Given the formidable difficulties of the task, and in order to obtain results free from uncontrolled approximations, we used the IMM, which allowed us to achieve exact results. The final expressions apply to any value of the coefficient of normal restitution α, any dimensionality d, and any value of the "interaction" parameter γ. As a bonus, in the elastic limit

($\alpha = 1$), our results provide the explicit forms of the Burnett transport coefficients of a classical gas for any number of dimensions (see Table II), which, to the best of our knowledge, had not been derived before.

It is interesting to remark that the structure of the inelastic Burnett constitutive equations is more general than that of the elastic counterpart. While the terms involving second-order gradients are the same in both situations, some Burnett coefficients that are degenerate in the elastic case [$\varpi_2 = \varpi_2' = \varpi_2''$ in Eq. (5.2) and $\theta_2 = \theta_2'$ in Eq. (5.5)] become different when $\alpha \neq 1$. In general, the dependence of the Burnett coefficients on inelasticity is far from being trivial (see Figs. 1 and 2): While some coefficients tend to increase or decrease with increasing inelasticity, other ones are hardly sensitive to α.

We also exploited the formal structure of the results for the IMM to obtain reasonable estimates of the Burnett coefficients for the IHSM. We plan to derive expressions for those coefficients by starting from the genuine Boltzmann equation for the IHSM and using similar Sonine approximations, as made before in the case of the NS coefficients. It will be instructive to use those expressions to assess the degree of reliability of the ones estimated here. Previous studies [37] considered a double expansion in the spatial gradients and in the degree of inelasticity up to second order, so that the Burnett transport coefficients coincided with their elastic forms.

Finally, it is worthwhile noting the potential usefulness of the Burnett-order hydrodynamic equations, as compared to the NS equations, to describe physical problems where gradients are not small. In fact, this is the typical situation in granular fluids due to the coupling between inelasticity and gradients [8]. On the other hand, some care must be taken since the Burnett equations, as noted in Sec. I, need some kind of regularization to avoid artificial instabilities [38,49–54].

APPENDIX A: IHSM EXPRESSIONS FOR c, ζ^*, $v_{2|0}^*$, AND $v_{2|1}^*$

In the case of the IHSM, accurate estimates are [64,65]

$$c = \frac{16(1-\alpha)(1-2\alpha^2)}{25 + 24d - \alpha(57-8d) - 2(1-\alpha)\alpha^2}, \quad \text{(A1)}$$

$$\zeta^* = \frac{d+2}{4d}(1-\alpha^2)\left(1 + \frac{3c}{16}\right), \quad \text{(A2)}$$

$$v_{0|2}^* = \frac{(1+\alpha)\left[d + \frac{3}{2}(1-\alpha)\right]}{2d}\left(1 + \frac{7c}{16}\right), \quad \text{(A3)}$$

$$v_{2|1}^* = \frac{1+\alpha}{8d}\left[\frac{11}{2}d + 8 - \frac{3}{2}\alpha(d+8)\right.$$
$$\left. + \frac{296 + 217d - 3(160 + 11d)\alpha}{32}c\right]. \quad \text{(A4)}$$

These expressions are employed in Sec. V D to estimate the Burnett coefficients in the IHSM.

APPENDIX B: EVALUATION OF $P_{ij}^{(2)}$

Multiplying both sides of Eq. (4.1) by $m V_i V_j$ and integrating over $\mathbf{v}$ one gets

$$\left(\partial_t^{(0)} + v_{0|2}\right) P_{ij}^{(2)} = -m \int d\mathbf{v}\, V_i V_j \left(\partial_t^{(1)} + \mathbf{v} \cdot \boldsymbol{\nabla}\right) f^{(1)}$$
$$+ \delta_{ij}\frac{2}{d}\left(\boldsymbol{\nabla} \cdot \mathbf{q}^{(1)} + P_{k\ell}^{(1)}\nabla_\ell u_k\right), \quad \text{(B1)}$$

where use has been made of the relation (4.6) and the collisional moment [42],

$$m \int d\mathbf{v}\, V_i V_j J^{(2)}[f,f] = -v_{0|2} P_{ij}^{(2)}, \quad \text{(B2)}$$

where $v_{r|s} = v_{r|s}^* v_0$ and $v_{0|2}^*$ is given by Eq. (3.25). The first term on the right-hand side of Eq. (B1) can be easily evaluated with the result,

$$m \int d\mathbf{v}\, V_i V_j \left(\partial_t^{(1)} + \mathbf{v} \cdot \boldsymbol{\nabla}\right) f^{(1)}$$
$$= D_t^{(1)} P_{ij}^{(1)} + \nabla_k Q_{ijk}^{(1)} + P_{ij}^{(1)}\boldsymbol{\nabla} \cdot \mathbf{u} + P_{kj}^{(1)}\nabla_k u_i + P_{ki}^{(1)}\nabla_k u_j, \quad \text{(B3)}$$

where $D_t^{(1)} = \partial_t^{(1)} + \mathbf{u} \cdot \boldsymbol{\nabla}$ is the material derivative and the tensor $Q_{ijk}^{(1)}$ is defined as

$$Q_{ijk}^{(1)} = m \int d\mathbf{v}\, V_i V_j V_k f^{(1)}. \quad \text{(B4)}$$

We now evaluate separately $\nabla_k Q_{ijk}^{(1)}$ and $D_t^{(1)} P_{ij}^{(1)}$.

The NS quantity $Q_{ijk}^{(1)}$ can be evaluated in a way similar to the evaluation of $\mathbf{q}^{(1)}$. First, we multiply Eq. (3.13) by $V_i V_j V_k$ and integrate over velocity. The result is

$$\partial_t^{(0)} Q_{ijk}^{(1)} + \nabla_\ell M_{ijk\ell}^{(0)} - \frac{p}{\rho}(\delta_{ij}\nabla_k p + \delta_{jk}\nabla_i p + \delta_{ik}\nabla_j p)$$
$$= m \int d\mathbf{v}\, V_i V_j V_k J^{(1)}[f,f], \quad \text{(B5)}$$

where

$$M_{ijk\ell}^{(0)} = m \int d\mathbf{v}\, V_i V_j V_k V_\ell f^{(0)}$$
$$= \frac{pT}{m}(1+c)(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}). \quad \text{(B6)}$$

The right-hand side of Eq. (B5) can be explicitly evaluated as [42]

$$m \int d\mathbf{v}\, V_i V_j V_k J^{(1)}[f,f]$$
$$= -\frac{3}{2}v_{0|2} Q_{ijk}^{(1)} + \frac{2}{d+2}\left(\frac{3}{2}v_{0|2} - v_{2|1}\right)$$
$$\times \left(\delta_{ij}q_k^{(1)} + \delta_{jk}q_i^{(1)} + \delta_{ik}q_j^{(1)}\right), \quad \text{(B7)}$$

where $\nu_{2|1}^*$ is given by Eq. (3.26). Substitution of Eq. (B7) into (B5) yields

$$
\left(\partial_t^{(0)} + \frac{3}{2}\nu_{0|2}\right)Q_{ijk}^{(1)}
$$

$$
= -\nabla_\ell M_{ijk\ell}^{(0)} + \frac{p}{\rho}(\delta_{ij}\nabla_k p + \delta_{jk}\nabla_i p + \delta_{ik}\nabla_j p)
$$

$$
+ \frac{2}{d+2}\left(\frac{3}{2}\nu_{0|2} - \nu_{2|1}\right)(\delta_{ij}q_k^{(1)} + \delta_{jk}q_i^{(1)} + \delta_{ik}q_j^{(1)}). \tag{B8}
$$

The solution to Eq. (B8) has the form

$$
Q_{ijk}^{(1)} = -a_Q(\delta_{ij}\nabla_k \ln p + \delta_{jk}\nabla_i \ln p + \delta_{ik}\nabla_j \ln p)
$$

$$
- b_Q(\delta_{ij}\nabla_k \ln T + \delta_{jk}\nabla_i \ln T + \delta_{ik}\nabla_j \ln T), \tag{B9}
$$

where the coefficients a_Q and b_Q are determined by consistency. They can be easily obtained by taking into account the identity $q_k^{(1)} = \frac{1}{2}Q_{iik}^{(1)}$ with the result

$$
a_Q = \frac{2}{d+2}n\mu, \quad b_Q = \frac{2}{d+2}(T\kappa - n\mu). \tag{B10}
$$

Thus, the gradient of $Q_{ijk}^{(1)}$ is

$$
\nabla_k Q_{ijk}^{(1)} = -\frac{2}{d+2}\frac{n\mu}{p}(\delta_{ij}\nabla^2 p + 2\nabla_i\nabla_j p) - \frac{2}{d+2}\left(\kappa - \frac{n\mu}{T}\right)(\delta_{ij}\nabla^2 T + 2\nabla_i\nabla_j T)
$$

$$
- \frac{2}{d+2}\frac{n\mu}{pT}\left\{(2-\gamma)\delta_{ij}(\boldsymbol{\nabla}\,p)\cdot(\boldsymbol{\nabla}\,T) + (2-\gamma)[(\nabla_i p)(\nabla_j T) + (\nabla_j p)(\nabla_i T)] - \frac{T}{p}\delta_{ij}(\boldsymbol{\nabla}\,p)^2 - 2\frac{T}{p}(\nabla_i p)(\nabla_j p)\right\}
$$

$$
- \frac{2}{d+2}T^{-1}\left(\kappa - \frac{n\mu}{T}\right)(1-\gamma)[\delta_{ij}(\boldsymbol{\nabla}\,T)^2 + 2(\nabla_i T)(\nabla_j T)]. \tag{B11}
$$

Now we turn to the evaluation of $D_t^{(1)}P_{ij}^{(1)}$. Using Eq. (3.19) for the pressure tensor at NS order and the balance equations (3.15)–(3.17), one finds

$$
D_t^{(1)}P_{ij}^{(1)} = \frac{2}{d}(1-\gamma)\eta(\boldsymbol{\nabla}\cdot\mathbf{u})\left(\nabla_i u_j + \nabla_j u_i - \frac{2}{d}\delta_{ij}\boldsymbol{\nabla}\cdot\mathbf{u}\right) + \eta\left\{\nabla_i\left(\frac{1}{\rho}\nabla_j p\right) + \nabla_j\left(\frac{1}{\rho}\nabla_i p\right) + (\nabla_i u_k)(\nabla_k u_j)\right.
$$

$$
\left. + (\nabla_j u_k)(\nabla_k u_i) - \frac{2}{d}\delta_{ij}\left[\boldsymbol{\nabla}\cdot\left(\frac{1}{\rho}\boldsymbol{\nabla}\,p\right) + (\nabla_\ell u_k)(\nabla_k u_\ell)\right]\right\}. \tag{B12}
$$

Substitution of Eqs. (B11) and (B12) into Eq. (B1) yields

$$
\left(\partial_t^{(0)} + \nu_{0|2}\right)P_{ij}^{(2)} = c_{P,1}\left(\nabla_i\nabla_j T - \frac{1}{d}\delta_{ij}\nabla^2 T\right) + c_{P,2}\left(\nabla_i\nabla_j p - \frac{1}{d}\delta_{ij}\nabla^2 p\right) + c_{P,3}\left[(\nabla_i T)(\nabla_j T) - \frac{1}{d}\delta_{ij}(\boldsymbol{\nabla}\,T)^2\right]
$$

$$
+ c_{P,4}\left[(\nabla_i p)(\nabla_j p) - \frac{1}{d}\delta_{ij}(\boldsymbol{\nabla}\,p)^2\right] + c_{P,5}\left[(\nabla_i T)(\nabla_j p) + (\nabla_i p)(\nabla_j T) - \frac{2}{d}\delta_{ij}(\boldsymbol{\nabla}\,p)\cdot(\boldsymbol{\nabla}\,T)\right]
$$

$$
+ c_{P,6}D\left(D_{ij} - \frac{1}{d}\delta_{ij}D\right) + c_{P,7}\left[D_{ik}D_{kj} - \omega_{ik}\omega_{kj} - \frac{1}{d}\delta_{ij}(D_{\ell k}D_{k\ell} - \omega_{\ell k}\omega_{k\ell}) - D_{ik}\omega_{kj} - D_{jk}\omega_{ki}\right], \tag{B13}
$$

where the coefficients $c_{P,i}$ are

$$
c_{P,1} = \frac{4}{d+2}\left(\kappa - \frac{n\mu}{T}\right), \tag{B14}
$$

$$
c_{P,2} = -pc_{P,4} = \frac{4}{d+2}\frac{n\mu}{p} - \frac{2\eta}{\rho}, \tag{B15}
$$

$$
c_{P,3} = -\frac{4}{d+2}(\gamma-1)T^{-1}\left(\kappa - \frac{n\mu}{T}\right), \tag{B16}
$$

$$
c_{P,5} = \frac{2}{d+2}(2-\gamma)\frac{n\mu}{Tp} - \eta\rho^{-1}T^{-1}, \tag{B17}
$$

$$
c_{P,6} = -\frac{2}{d}\eta(4 - d - 2\gamma), \tag{B18}
$$

$$
c_{P,7} = 2\eta. \tag{B19}
$$

The structure of Eq. (B13) shows that the constitutive equation for $P_{ij}^{(2)}$ has the form (4.7), where the dimensionless coefficients a_i can be determined by inserting Eq. (4.7) into Eq. (B13) and equating coefficients of the same type of gradients. After tedious algebra one finally gets Eqs. (4.11)–(4.16).

It is interesting to remark that, while $c_{P,2} = -pc_{P,4}$, one has $a_2 \neq -a_4$ (except in the elastic limit). This is due to the different action of the operator $\partial_t^{(0)}$ on $p^{-1}(\nabla_i p)(\nabla_j p)$ and $\nabla_i\nabla_j p$.

APPENDIX C: EVALUATION OF $\mathbf{q}^{(2)}$

The evaluation of $\mathbf{q}^{(2)}$ proceeds along similar lines as in the case of $P_{ij}^{(2)}$. First, by multiplying both sides of Eq. (4.1) by

$\frac{m}{2}V^2 V_i$ and integrating over velocity, one obtains

$$\left(\partial_t^{(0)} + \nu_{2|1}\right) q_i^{(2)} = -\frac{m}{2} \int d\mathbf{v}\, V^2 V_i \left(\partial_t^{(1)} + \mathbf{v}\cdot\boldsymbol{\nabla}\right) f^{(1)}$$
$$+ \frac{d+2}{2}\frac{p}{\rho}\nabla_j P_{ij}^{(1)}, \tag{C1}$$

where use has been made of the relation (4.5) and [42]

$$\frac{m}{2}\int d\mathbf{v}\, V^2 V_i J^{(2)}[f,f] = -\nu_{2|1} q_i^{(2)}. \tag{C2}$$

The first term on the right-hand side of Eq. (C1) becomes

$$\frac{m}{2}\int d\mathbf{v} V^2 V_i \left(\partial_t^{(1)} + \mathbf{v}\cdot\boldsymbol{\nabla}\right) f^{(1)}$$
$$= D_t^{(1)} q_i^{(1)} + \nabla_j R_{ij}^{(1)} + Q_{ijk}^{(1)}\nabla_k u_j$$
$$+ q_i^{(1)}\boldsymbol{\nabla}\cdot\mathbf{u} + q_j^{(1)}\nabla_j u_i - \rho^{-1} P_{ij}^{(1)}\nabla_j p, \tag{C3}$$

where

$$R_{ij}^{(1)} = \frac{m}{2}\int d\mathbf{v}\, V^2 V_i V_j f^{(1)}. \tag{C4}$$

In order to evaluate $R_{ij}^{(1)}$, let us multiply both sides of Eq. (3.13) by $\frac{m}{2}V^2 V_i V_j$ and integrate over velocity to obtain

$$\partial_t^{(0)} R_{ij}^{(1)} = -\frac{pT}{m}(d+4)(1+c)\left(D_{ij} - \frac{D}{d}\delta_{ij}\right)$$
$$+ \frac{m}{2}\int d\mathbf{v}\, V^2 V_i V_j J^{(1)}[f,f]. \tag{C5}$$

The collision integral is [42]

$$\frac{m}{2}\int d\mathbf{v}\, V^2 V_i V_j J^{(1)}[f,f]$$
$$= -\nu_{2|2} R_{ij}^{(1)} + \frac{dp}{2\rho}\lambda P_{ij}^{(1)} + \frac{m}{2}\delta_{ij}(\nu_{2|2} - \nu_{4|0})M_{4|0}^{(1)}, \tag{C6}$$

where $\nu_{2|2}^* = \nu_{2|2}/\nu_0$ and $\lambda^* = \lambda/\nu_0$ are given by Eqs. (4.23) and (4.24), respectively, $\nu_{4|0}$ can be found in Ref. [42] but will not be needed here, and

$$M_{4|0}^{(1)} = \int d\mathbf{v}\, V^4 f^{(1)}(\mathbf{v}). \tag{C7}$$

However, as said in Sec. III B, $M_{4|0}^{(1)} = 0$ [see Eq. (3.18)]. Consequently, Eq. (C5) becomes

$$\left(\partial_t^{(0)} + \nu_{2|2}\right) R_{ij}^{(1)} = -\frac{pT}{m}\left[(d+4)(1+c) + \frac{d}{p}\lambda\eta\right]$$
$$\times \left(D_{ij} - \frac{D}{d}\delta_{ij}\right). \tag{C8}$$

Its solution is

$$R_{ij}^{(1)} = -\frac{T\eta_0}{m}\psi\left(D_{ij} - \frac{D}{d}\delta_{ij}\right), \tag{C9}$$

ψ being given by Eq. (4.22). The divergence of the tensor $R_{ij}^{(1)}$ is

$$\nabla_j R_{ij}^{(1)} = -\frac{\eta_0}{m}(2-\gamma)\psi\left(D_{ij} - \frac{1}{d}\delta_{ij}D\right)\nabla_j T$$
$$- \frac{T\eta_0}{2m}\psi\left(\nabla^2 u_i + \frac{d-2}{d}\nabla_i D\right). \tag{C10}$$

From Eq. (3.20) and the balance equations (3.15)–(3.17), one gets

$$D_t^{(1)} q_i^{(1)} = \frac{2}{d}(2-\gamma)\left[\frac{n\mu}{p}D\nabla_i p + \left(\kappa - \frac{n\mu}{T}\right)D\nabla_i T\right]$$
$$+ \left(\frac{2}{d}\kappa + \frac{n\mu}{T}\right)T\nabla_i D + \frac{n\mu}{p}(\nabla_i u_j)(\nabla_j p)$$
$$+ \left(\kappa - \frac{n\mu}{T}\right)(\nabla_i u_j)(\nabla_j T). \tag{C11}$$

Using Eqs. (C10) and (C11), Eq. (C1) reduces to

$$\left(\partial_t^{(0)} + \nu_{2|1}\right) q_i^{(2)} = c_{q,1}\nabla^2 u_i + c_{q,2}\nabla_i D + c_{q,3}D_{ij}\nabla_j T$$
$$+ c_{q,4}D_{ij}\nabla_j p + c_{q,5}\omega_{ij}\nabla_j T$$
$$+ c_{q,6}\omega_{ij}\nabla_j p + c_{q,7}D\nabla_i T + c_{q,8}D\nabla_i p, \tag{C12}$$

where the coefficients $c_{q,i}$ are

$$c_{q,1} = \frac{T\eta_0}{2m}[\psi - (d+2)\eta^*], \tag{C13}$$

$$c_{q,2} = \frac{d-2}{d}\frac{T\eta_0}{2m}[\psi - (d+2)\eta^*] - \frac{2}{d}T\kappa - n\mu, \tag{C14}$$

$$c_{q,3} = (d+2)\frac{\eta_0}{m}\left[\frac{2-\gamma}{d+2}\psi - (1-\gamma)\eta^*\right]$$
$$+ \frac{4}{d+2}\left(\kappa - \frac{n\mu}{T}\right), \tag{C15}$$

$$c_{q,4} = \frac{4}{d+2}\frac{n\mu}{p} - \frac{2\eta}{\rho}, \tag{C16}$$

$$c_{q,5} = 2\left(\kappa - \frac{n\mu}{T}\right), \quad c_{q,6} = 2\frac{n\mu}{p}, \tag{C17}$$

$$c_{q,7} = -(d+2)\frac{\eta_0}{dm}\left[\frac{2-\gamma}{d+2}\psi - (1-\gamma)\eta^*\right]$$
$$+ \frac{d^2 - 8 + 2\gamma(d+2)}{d(d+2)}\left(\kappa - \frac{n\mu}{T}\right), \tag{C18}$$

$$c_{q,8} = \frac{2\eta}{d\rho} + \frac{n\mu}{p}\frac{d^2 - 8 + 2\gamma(d+2)}{d(d+2)}. \tag{C19}$$

Equation (C12) shows that the constitutive equation for $q_i^{(2)}$ has the structure (4.19). The dimensionless coefficients b_i are obtained by inserting Eq. (4.19) into Eq. (C12). The final results are displayed by Eqs. (4.20), (4.21), and (4.25)–(4.27).

APPENDIX D: NON-NEWTONIAN UNIFORM SHEAR AND UNIFORM LONGITUDINAL FLOWS

1. Unsteady uniform shear flow

The uniform shear flow (USF) is an incompressible flow characterized by uniform density and temperature ($\nabla n = \nabla T = 0$) and a uniform shear rate, i.e., $\nabla_i u_j = a^* v_0 \delta_{iy} \delta_{jx}$ with $\nabla a^* = 0$.

Proceeding in a way analogous to the case of the Boltzmann equation for the IHSM in the Grad approximation [31] or with a simple kinetic model [71–74], it is possible to eliminate time in favor of the reduced shear rate a^* to obtain a coupled set of two equations for the rheological dependence of P_{xy}^* and P_{yy}^* on a^*, where $P_{ij}^* = P_{ij}/p$. In the case of our generalized IMM model, those two equations are exact and read [46]

$$\gamma \left(\frac{2a^*}{d} P_{xy}^* + \zeta^* \right) \frac{\partial P_{xy}^*}{\partial a^*} = -P_{yy}^* + \frac{2}{d} P_{xy}^{*2} - \frac{v_{0|2}^* - \zeta^*}{a^*} P_{xy}^*,$$
(D1)

$$\gamma \left(\frac{2a^*}{d} P_{xy}^* + \zeta^* \right) \frac{\partial P_{yy}^*}{\partial a^*} = \frac{2}{d} P_{yy}^* P_{xy}^* - \frac{v_{0|2}^* - \zeta^*}{a^*} \left(P_{yy}^* - 1 \right).$$
(D2)

The numerical solution of this set of equations provides $P_{xy}^*(a^*)$ and $P_{yy}^*(a^*)$ for any a^* [47]. Here, however, we are interested in the analytical results to second order in a^*. In that case,

$$P_{xy}^*(a^*) = -\eta^* a^* + \mathcal{O}(a^{*3}),$$
(D3)

$$P_{yy}^*(a^*) = 1 + \frac{\Psi}{d} a^{*2} + \mathcal{O}(a^{*4}),$$
(D4)

where η^* is the (reduced) NS shear viscosity and Ψ is a viscometric function. Inserting Eqs. (D3) and (D4) into Eqs. (D1) and (D2), and equating terms of the same order in a^* one recovers Eq. (3.22) and obtains $\Psi = a_7$, where a_7 is given by Eq. (4.12). It is straightforward to check that Eq. (4.7), when particularized to the USF conditions, indeed yields Eq. (D4) with $\Psi = a_7$.

2. Unsteady uniform longitudinal flow

The uniform longitudinal flow (ULF) shares with the USF the conditions $\nabla n = \nabla T = 0$, but it is a compressible flow because now $\nabla_i u_j = a^* v_0 \delta_{ix} \delta_{jx}$, $\nabla a^* = 0$ [72–75]. Eliminating again time in favor of the reduced strain (or longitudinal rate) a^* it is possible to find a closed equation for $P_{xx}^*(a^*)$:

$$\gamma \left(\frac{2a^*}{d} P_{xx}^* + \zeta^* \right) \frac{\partial P_{xx}^*}{\partial a^*}$$
$$= -2 P_{xx}^* \left(1 - \frac{P_{xx}^*}{d} \right) - \frac{v_{0|2}^* - \zeta^*}{a^*} (P_{xx}^* - 1).$$
(D5)

To second order in a^* one can write

$$P_{xx}^* = 1 - 2 \frac{d-1}{d} \eta^* a^* + \frac{d-1}{d} \Phi a^{*2} + \mathcal{O}(a^{*3}),$$
(D6)

where η^* is again the (reduced) NS shear viscosity but Φ is a viscometric function different from Ψ. Substitution of Eq. (D6) into Eq. (D5) allows one to recover Eq. (3.22) and obtain $\Phi = a_6 + a_7$, where a_6 and a_7 are given by Eqs. (4.11) and (4.12), respectively. As before, it can be checked that Eq. (D6) is indeed equivalent to Eq. (4.7) particularized to the ULF conditions.

APPENDIX E: DIVERGENCE OF THE BURNETT COEFFICIENTS

In this appendix we analyze the regions in the (α, γ) plane where the Burnett coefficients for the IMM diverge.

It can be checked that $v_{0|2}^* - \zeta^* > 0$ for all α, so that η^*, a_6, and a_7 are finite [cf. Eqs. (3.22), (4.11), and (4.12)]. Next, from Eqs. (4.13) and (4.14) one finds that the divergence threshold $\alpha_{\text{th}}^{(a_1, a_2)}$ for a_1 and a_2 takes place either at $\alpha = (4 - d)/3d$ or when $\Delta = 0$, i.e., $v_{0|2}^* - (2 - \gamma)\zeta^* = 0$, whatever comes first. Therefore,

$$\alpha_{\text{th}}^{(a_1, a_2)} = \begin{cases} \frac{4-d}{3d}, & \frac{d-2}{2(d-1)} \leqslant \gamma \leqslant \frac{1}{2}, \\ \frac{1-(d+2)\gamma/2}{d+1-(d+2)\gamma/2}, & 0 \leqslant \gamma \leqslant \frac{d-2}{2(d-1)}. \end{cases}$$
(E1)

Regarding the coupled coefficients a_3, a_4, and a_5, the determinant of the matrix L is $(v_{0|2}^* - 2\zeta^*)\Delta$, so that the divergence of κ^* and μ^* is preempted by $v_{0|2}^* - 2\zeta^* = 0$. Thus,

$$\alpha_{\text{th}}^{(a_3, a_4, a_5)} = \frac{1}{d+1}.$$
(E2)

Now we turn to the heat flux Burnett coefficients. It can be checked that $v_{2|2}^* - 2\zeta^* > 0$ for all α, so that ψ is finite. As a consequence, the threshold value $\alpha_{\text{th}}^{(b_1)}$ for b_1 occurs when $v_{2|1}^* - 2(1 - \gamma)\zeta^* = 0$ [cf. Eq. (4.20)]. This implies

$$\alpha_{\text{th}}^{(b_1)} = \frac{4 - d - 4(d+2)\gamma}{3d - 4(d+2)\gamma}.$$
(E3)

In the case of b_2 [cf. Eq. (4.21)], its divergence is due to that of κ^* and μ^*, i.e.,

$$\alpha_{\text{th}}^{(b_2)} = \frac{4 - d}{3d}.$$
(E4)

Finally, the determinant of the matrix M is $[v_{2|1}^* - (2 - \gamma)\zeta^*][v_{2|1}^* - 2(1 - \gamma)\zeta^*]$, so that the divergence of b_3–b_8 is again due that of κ^* and μ^*:

$$\alpha_{\text{th}}^{(b_3 - b_8)} = \frac{4 - d}{3d}.$$
(E5)

The regions of divergence of the coefficients for $d = 2$ and $d = 3$ are depicted in Fig. 3.

[1] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases*, 3rd ed. (Cambridge University Press, Cambridge, 1970).

[2] P. Résibois and M. de Leener, *Classical Kinetic Theory of Fluids* (John Wiley and Sons, New York, 1977).

[3] A. V. Bobylev, J. A. Carrillo, and I. M. Gamba, J. Stat. Phys. **98**, 743 (2000).

[4] E. Ben-Naim and P. L. Krapivsky, Phys. Rev. E **61**, R5 (2000).

[5] M. H. Ernst and R. Brito, J. Stat. Phys. **109**, 407 (2002).

[6] E. Ben-Naim and P. L. Krapivsky, in *Granular Gas Dynamics*, edited by T. Pöschel and S. Luding (Springer, Berlin, 2003), Vol. 624 of *Lecture Notes in Physics*, pp. 65–94.

[7] V. Garzó and A. Santos, Math. Model. Nat. Phenom. **6**, 37 (2011).

[8] I. Goldhirsch, Annu. Rev. Fluid Mech. **35**, 267 (2003).

[9] N. V. Brilliantov and T. Pöschel, *Kinetic Theory of Granular Gases* (Oxford University Press, Oxford, 2004).

[10] J. T. Jenkins and M. W. Richman, Phys. Fluids **28**, 3485 (1985).

[11] P. Zamankhan, H. V. Tafreshi, W. Polashenski, P. Sarkomaa, and C. L. Hyndman, J. Chem. Phys. **109**, 4487 (1998).

[12] I. Goldhirsch, S. H. Noskowicz, and O. Bar-Lev, Phys. Rev. Lett. **95**, 068002 (2005).

[13] A. Zippelius, Physica A **369**, 143 (2006).

[14] N. V. Brilliantov, T. Pöschel, W. T. Kranz, and A. Zippelius, Phys. Rev. Lett. **98**, 128001 (2007).

[15] A. Santos, G. M. Kremer, and V. Garzó, Prog. Theor. Phys. Suppl. **184**, 31 (2010).

[16] A. Santos, G. M. Kremer, and M. dos Santos, Phys. Fluids **23**, 030604 (2011).

[17] N. V. Brilliantov and T. Pöschel, in *Granular Gases*, edited by T. Pöschel and S. Luding (Springer, Berlin, 2001), Vol. 564 of *Lecture Notes in Physics*, pp. 100–124.

[18] N. Brilliantov and T. Pöschel, Phys. Rev. E **67**, 061304 (2003).

[19] V. Garzó, in *Theory and Simulation of Hard-Sphere Fluids and Related Systems*, edited by A. Mulero (Springer-Verlag, Berlin, 2008), Vol. 753 of *Lectures Notes in Physics*, pp. 493–540.

[20] D. L. Koch and R. J. Hill, Ann. Rev. Fluid Mech. **33**, 619 (2001).

[21] V. Garzó, S. Tenneti, S. Subramaniam, and C. M. Hrenya, J. Fluid Mech. **712**, 129 (2012).

[22] V. Garzó, M. G. Chamorro, and F. Vega Reyes, Phys. Rev. E **87**, 032201 (2013); **87**, 059906(E) (2013).

[23] N. Khalil and V. Garzó, Phys. Rev. E **88**, 052201 (2013).

[24] J. J. Brey, J. W. Dufty, C. S. Kim, and A. Santos, Phys. Rev. E **58**, 4638 (1998).

[25] V. Garzó and J. W. Dufty, Phys. Fluids **14**, 1476 (2002).

[26] V. Garzó and J. W. Dufty, Phys. Rev. E **59**, 5895 (1999).

[27] V. Garzó, J. W. Dufty, and C. M. Hrenya, Phys. Rev. E **76**, 031303 (2007).

[28] V. Garzó, C. M. Hrenya, and J. W. Dufty, Phys. Rev. E **76**, 031304 (2007).

[29] T. P. C. van Noije, M. H. Ernst, and R. Brito, Physica A **251**, 266 (1998).

[30] J. J. Brey, M. J. Ruiz-Montero, and F. Moreno, Phys. Rev. E **55**, 2846 (1997).

[31] A. Santos, V. Garzó, and J. W. Dufty, Phys. Rev. E **69**, 061303 (2004).

[32] J. F. Lutsko, Phys. Rev. E **73**, 021302 (2006).

[33] V. Garzó, Phys. Rev. E **73**, 021304 (2006).

[34] F. Vega Reyes, A. Santos, and V. Garzó, J. Stat. Mech. (2011) P07005.

[35] J. J. Brey, D. Cubero, F. Moreno, and M. J. Ruiz-Montero, Europhys. Lett. **53**, 432 (2001).

[36] J. J. Brey, N. Khalil, and M. J. Ruiz-Montero, J. Stat. Mech. (2009) P08019.

[37] N. Sela and I. Goldhirsch, J. Fluid Mech. **361**, 41 (1998).

[38] I. Goldhirsch, Powder Technol. **182**, 130 (2008).

[39] C. Truesdell and R. G. Muncaster, *Fundamentals of Maxwell's Kinetic Theory of a Simple Monatomic Gas* (Academic Press, New York, 1980).

[40] V. Garzó and A. Santos, *Kinetic Theory of Gases in Shear Flows. Nonlinear Transport* (Kluwer Academic, Dordrecht, 2003).

[41] M. H. Ernst, Phys. Rep. **78**, 1 (1981).

[42] V. Garzó and A. Santos, J. Phys. A: Math. Theor. **40**, 14927 (2007).

[43] A. Santos, Physica A **321**, 442 (2003).

[44] V. Garzó, J. Stat. Phys. **112**, 657 (2003).

[45] V. Garzó and A. Astillero, J. Stat. Phys. **118**, 935 (2005).

[46] V. Garzó, J. Phys. A: Math. Theor. **40**, 10729 (2007).

[47] A. Santos and V. Garzó, J. Stat. Mech. (2007) P08021.

[48] K. Kohlstedt, A. Snezhko, M. V. Sapozhnikov, I. S. Aranson, J. S. Olafsen, and E. Ben-Naim, Phys. Rev. Lett. **95**, 068001 (2005).

[49] A. V. Bobylev, Sov. Phys. Dokl. **27**, 29 (1982).

[50] F. J. Uribe, R. M. Velasco, and L. S. García-Colín, Phys. Rev. E **62**, 5835 (2000).

[51] S. Jin and M. Slemrod, Physica D **150**, 207 (2001).

[52] R. Balakrishnan, J. Fluid Mech. **503**, 201 (2004).

[53] A. V. Bobylev, J. Stat. P **124**, 371 (2006).

[54] M. Colangeli, I. V. Karlin, and M. Kröger, Phys. Rev. E **75**, 051204 (2007).

[55] I. Goldhirsch and G. Zanetti, Phys. Rev. Lett. **70**, 1619 (1993).

[56] S. McNamara, Phys. Fluids A **5**, 3056 (1993).

[57] M. H. Ernst, E. Trizac, and A. Barrat, Europhys. Lett. **76**, 56 (2006).

[58] E. Trizac, A. Barrat, and M. H. Ernst, Phys. Rev. E **76**, 031305 (2007).

[59] R. Yano, J. Phys. A: Math. Theor. **46**, 375502 (2013).

[60] J. H. Ferziger and G. H. Kaper, *Mathematical Theory of Transport Processes in Gases* (North-Holland, Amsterdam, 1972).

[61] A. Goldshtein and M. Shapiro, J. Fluid Mech. **282**, 75 (1995).

[62] J. J. Brey and D. Cubero, in *Granular Gases*, edited by T. Pöschel and S. Luding (Springer, Berlin, 2001), Vol. 564 of *Lectures Notes in Physics*, pp. 59–78.

[63] T. P. C. van Noije and M. H. Ernst, Granular Matter **1**, 57 (1998).

[64] J. M. Montanero and A. Santos, Granular Matter **2**, 53 (2000).

[65] V. Garzó, A. Santos, and J. M. Montanero, Physica A **376**, 94 (2007).

[66] J. A. McLennan, *Introduction to Non-Equilibrium Statistical Mechanics* (Prentice Hall, Englewood Cliffs, 1989).

[67] M. Slemrod, Arch. Rat. Mech. Anal. **161**, 339 (2002).

[68] J. J. Brey, M. I. García de Soria, and P. Maynar, Phys. Rev. E **82**, 021303 (2010).

[69] E. Ben-Naim and P. L. Krapivsky, Phys. Rev. E **66**, 011309 (2002).

[70] M. H. Ernst and R. Brito, Europhys. Lett. **58**, 182 (2002).

[71] A. Astillero and A. Santos, Europhys. Lett. **78**, 24002 (2007).

[72] A. Santos, Phys. Rev. Lett. **100**, 078003 (2008).

[73] A. Santos, in *The XVth International Congress on Rheology*, edited by A. Co, G. Leal, R. Colby, and A. J. Giacomin (AIP Conference Proceedings, Melville, 2008), Vol. 1027, pp. 914–916.

[74] A. Astillero and A. Santos, Phys. Rev. E **85**, 021302 (2012).

[75] A. Santos, in *Rarefied Gas Dynamics: Proceedings of the 26th International Symposium on Rarefied Gas Dynamics*, edited by T. Abe (AIP Conference Proceedings, Melville, 2009), Vol. 1084, pp. 93–98.