Interactions between solitons and other nonlinear Schrödinger waves

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The nonlinear Schrödinger (NLS) equation is widely used in natural science. Various nonlinear excitations of the NLS equation have been found by many methods. However, except for the soliton-soliton interactions, it is very difficult to find interaction solutions between different types of nonlinear excitations. In this paper, the symmetry reduction method is further developed to find interaction solutions between solitons and other types of NLS waves. Especially, the soliton-cnoidal wave interaction solutions are explicitly studied in terms of the Jacobi elliptic functions and the third type of incomplete elliptic integrals. Some special concrete interaction solutions and their asymptotic behaviors are discussed both in analytical and graphical ways.

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I. INTRODUCTION

The soliton and/or solitary wave equations connect rich histories of exactly solvable systems constructed in mathematical, statistical, and many-body physics and powerfully demonstrate the unity of nonlinear concepts across disciplines and scales from micro-physics and biology to cosmology [1]. Among these equations, the nonlinear Schrödinger (NLS) equation,

$$p_t + \tfrac{1}{2} i b p_{xx} - i|p|^2 p = 0, \quad i \equiv \sqrt{-1}, \quad b \equiv \pm 1, \tag{1}$$

is most ubiquitous [2]. Originally, the NLS equation is derived to describe the envelope dynamics of a quasimonochromatic plane wave propagating in a weakly nonlinear dispersive medium when dissipative processes are negligible (see, for instance, Ref. [3]). The NLS equation finds an important application in plasma physics, where it describes electron (Langmuir) waves [4]. The NLS equation in nonlinear optics is also well known to describe self-modulation and self-focusing of light in a Kerr-type nonlinear medium [5]. The great current interest in the NLS application is initiated by the prediction of solitons in nonlinear optical fibers [6] and the concept of the soliton laser [7]. Furthermore, the NLS equation is widely used in ferromagnets with easy-axis anisotropy, molecular chains, nonideal Bose gas, nuclear matter, solid-state medium, gravity waves, optical lattice, Bose-Einstein condensations, and so on [8].

The multiple soliton solutions of the NLS equation have been obtained by many authors via different methods, such as Hirota's bilinear method [9], the Darboux transformation (DT) [10], and the Bäcklund transformation (BT) [11]. Using the DT and BT, in principle, one can obtain a new solution from a known one. However, in practice, one can only find multiple soliton solutions stemming from simple constant solutions. It is rather difficult to find new explicit solutions starting from nonconstant nonlinear waves such as the cnoidal waves and Painlevé waves via the DT and BT. In Refs. [12,13], the mutisoliton complexes on a cnoidal wave background have been studied by the DTs for the multicomponent NLS equations, the sine-Gordon (SG) equation, and the Toda lattice. Recently, it is found that combining the symmetry reduction method and the DT- or BT-related nonlocal symmeries [14], one can readily find the interaction solutions among solitons and other nonlinear excitations, including the cnoidal waves for the KdV [15] and KP [16] equations.

In Sec. II of this paper, we review the known local and nonlocal symmetries for the Ablowitz-Kaup-Newell-Segur (AKNS) system, which is a general form of the NLS equation. In Sec. III, to find the finite transformation related to a special nonlocal symmetry, the nonlocal symmetry for the original AKNS system is localized for an extended AKNS system. Thus, the finite Darboux-Bäcklund transformation (DBT) theorem is naturally obtained by Lie's first principle. In Sec. IV, thanks to the localization procedure of the last section, the group invariant solutions related to the nonlocal symmetries of the AKNS system are obtained by means of the symmetry reduction method. In Sec. V, some special exact solutions on the dark or gray solitons dressed by the cnoidal periodic waves are explicitly given by means of the Jacobi elliptic functions and the third type of incomplete elliptic integrals. The last section is a short summary and discussion.

II. INFINITELY MANY LOCAL AND NONLOCAL SYMMETRIES OF THE AKNS SYSTEM

It is known that an integrable system possesses infinitely many symmetries. Using these symmetries, one can obtain various interesting results of the model especially to study its exact solutions [17] via symmetry reduction methods and even to find its ALL possible solutions [18]. However, one usually uses the local symmetries to find symmetry reductions while the existence of infinitely many nonlocal symmetries [19] is ignored by most of scientists. Recently, we have demonstrated that the nonlocal symmetries can be successfully used to discover some types of important interaction solutions that are difficult to be found by other approaches [14,15]. Though we will only use one nonlocal symmetry and five local symmetries to obtain some types of exact solutions, we list as many as possible symmetries for completeness and further studies in future.

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For simplicity and generality, we consider the NLS equation as a special case of the AKNS system:

$$p_t + \tfrac{1}{2} i b p_{xx} - i p^2 q = 0, \tag{2a}$$

$$q_t - \tfrac{1}{2} i b q_{xx} + i q^2 p = 0, \tag{2b}$$

with q being a complex conjugate of p; i.e., $q = p^*$.

A symmetry σ,

$$\sigma \equiv \begin{pmatrix} \sigma^p \\ \sigma^q \end{pmatrix}, \tag{3}$$

is defined as a solution of the linearized equation of the AKNS system Eq. (2):

$$\sigma_t^p + \tfrac{1}{2} i b \sigma_{xx}^p - 2ipq\sigma^p - ip^2\sigma^q = 0, \tag{4a}$$

$$\sigma_t^q - \tfrac{1}{2} i b \sigma_{xx}^q + 2ipq\sigma^q + iq^2\sigma^p = 0, \tag{4b}$$

which means the AKNS system Eq. (2) is form invariant under the transformation

$$\begin{pmatrix} p \\ q \end{pmatrix} \to \begin{pmatrix} p \\ q \end{pmatrix} + \epsilon \begin{pmatrix} \sigma^p \\ \sigma^q \end{pmatrix}, \tag{5}$$

with an infinitesimal parameter ϵ.

The infinitely many local K symmetries,

$$K_n = \Phi^n \begin{pmatrix} -ip \\ iq \end{pmatrix}, \quad n = 0, 1, 2, \ldots, \tag{6}$$

and τ symmetries,

$$\tau_n = \Phi^n \tau_0, \quad \tau_0 = \begin{pmatrix} tp_x + ibxp \\ tq_x - ibxq \end{pmatrix}, \quad n = 0, 1, 2, \ \ldots, \tag{7}$$

are known in literature [28]. In Eqs. (6) and (7), the recursion operator Φ is defined as

$$\Phi = \begin{pmatrix} -\partial + 2bp\partial^{-1}q & 2bp\partial^{-1}p \\ -2bq\partial^{-1}q & \partial - 2bq\partial^{-1}p \end{pmatrix}, \tag{8}$$

with $\partial = \partial/\partial_x$, $\partial^{-1} = \int_{-\infty}^{x} dx$.

The K_n symmetries constitute a commute symmetry algebra and the τ_n symmetries constitute the centerless Virasoro symmetry algebra.

Remark 1. For the K_n symmetry, n can only be nonnegative because K_0 is a kernel of the inverse recursion operator; i.e., $\Phi^{-1}K_0 = 0$. For the τ_n symmetry, n can be extended to both positive and negative because τ_0 is not a kernel of the inverse recursion operator. However, τ_n for negative n are nonlocal symmetries. Whence the negative set of the τ_n is considered, the τ_n symmetries constitute a full centerless Virasoro algebra.

Especially, the symmetries

$$K_0 = \begin{pmatrix} -ip \\ iq \end{pmatrix}, \quad K_1 = \begin{pmatrix} p_x \\ q_x \end{pmatrix}, \quad K_2 = -2ib \begin{pmatrix} p_t \\ q_t \end{pmatrix},$$

and

$$\tau_0 = \begin{pmatrix} tp_x + ibxp \\ tq_x - ibxq \end{pmatrix}, \quad \tau_1 = -ib \begin{pmatrix} 2tp_t + (xp)_x \\ 2tq_t + (xq)_x \end{pmatrix}$$

constitute a five-dimensional Lie point symmetry algebra with nonzero commutators

$$[\tau_0, K_1] = ibK_0, \quad [\tau_0, K_2] = 2ibK_1, \quad [\tau_1, K_1] = ibK_1,$$
$$[\tau_1, K_2] = 2ibK_2, \ [\tau_1, \tau_0] = -ib\tau_0, \tag{9}$$

where the definition of the commutator $[A, B]$ is standard [17].

To find infinitely many nonlocal symmetries, one can use some different approaches, for instance, the inverse recursion operator method [19], the infinitesimal forms of the Darboux transformations [10] or Bäcklund transformations [11], the conformal invariance of the Schwarzian forms [20], the residual of the truncated Painlevé analysis [21], the derivatives of the inner parameters [14] and the higher-order Lax operators [22], or the infinitely many nonhomogeneous Lax pairs [23]. In this paper, we write down only the nonlocal symmetries related to the infinitesimal forms of the Darboux transformation.

For the AKNS system Eq. (2), its Lax pair possesses the following form [24]:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_x = \begin{pmatrix} -i\lambda & \frac{p}{\sqrt{b}} \\ \frac{q}{\sqrt{b}} & i\lambda \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{10}$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_t = \begin{pmatrix} ib\lambda^2 + \frac{1}{2}ipq & -\frac{\sqrt{b}}{2}(ip_x + 2p\lambda) \\ \frac{\sqrt{b}}{2}(iq_x - 2q\lambda) & -ib\lambda^2 - \frac{1}{2}ipq \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{11}$$

A simple nonlocal symmetry of the AKNS system related to the Lax pair is the so-called square eigenfunction symmetry [25]

$$N_0 = \begin{pmatrix} \phi_1^2 \\ \phi_2^2 \end{pmatrix}, \tag{12}$$

which is related to the infinitesimal form of the usual Darboux transformation [25]. N_0 is a kernel of the recursion operator Φ. Thus, $\Phi^n N_0$ is zero for positive n.

In addition to the infinitely many nonlocal symmetries,

$$N_n' = \Phi^{-n} N_0, \quad n = 0, 1, 2, \ \ldots, \infty, \tag{13}$$

for a *fixed* spectral parameter λ, one can find infinitely many nonlocal symmetries via some different ways, say, by expanding N_0 as a series near a *given* $\lambda_1 = \lambda + \delta$,

$$N_0 = \sum_{n=0}^{\infty} N_n'' \delta^n, \tag{14}$$

where N_n'' are also symmetries of the AKNS system for an arbitrary n.

More interestingly, the *arbitrariness* of the spectral parameter λ implies that N_0 itself expresses infinitely many symmetries and thus we can sum up any number of square eigenfunction symmetries with *different eigenvalues*

$$N_n = \sum_{i=0}^{n} c_i \begin{pmatrix} \phi_{1i}^2 \\ \phi_{2i}^2 \end{pmatrix}, \tag{15}$$

where

$$\begin{pmatrix} \phi_{1i} \\ \phi_{2i} \end{pmatrix}_x = \begin{pmatrix} -i\lambda_i & \frac{p}{\sqrt{b}} \\ \frac{q}{\sqrt{b}} & i\lambda_i \end{pmatrix} \begin{pmatrix} \phi_{1i} \\ \phi_{2i} \end{pmatrix}, \quad i = 0, 1, 2, \ \ldots, n, \tag{16a}$$

$$\begin{pmatrix} \phi_{1i} \\ \phi_{2i} \end{pmatrix}_t = \begin{pmatrix} ib\lambda_i^2 + \frac{1}{2}ipq & -\frac{\sqrt{b}}{2}(ip_x + 2p\lambda_i) \\ \frac{\sqrt{b}}{2}(iq_x - 2q\lambda_i) & -ib\lambda_i^2 - \frac{1}{2}ipq \end{pmatrix} \begin{pmatrix} \phi_{1i} \\ \phi_{2i} \end{pmatrix}. \tag{16b}$$

It should be mentioned that the last set of infinitely many nonlocal symmetries Eq. (15) can be used to find algebra-geometric solutions by using the so-called nonlinearization approach [26]. They can also be localized to a Lie point symmetry for a suitable extended system and then the n-times Darboux transformations by using the method of the next section. However, as it is not the purpose of the present paper, we will leave this problem in our future study similar to the KdV equation [27].

III. LOCALIZATION OF NONLOCAL SYMMETRIES

Now, one of the important questions is what kind of finite transformations are related to the nonlocal symmetries. In this section, we only concentrate on the finite transformation of the square eigenfunction symmetry N_0 given in Eq. (12). According to Lie's first principle, to find the finite transformation of N_0, one has to solve the following "initial value" problem:

$$\frac{dp(\epsilon)}{d\epsilon} = \phi_1^2(\epsilon),\ p(0) = p, \tag{17a}$$

$$\frac{dq(\epsilon)}{d\epsilon} = \phi_2^2(\epsilon),\ q(0) = q, \tag{17b}$$

with

$$\begin{pmatrix}\phi_1(\epsilon)\\ \phi_2(\epsilon)\end{pmatrix}_x = \begin{pmatrix}-i\lambda & \frac{p(\epsilon)}{\sqrt{b}}\\ \frac{q(\epsilon)}{\sqrt{b}} & i\lambda\end{pmatrix}\begin{pmatrix}\phi_1(\epsilon)\\ \phi_2(\epsilon)\end{pmatrix}, \tag{18a}$$

$$\begin{pmatrix}\phi_1(\epsilon)\\ \phi_2(\epsilon)\end{pmatrix}_t = \begin{pmatrix} ib\lambda^2 + \frac{1}{2}ip(\epsilon)q(\epsilon) & -\frac{\sqrt{b}}{2}(ip_x(\epsilon) + p(\epsilon)\lambda)\\ \frac{\sqrt{b}}{2}(iq_x(\epsilon) - 2q(\epsilon)\lambda) & -ib\lambda_n^2 - \frac{1}{2}ip(\epsilon)q(\epsilon)\end{pmatrix} \times \begin{pmatrix}\phi_1(\epsilon)\\ \phi_2(\epsilon)\end{pmatrix}. \tag{18b}$$

Because of the presence of $\phi_1(\epsilon)$ and $\phi_2(\epsilon)$ in the initial value problem Eq. (17), we have to study the corresponding symmetry transformation for the spectral functions ϕ_1 and ϕ_2 related to the symmetry N_0 for p and q. In other words, we have to solve the symmetry equations

$$\begin{pmatrix}\sigma_x^{\phi_1}\\ \sigma_x^{\phi_2}\end{pmatrix} = \begin{pmatrix}-i\lambda & \frac{p}{\sqrt{b}}\\ \frac{q}{\sqrt{b}} & i\lambda\end{pmatrix}\begin{pmatrix}\sigma^{\phi_1}\\ \sigma^{\phi_2}\end{pmatrix} + \begin{pmatrix}\frac{\phi_2}{\sqrt{b}} & 0\\ 0 & \frac{\phi_1}{\sqrt{b}}\end{pmatrix}\begin{pmatrix}\sigma^p\\ \sigma^q\end{pmatrix} + i\sigma^\lambda\begin{pmatrix}-\phi_1\\ \phi_2\end{pmatrix}, \tag{19}$$

and

$$\begin{pmatrix}\sigma_t^{\phi_1}\\ \sigma_t^{\phi_2}\end{pmatrix} = \begin{pmatrix} ib\lambda^2 + \frac{1}{2}ipq & -\frac{1}{2}\sqrt{b}(ip_x + 2p\lambda)\\ \frac{1}{2}\sqrt{b}(iq_x - 2q\lambda) & -ib\lambda^2 - \frac{1}{2}ipq\end{pmatrix}\begin{pmatrix}\sigma^{\phi_1}\\ \sigma^{\phi_2}\end{pmatrix} + \sqrt{b}\sigma^\lambda\begin{pmatrix}2i\sqrt{b}\lambda\phi_1 - p\phi_2\\ -2i\sqrt{b}\lambda\phi_2 - q\phi_1\end{pmatrix} + \begin{pmatrix}\frac{i}{2}q\phi_1 - \frac{\sqrt{b}}{2}\phi_2(2\lambda + i\partial_x) & \frac{i}{2}p\phi_1\\ -\frac{i}{2}q\phi_2 & -\frac{i}{2}p\phi_2 + \frac{\sqrt{b}}{2}\phi_1(i\partial_x - 2\lambda)\end{pmatrix}\begin{pmatrix}\sigma^p\\ \sigma^q\end{pmatrix}, \tag{20}$$

with

$$\sigma^p = \phi_1^2, \quad \sigma^q = \phi_2^2. \tag{21}$$

It is not difficult to verify that Eqs. (19), (20), and (21) have the solution

$$\sigma^{\phi_1} = \phi\phi_1, \quad \sigma^{\phi_2} = \phi\phi_2, \tag{22}$$

with

$$\phi_x = \frac{1}{\sqrt{b}}\phi_1\phi_2, \tag{23a}$$

$$\phi_t = \frac{i}{2}(q\phi_1^2 - p\phi_2^2) - 2\sqrt{b}\lambda\phi_1\phi_2. \tag{23b}$$

It is easy to prove that the consistent condition of Eq. (23), namely, $\phi_{xt} = \phi_{tx}$, is identically satisfied. According to the definition Eq. (23), it is not difficult to verify that ϕ is just a solution of the Schwarzian AKNS system:

$$\left(\frac{\phi_t}{\phi_x}\right)_t = \left(\frac{3\phi_t^2}{2\phi_x^2} - \frac{1}{4}\{\phi;\ x\} + 4\lambda b\frac{\phi_t}{\phi_x}\right)_x,$$

$$\{\phi;\ x\} \equiv \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\frac{\phi_{xx}^2}{\phi_x^2}. \tag{24}$$

Similarly, due to the entrance of ϕ in Eq. (22), we have to study the solution of the symmetry equation for the field ϕ:

$$\sigma_x^\phi = \frac{1}{\sqrt{b}}(\sigma^{\phi_1}\phi_2 + \sigma^{\phi_2}\phi_1), \tag{25a}$$

$$\sigma_t^\phi = \frac{i}{2}(\sigma^q\phi_1^2 - \sigma^p\phi_2^2 + 2q\phi_1\sigma^{\phi_1} - 2p\phi_2\sigma^{\phi_2}) - 2\sqrt{b}\lambda(\sigma^{\phi_1}\phi_2 + \sigma^{\phi_2}\phi_1) - 2\sqrt{b}\phi_1\phi_2\sigma^\lambda, \tag{25b}$$

with Eqs. (21) and (22).

It is straightforward to find that the meaningful solution of Eq. (25) with Eqs. (21) and (22) has the form

$$\sigma^\phi = \phi^2, \tag{26}$$

which is just the infinitesimal form of the Möbious transformation.

Therefore, the nonlocal symmetry N_0 of the AKNS system is localized for the extended AKNS system (EAKNS) of Eqs. (2), (10), (11), and (23).

The nonlocal symmetry N_0 of the original AKNS system is localized to a *Lie point symmetry*in the vector form

$$V = \phi_1^2\frac{\partial}{\partial p} + \phi_2^2\frac{\partial}{\partial q} + \phi\phi_1\frac{\partial}{\partial\phi_1} + \phi\phi_2\frac{\partial}{\partial\phi_2} + \phi^2\frac{\partial}{\partial\phi} \tag{27}$$

for the EAKNS system. In other words, the vector V expressed by Eq. (27) of the EAKNS system is a special closed prolongation related to the nonlocal symmetry. So the localization of the nonlocal symmetry is equivalent to finding a closed prolongation structure and the introduced related equations constitute an extended system such that the nonlocal symmetry becomes a Lie point symmetry.

According to the vector Eq. (27), the initial value problem Eq. (17) is changed to

$$\frac{dp(\epsilon)}{d\epsilon} = \phi_1^2(\epsilon),\ p(0) = p, \tag{28a}$$

$$\frac{dq(\epsilon)}{d\epsilon} = \phi_2^2(\epsilon),\ q(0) = q, \tag{28b}$$

$$\frac{d\phi_1(\epsilon)}{d\epsilon} = \phi(\epsilon)\phi_1(\epsilon),\ \phi_1(0) = \phi_1, \tag{28c}$$

$$\frac{d\phi_2(\epsilon)}{d\epsilon} = \phi(\epsilon)\phi_2(\epsilon),\ \phi_2(0) = \phi_2, \tag{28d}$$

$$\frac{d\phi(\epsilon)}{d\epsilon} = \phi^2(\epsilon),\ \phi(0) = \phi. \tag{28e}$$

After solving out the initial value problem Eq. (28), we have the following Binary Darboux-Bäcklund transformation (BDBT) theorem:

Theorem 1 (BDBT theorem). If $\{p, q, \phi_1, \phi_2, \phi\}$ is a solution of the extended AKNS system of Eqs. (2), (10), (11), and (23), so is $\{p(\epsilon), q(\epsilon), \phi_1(\epsilon), \phi_2(\epsilon), \phi(\epsilon)\}$, with

$$p(\epsilon) = p + \frac{\epsilon\phi_1^2}{1-\epsilon\phi}, \tag{29a}$$

$$q(\epsilon) = q + \frac{\epsilon\phi_2^2}{1-\epsilon\phi}, \tag{29b}$$

$$\phi_1(\epsilon) = \frac{\phi_1}{1-\epsilon\phi}, \tag{29c}$$

$$\phi_2(\epsilon) = \frac{\phi_2}{1-\epsilon\phi}, \tag{29d}$$

$$\phi(\epsilon) = \frac{\phi}{1-\epsilon\phi}. \tag{29e}$$

In this section, the localization procedure is only applied to the nonlocal symmetry N_0. Actually, the nonlocal symmetries N_n expressed in Eq. (15) for arbitrary n can be accomplished in the same way. Similar work has been finished for the KdV equation [27].

IV. SYMMETRY REDUCTIONS OF THE AKNS SYSTEM WITH NONLOCAL SYMMETRIES

Symmetry reduction is one of the most powerful methods of studying exact solutions of nonlinear systems [17]. However, usually only the local symmetries are utilized to explore symmetry reductions, while the infinitely many nonlocal symmetries [19] are mostly ignored. Recently, we have found that the nonlocal symmetries can also be successfully used to obtain some types of important interaction solutions that are difficult to find by other approaches [14,15].

In this section, we study the symmetry reductions of the AKNS system Eq. (2) under the local symmetries $\{K_0, K_1, K_2, \tau_0, \tau_1\}$ and the nonlocal symmetry N_0, which is corresponding to the infinitesimal form of the Darboux transformation. As in the KdV case [15], to find symmetry reductions related to the nonlocal symmetry, we have to extend the original system such that the nonlocal symmetry can be localized to a Lie point symmetry for the extended system.

For the AKNS system, its extended system (the EAKNS system) has been given in the last section. It is easy to demonstrate that the general Lie point symmetry solution of the EAKNS system of Eqs. (2), (10), (11), and (23) has the form

$$\sigma_{nl} = \begin{pmatrix} \sigma^p \\ \sigma^q \\ \sigma^{\phi_1} \\ \sigma^{\phi_2} \\ \sigma^{\phi} \\ \sigma^{\lambda} \end{pmatrix} = \begin{pmatrix} (c_5x + c_4t + c_2)p_x + (2c_5t + c_3)p_t + (-c_1 + c_5 + c_4bix)p + c_6\phi_1^2 \\ (c_5x + c_4t + c_2)q_x + (2c_5t + c_3)q_t + (c_1 + c_5 - c_4bix)q + c_6\phi_2^2 \\ (c_5x + c_4t + c_2)\phi_{1x} + (2c_5t + c_3)\phi_{1t} + [(c_7 - \frac{c_1}{2}) + c_6\phi + \frac{1}{2}c_4bix]\phi_1 \\ (c_5x + c_4t + c_2)\phi_{2x} + (2c_5t + c_3)\phi_{2t} + [(c_7 + \frac{c_1}{2}) + c_6\phi - \frac{1}{2}c_4bix]\phi_2 \\ (c_5x + c_4t + c_2)\phi_x + (2c_5t + c_3)\phi_t + (2c_7 - c_5 + c_6\phi)\phi + c_8 \\ c_5\lambda - \frac{b}{2}c_4 \end{pmatrix}$$

$$\equiv c_1K_0 + c_2K_1 + c_3K_2 + c_4\tau_0 + c_5\tau_1 + c_6N_0 + c_7S_0 + c_8T_0. \tag{30}$$

The last component of the symmetry Eq. (30) implies that for the EAKNS system, the scaling and Galileo invariance must accompany the transformation of the spectral parameter λ. In Eq. (30), two additional symmetries, S_0 and T_0, correspond to the ϕ-scaling and ϕ-translation invariance, respectively. From the fifth component of the symmetry Eq. (30), it is not difficult to find that the symmetries N_0, S_0, and T_0 all commute with others, $\{K_0, K_1, K_2, \tau_0, \tau_1\}$.

To find the symmetry reductions related to the nonlocal symmetry, i.e., to find group invariant solutions related to the symmetry Eq. (30) with $c_6 \neq 0$, four nontrivial cases should be considered. Generally, the group invariant solutions can be solved from the invariant condition $\sigma_{nl} = 0$. For $c_6 \neq 0$, we can simply take $c_6 \equiv 1$ without loss of generality.

Case 1. $c_5 = 0$, $c_8 \neq c_7^2$. In this case, $c_4 = 0$ should be held at the same time because of the last component of Eq. (30). The final group invariant solution obtained from $\sigma_{nl} = 0$ can be written as $[{c_8'}^2 \equiv (c_7^2 - c_8)c_3^{-2}, c_2 = cc_3]$,

$$\phi = -c_7 + c_8'c_3\tanh\{c_8'[F(\eta) + t]\}, \quad \eta = x - ct, \tag{31a}$$

$$\phi_1 = \Phi_1(\eta)\operatorname{sech}[c_8'F(\eta) + c_8't]\exp\left(+\frac{c_1t}{2c_3}\right), \tag{31b}$$

$$\phi_2 = \Phi_2(\eta)\operatorname{sech}[c_8'F(\eta) + c_8't]\exp\left(-\frac{c_1t}{2c_3}\right), \tag{31c}$$

$$p = \{c_3 c_8' P(\eta) - \Phi_1^2(\eta)\tanh[c_8' F(\eta) + c_8' t]\}\exp\left(\frac{c_1}{c_3}t\right), \tag{31d}$$

$$q = \{c_3 c_8' Q(\eta) - \Phi_2^2(\eta)\tanh[c_8' F(\eta) + c_8' t]\}\exp\left(-\frac{c_1}{c_3}t\right). \tag{31e}$$

The group invariant functions $F(\eta)$, $P(\eta)$, $Q(\eta)$, $\Phi_1(\eta)$, and $\Phi_2(\eta)$ satisfy the following reduction equations:

$$F_\eta = \frac{\Phi_1\Phi_2}{\sqrt{b}c_3 {c_8'}^2}, \tag{32a}$$

$$Q = \frac{2i(c-2b\lambda)\Phi_2}{\sqrt{b}\Phi_1} + \frac{P\Phi_2^2}{\Phi_1^2} - \frac{2i{c_8'}^2 c_3}{\Phi_1^2}, \tag{32b}$$

$$\Phi_{1\eta} = \frac{P\Phi_2}{\sqrt{b}} - i\lambda\Phi_1, \tag{32c}$$

$$\Phi_{2\eta} = \frac{Q\Phi_1}{\sqrt{b}} + i\lambda\Phi_2, \tag{32d}$$

$$P_\eta = \frac{P^2\Phi_2}{\sqrt{b}\Phi_1} - 2iP\left(\lambda + \frac{{c_8'}^2 c_3}{\sqrt{b}\Phi_1\Phi_2}\right) + \frac{\Phi_2\Phi_1^3}{\sqrt{b}{c_8'}^2 c_3^2} + \frac{(ic_1 - 2c\lambda c_3 + 2bc_3\lambda^2)\Phi_1}{\sqrt{b}c_3\Phi_2}. \tag{32e}$$

After finishing some simple calculations, the reduction system Eq. (32) can be simplified to the following single equation ($\{F;\eta\} \equiv \frac{F_{\eta\eta\eta}}{F_\eta} - \frac{3}{2}\frac{F_{\eta\eta}^2}{F_\eta^2}$),

$$\begin{aligned}\{F;\eta\} &= 2{c_8'}^2 F_\eta^2 + \frac{8(2b\lambda - c)}{F_\eta} + \frac{6}{F_\eta^2} + 2c^2 + 12\lambda^2 - 12bc\lambda + \frac{2ibc_1}{c_3}\\ &\equiv 2{c_8'}^2 F_\eta^2 + 2C_1 - \frac{8C}{F_\eta} + \frac{6}{F_\eta^2},\end{aligned} \tag{33}$$

with

$$\Phi_1(\eta) = \alpha\sqrt{F_\eta}\exp[i\theta(\eta)], \quad \theta(\eta) \equiv (\lambda - bc)\eta + b\int F_\eta^{-1}\mathrm{d}\eta, \tag{34a}$$

$$\Phi_2(\eta) = -\frac{c_3}{\alpha}\sqrt{bF_\eta}\exp[-i\theta(\eta)], \tag{34b}$$

$$P(\eta) = \frac{\Phi_1^2(\eta)\{bF_{\eta\eta} + 2i[1 + (2b\lambda - c)F_\eta]\}}{bc_3{c_8'}^2 F_\eta^2}, \tag{34c}$$

$$Q(\eta) = \frac{c_3{c_8'}^2\{bF_{\eta\eta} - 2i[1 + (2b\lambda - c)F_\eta]\}}{2\Phi_1^2(\eta)}, \tag{34d}$$

where α is an arbitrary constant.

The reduction Eq. (33) can be further integrated as

$$F_{1\eta}^2 = -4F_1^4 + 8(c - 2b\lambda)F_1^3 - 4\left(c^2 + 6\lambda^2 - 6bc\lambda + \frac{ibc_1}{c_3}\right)F_1^2 + \gamma_1 F_1 + 4{c_8'}^2, \tag{35}$$

with

$$F(\eta) = \int \frac{1}{F_1(\eta)}d\eta + \gamma_2, \tag{36}$$

γ_1 and γ_2 being arbitrary integral constants.

In this case, the original physical quantities, p, q, and $I \equiv pq$ are related to $F(\eta)$ by

$$p = \frac{\alpha^2\{F_{\eta\eta} - 2c_8' F_\eta^2\tanh[c_8'(t+F)] - 2ib(CF_\eta - 1)\}}{2c_3 c_8'^2 F_\eta}\exp(-ib\theta), \tag{37}$$

$$q = \frac{bc_3 c_8'^2\{F_{\eta\eta} - 2c_8' F_\eta^2\tanh[c_8'(t+F)] + 2ib(CF_\eta - 1)\}}{2\alpha^2 F_\eta}\exp(ib\theta), \tag{38}$$

and

$$I = \frac{b}{4F_\eta^2}\left(\{2c_8' F_\eta^2\tanh[c_8'(t+F)] - F_{\eta\eta}\}^2 + (1 - CF_\eta)^2\right) \equiv J + K, \tag{39}$$

$$K \equiv \begin{cases} bc'_8 F_\eta\{1 - \tanh[c'_8(t+F)]\}(\ln F_\eta)_\eta - bc_8'^2 F_\eta^2\mathrm{sech}^2[c'_8(t+F)] + I_0, & x - v_s t \geqslant 0,\\ -bc'_8 F_\eta\{1 + \tanh[c'_8(t+F)]\}(\ln F_\eta)_\eta - bc_8'^2 F_\eta^2\mathrm{sech}^2[c'_8(t+F)] + I_0, & x - v_s t < 0,\end{cases} \tag{40}$$

$$J \equiv \begin{cases} b\{[c'_8 F_\eta - \frac{1}{2}(\ln F_\eta)_\eta]^2 + (F_\eta^{-1} - C)^2\} - I_0, & x - v_s t \geqslant 0,\\ b\{[c'_8 F_\eta - \frac{1}{2}(\ln F_\eta)_\eta]^2 + (F_\eta^{-1} - C)^2\} - I_0, & x - v_s t < 0,\end{cases} \tag{41}$$

where

$$\theta \equiv 2(C + b\lambda)\eta + (C_1 + 2bC\lambda + 2\lambda^2 - C^2)t - 2\int F_\eta^{-1}\mathrm{d}\eta.$$

Here, K and J are introduced for convenience later to discuss the limiting procedures $x \to \pm\infty$ for any fixed time. In Eqs. (40) and (41), I_0 is an arbitrary constant for convenience later and it can be simply taken as zero. v_s is the velocity of the soliton, which will be determined in the next section.

Remark 2. For the NLS case, if we accept that all the parameters in Eq. (39) are real, then the result is valid only for $b=1$ because $I = pq = |p|^2 > 0$. In other words, if one tries to find some significant results for the $b=-1$ case, the parameters appearing in the solutions have to be complex, which makes it difficult to find nonsingular solutions. Thus, in the Sec. V we discuss only explicit solutions for the $b=1$ case.

There is more about the solutions of Eq. (33) [or equivalently Eq. (35)] and then the quantity Eqs. (39), (40), and (41) of the AKNS system presented in the next section.

Case 2. $c_5 = 0$, $c_8 = c_7^2$. In this case, we have

$$\phi = -c_7 + \frac{c_3}{t+F(\eta)}, \quad \eta = x - ct,\, c_2 = cc_3, \tag{42a}$$

$$\phi_1 = \frac{\Phi_1(\eta)}{[F(\eta)+t]} \exp\left(\frac{c_1 t}{2c_3}\right), \tag{42b}$$

$$\phi_2 = \frac{\Phi_2(\eta)}{[F(\eta)+t]} \exp\left(-\frac{c_1 t}{2c_3}\right), \tag{42c}$$

$$p = \left\{P(\eta) + \frac{\Phi_1^2(\eta)}{c_3[F(\eta)+t]}\right\} \exp\left(\frac{c_1}{c_3}t\right), \tag{42d}$$

$$q = \left\{Q(\eta) + \frac{\Phi_1^2(\eta)}{c_3[F(\eta)+t]}\right\} \exp\left(-\frac{c_1}{c_3}t\right). \tag{42e}$$

The functions $F(\eta)$, $P(\eta)$, $Q(\eta)$, $\Phi_1(\eta)$, and $\Phi_2(\eta)$ satisfy the reduction equations

$$F_\eta = -\frac{\Phi_1\Phi_2}{\sqrt{b}c_3}, \tag{43a}$$

$$Q = \frac{2i(c-2b\lambda)\Phi_2}{\sqrt{b}\Phi_1} + \frac{P\Phi_2^2}{\Phi_1^2} + \frac{2ic_3}{\Phi_1^2}, \tag{43b}$$

$$\Phi_{1\eta} = \frac{P\Phi_2}{\sqrt{b}} - i\lambda\Phi_1, \tag{43c}$$

$$\Phi_{2\eta} = \frac{Q\Phi_1}{\sqrt{b}} + i\lambda\Phi_2, \tag{43d}$$

$$P_\eta = \frac{P^2\Phi_2}{\sqrt{b}\Phi_1} - 2iP\left(\lambda - \frac{c_3}{\sqrt{b}\Phi_1\Phi_2}\right) + \frac{(ic_1 - 2c\lambda c_3 + 2bc_3\lambda^2)\Phi_1}{\sqrt{b}c_3\Phi_2}. \tag{43e}$$

The reduction system Eq. (43) can also be solved by Eq. (35) with Eq. (36), ${c_8'}^2 = 0$, and

$$\Phi_1(\eta) = \alpha\sqrt{F_\eta}\exp[i\theta(\eta)], \qquad \theta(\eta) \equiv (\lambda - bc)\eta + b\int F_\eta^{-1}\mathrm{d}\eta, \tag{44a}$$

$$\Phi_2(\eta) = -\frac{c_3}{\alpha}\sqrt{bF_\eta}\exp[-i\theta(\eta)], \tag{44b}$$

$$P(\eta) = \frac{\Phi_1^2(\eta)\{-bF_{\eta\eta} + 2i[1+(2b\lambda - c)F_\eta]\}}{2bc_3F_\eta^2}, \tag{44c}$$

$$Q(\eta) = \frac{c_3\{-bF_{\eta\eta} - 2i[1+(2b\lambda - c)F_\eta]\}}{2\Phi_1^2(\eta)}. \tag{44d}$$

Case 3. $c_5 \neq 0$, $c_8 \neq (c_7 - \frac{c_5}{2})^2$. In this case, the group invariant solution obtained from the invariant condition $\sigma_{nl} = 0$ has the form

$$\phi = c_5c_8''\tanh\{c_8''[F(\xi)+\ln(\tau)]\} - c_7',$$
$$\xi = \frac{x+c_5c_2'}{\tau} - \frac{c_4\tau}{2c_5^2}, \quad \tau = \sqrt{2c_5t + c_3}, \tag{45a}$$

$$\phi_1 = \Phi_1(\xi)\mathrm{sech}\{c_8''[F(\xi)+\ln(\tau)]\}\tau^{\frac{1}{2c_5}(c_1-c_5+ibc_4c_5c_2')} \times \exp\left[-\frac{ibc_4\tau}{8c_5^3}(c_4\tau + 4c_5^2\xi)\right], \tag{45b}$$

$$\phi_2 = \Phi_2(\xi)\mathrm{sech}\{c_8''[F(\xi)+\ln(\tau)]\}\tau^{-\frac{1}{2c_5}(c_1+c_5+ibc_4c_5c_2')} \times \exp\left[\frac{ibc_4\tau}{8c_5^3}(c_4\tau + 4c_5^2\xi)\right], \tag{45c}$$

$$p = \left[c_5c_8''P(\xi) - \Phi_1^2(\xi)\tanh(\{c_8''[F(\xi)+\ln(\tau)]\})\right] \times \tau^{\frac{(c_1-c_5+ibc_4c_5c_2')}{c_5}} \exp\left[-\frac{ibc_4\tau}{4c_5^3}(c_4\tau + 4c_5^2\xi)\right], \tag{45d}$$

$$q = \left(c_5c_8''Q(\xi) - \Phi_2^2(\xi)\tanh\{c_8''[F(\xi)+\ln(\tau)]\}\right) \times \tau^{\frac{-ibc_4c_2'-c_5-c_1}{c_5}} \exp\left[\frac{ibc_4\tau}{4c_5^3}(c_4\tau + 4c_5^2\xi)\right], \tag{45e}$$

where $c_7' = c_7 - \frac{c_5}{2}$, $c_8'' = \frac{\sqrt{{c_7'}^2 - c_8}}{c_5}$, $c_2' \equiv \frac{c_2}{c_5^2} - \frac{c_3c_4}{2c_5^3}$, while the group invariant functions $F(\xi)$, $P(\xi)$, $Q(\xi)$, $\Phi_1(\xi)$, and $\Phi_2(\xi)$ should satisfy symmetry reduction equations, which can be obtained by substituting Eq. (45) into the EAKNS system of Eqs. (2), (10), (11), and (23). It is straightforward to find the final reduction equations are

$$F_\xi = \frac{\Phi_1\Phi_2}{c_5{c_8''}^2\sqrt{b}}, \tag{46a}$$

$$Q = \frac{2ic_5\xi\Phi_2}{\sqrt{b}\Phi_1} + \frac{P\Phi_2^2}{\Phi_1^2} - \frac{2ic_5^2{c_8''}^2}{\Phi_1^2}, \tag{46b}$$

$$\Phi_{1\xi} = \frac{P\Phi_2}{\sqrt{b}}, \tag{46c}$$

$$\Phi_{2\xi} = \frac{Q\Phi_1}{\sqrt{b}}, \tag{46d}$$

$$P_\xi = \frac{P^2\Phi_2}{\sqrt{b}\Phi_1} - \frac{2ic_5^2{c_8''}^2P}{\sqrt{b}\Phi_1\Phi_2} + \frac{\Phi_2\Phi_1^3}{\sqrt{b}c_5^2{c_8''}^2} + \frac{[bc_5c_4c_2' + i(c_5-c_1)]\Phi_1}{\sqrt{b}\Phi_2}, \tag{46e}$$

which can be solved via the fourth Painlevé equation. The results read

$$\Phi_1(\xi) = \alpha\sqrt{F_\xi}\exp\left[\frac{1}{2}ibc_5\left(2\int F_\xi^{-1}d\xi - \xi^2\right)\right], \quad (47a)$$

$$\Phi_2(\xi) = \alpha^{-1}F_\xi^{-1/2}\sqrt{b}c_5c_8''^2 \times \exp\left[-\frac{1}{2}ibc_5\left(2\int F_\xi^{-1}d\xi - \xi^2\right)\right], \quad (47b)$$

$$P(\xi) = -\frac{\alpha^2}{2c_5c_8''^2}\exp\left[ibc_5\left(2\int F_\xi^{-1}d\xi - \xi^2\right)\right] \times [2ic_5b(\xi F_\xi - 1) - F_{\xi\xi}], \quad (47c)$$

$$Q(\xi) = \frac{c_5c_8''^2 b}{2\alpha^2}\exp\left[-ibc_5\left(2\int F_\xi^{-1}d\xi - \xi^2\right)\right] \times [F_{\xi\xi} + 2ic_5b(\xi F_\xi - 1)], \quad (47d)$$

while $F = F(\xi)$ is given by the following equation

$$2F_\xi\left(F_{\xi\xi\xi} + 8c_5^2\xi\right) - 3F_{\xi\xi}^2 - 4c_8''^2 + 4\left(c_4c_5c_2' - ibc_1 - c_5^2\xi^2\right)F_\xi^2 - 12c_5^2 = 0. \quad (48)$$

Making the transformation

$$F_\xi = F_1^{-1}, \quad (49)$$

Eq. (48) becomes

$$F_{1\xi\xi} = \frac{1}{2}\frac{F_{1\xi}^2}{F_1} - 6c_5^2F_1^3 + 8c_5^2F_1^2 + 2\left(c_4c_5c_2' - ibc_1 - c_5^2\xi^2\right)F_1 - \frac{2c_8''^2}{F_1}, \quad (50)$$

which is completely same as the standard form of the fourth Painlevé (PIV) equation

$$y_{zz} = \frac{1}{2}\frac{y_z^2}{y} + \frac{3}{2}y^3 + 4zy^2 + 2(z^2 - \beta_1)y + \frac{\beta_2}{y}, \quad (51)$$

with the scaling relation

$$F_1 = -\tfrac{1}{2}\left(-c_5^2\right)^{-1/4}y\left[\left(-c_5^2\right)^{1/4}\xi\right], \quad (52)$$

and $\beta_1 = c_5^{-1}(c_1b + ic_4c_5c_2')$, $\beta_2 = -8c_8''^2$.

It is known that the general solutions of the Painlevé equations cannot be expressed by known simple functions. Here we write down a special class of solutions ($c_5' = -ic_5$)

$$F_1(\xi) = \xi + \frac{2\delta c_8'' - 1}{2c_5'\xi} - \frac{\alpha K_U\left(\delta c_8'' - \frac{1}{2}, \frac{3}{2}, c_5'\xi^2\right) + c_8''(c_8'' - \delta)K_M\left(\delta c_8'' - \frac{1}{2}, \frac{3}{2}, c_5'\xi^2\right)}{\alpha K_U\left(\delta c_8'' + \frac{1}{2}, \frac{3}{2}, c_5'\xi^2\right) + c_8''\delta K_M\left(\delta c_8'' + \frac{1}{2}, \frac{3}{2}, c_5'\xi^2\right)} \quad (53)$$

for the PIV Eq. (50) with a constant condition

$$c_1 = -ic_4bc_2'c_5 + c_5b - 2\delta c_5bc_8'', \ \delta = \pm 1, \quad (54)$$

where α is an arbitrary constant, $K_M(\mu, \nu, x)$ and $K_U(\mu, \nu, x)$ are two independent Kummer functions, which are solutions of the Kummer equation

$$xy_{xx} + (\nu - x)y_x - \mu y = 0. \quad (55)$$

It is remarkable that if $\delta c_8''$ is a negative half integer, the Kummer functions and then F_1 become rational functions. For instance, if $\delta = 1$, $b = -1$, and $c_8'' = -\frac{3}{2}$, we have

$$F_1 = \frac{3(2c_5'\xi^2 - 1)}{2c_5'(2c_5'\xi^2 - 3)\xi}, \quad F(\xi) = \frac{1}{3}c_5'\xi^2 - \frac{1}{3}\ln(2c_5'\xi^2 - 1), \quad (56a)$$

$$p = \frac{4i\alpha^2\left[(2c_5'\xi^2 - 1)\tau e^{c_5'\xi^2} - 1\right]}{c_5'\tau^3(1 + \tau e^{c_5'\xi^2})}e^{-\frac{(2\xi c_5'^2 - c_4\tau)^2}{4c_5'^3}}, \quad (56b)$$

$$q = -\frac{9ic_5'^2\tau^3}{\alpha^2(\tau^3e^{c_5'\xi^2} + 2c_5'\xi^2 - 1)}e^{\frac{(2\xi c_5'^2 - c_4\tau)^2}{4c_5'^3}}. \quad (56c)$$

Case 4. $c_5 \neq 0$, $c_8 = (c_7 - \frac{c_5}{2})^2$. In this case, the corresponding similarity reduction solution has the form

$$\phi = \frac{c_5}{F(\xi) + \ln(\tau)} - c_7', \quad \xi = \frac{x + c_5c_2'}{\tau} - \frac{c_4\tau}{2c_5^2}, \quad (57a)$$

$$\tau = \sqrt{2c_5t + c_3},$$

$$\phi_1 = \frac{\Phi_1(\xi)}{F(\xi) + \ln(\tau)}\tau^{\frac{1}{2c_5}(c_1 - c_5 + ibc_4c_5c_2')} \times \exp\left[-\frac{ibc_4\tau}{8c_5^3}\left(c_4\tau + 4c_5^2\xi\right)\right], \quad (57b)$$

$$\phi_2 = \frac{\Phi_2(\xi)}{F(\xi) + \ln(\tau)}\tau^{-\frac{1}{2c_5}(c_1 + c_5 + ibc_4c_5c_2')} \times \exp\left[\frac{ibc_4\tau}{8c_5^3}\left(c_4\tau + 4c_5^2\xi\right)\right], \quad (57c)$$

$$p = \left[P(\xi) + \frac{\Phi_1^2(\xi)}{c_5(F(\xi) + \ln(\tau))}\right]\tau^{\frac{(c_1 - c_5 + ibc_4c_5c_2')}{c_5}} \times \exp\left[-\frac{ibc_4\tau}{4c_5^3}\left(c_4\tau + 4c_5^2\xi\right)\right], \quad (57d)$$

$$q = \left[Q(\xi) + \frac{\Phi_2^2(\xi)}{c_5(F(\xi) + \ln(\tau))}\right]\tau^{\frac{-ibc_4c_2' - c_5 - c_1}{c_5}} \times \exp\left[\frac{ibc_4\tau}{4c_5^3}\left(c_4\tau + 4c_5^2\xi\right)\right]. \quad (57e)$$

Correspondingly, the group invariant functions $F(\xi)$, $P(\xi)$, $Q(\xi)$, $\Phi_1(\xi)$, and $\Phi_2(\xi)$ satisfy the following symmetry reduction equations

$$F_\xi = -\frac{\Phi_1\Phi_2}{c_5\sqrt{b}}, \tag{58a}$$

$$Q = \frac{2ic_5\xi\Phi_2}{\sqrt{b}\Phi_1} + \frac{P\Phi_2^2 + 2ic_5^2}{\Phi_1^2}, \tag{58b}$$

$$\Phi_{1\xi} = \frac{P\Phi_2}{\sqrt{b}}, \tag{58c}$$

$$\Phi_{2\xi} = \frac{Q\Phi_1}{\sqrt{b}}, \tag{58d}$$

$$P_\xi = \frac{P^2\Phi_2}{\sqrt{b}\Phi_1} + \frac{2ic_5^2 P}{\sqrt{b}\Phi_1\Phi_2} + \frac{[bc_5c_4c_2' + i(c_5 - c_1)]\Phi_1}{\sqrt{b}\Phi_2}. \tag{58e}$$

The solutions of the reduction system Eq. (58) can be expressed by the special PIV Eq. (50) with $c_8'' = 0$, which read

$$\Phi_1(\xi) = \phi_0\sqrt{F_\xi}\exp\left[\frac{1}{2}ibc_5\left(2\int F_\xi^{-1}d\xi - \xi^2\right)\right], \tag{59a}$$

$$\Phi_2(\xi) = -\phi_0^{-1}\sqrt{bF_\xi}c_5\exp\left[-\frac{1}{2}ibc_5\left(2\int F_\xi^{-1}d\xi - \xi^2\right)\right], \tag{59b}$$

$$P(\xi) = \frac{\phi_0^2}{2c_5F_\xi}\exp\left[ibc_5\left(2\int F_\xi^{-1}d\xi - \xi^2\right)\right] \times [2ic_5b(\xi F_\xi - 1) - F_{\xi\xi}], \tag{59c}$$

$$Q(\xi) = -\frac{c_5 b}{2\phi_0^2F_\xi}\exp\left[-ibc_5\left(2\int F_\xi^{-1}d\xi - \xi^2\right)\right] \times [F_{\xi\xi} + 2ic_5b(\xi F_\xi - 1)], \tag{59d}$$

where $F = F(\xi)$ is given by Eq. (48) with $c_8'' = 0$.

V. WAVE INTERACTION SOLUTIONS OF THE AKNS AND NLS SYSTEMS

Except for the first reduction given in the last section, all the solutions of the other three reductions are analytical only at some finite regions for both space and time variables. Thus, in this section, we discuss only concrete special examples of the first case to find the interacting wave solutions of the AKNS and NLS systems. It is noted that the key point is to write down the explicit solutions of the reduction Eq. (33) or equivalently Eq. (35) with Eq. (36). From Eq. (35), we know that the solution of Eq. (33) can be written as the function of the Jacobi elliptic functions. Therefore, we seek for solutions of Eq. (33) in the form

$$F(\eta) = \frac{A_1}{c_8'}\eta + \frac{A}{c_8'}\,E_\pi[B\,\mathrm{sn}(k\eta,m),n,\nu], \tag{60}$$

where A, A_1, B, k, m, and n are constants, $\mathrm{sn}(k\eta,m) \equiv S_n$, $\mathrm{cn}(k\eta,m) \equiv C_n$, $\mathrm{dn}(k\eta,m) \equiv D_n$, are the usual Jacobi elliptic functions, and $E_\pi(\zeta,\, n,\, \nu)$ is the third type of incomplete elliptic integral defined by

$$E_\pi(\zeta,\, n,\, \nu) = \int_0^\zeta \frac{\mathrm{d}t}{(1 - nt^2)\sqrt{(1 - t^2)(1 - \nu^2t^2)}}.$$

Substituting Eq. (60) into Eq. (33), one can find that three situations are important: (i) $B = 1$, $\nu = m$, (ii) $B = m$, $\nu = m^{-1}$, and (iii) $n = 0$, $\nu = 1$.

Case (i). $B = 1$, $\nu = m$. In this case, Eq. (60) becomes

$$F(\eta) = \frac{A_1}{c_8'}\eta + \frac{A}{c_8'}\,E_\pi(S_n,\, n,\, m). \tag{61}$$

Substituting Eq. (61) into Eq. (33) yields

$$\begin{aligned}
&\big[-C_1A_1^2n^4 + 4Cc_8'A_1n^4 - n^3\big(3c_8'^2n + A_1^4n + AA_1k^3m^2\big)\big]S_n^8 + \big[2A_1n^3(2A_1 + Ak)C_1 - 4c_8'n^3(Ak + 4A_1)C \\
&- 2n^2\big(A^2k^4m^2 - 6c_8'^2n - 2A_1^4n + Ak^3A_1m^2 - Ak^3A_1nm^2 - 2A_1^3nAk - Ak^3A_1n\big)\big]S_n^6 + \big[-n^2\big(6A_1Ak + A^2k^2 + 6A_1^2\big)C_1 \\
&+ 12c_8'n^2(2A_1 + Ak)C - n\big(18nc_8'^2 - A^2k^4n - A^2k^4nm^2 + 6A_1^2A^2nk^2 - 3Ak^3A_1m^2 - 3A^2k^4m^2 + 3AA_1k^3n^2 \\
&+ 12A_1^3nAk + 6A_1^4n\big)\big]S_n^4 + [2n(Ak + A_1)(2A_1 + Ak)C_1 - 4nc_8'(4A_1 + 3Ak)C + 2n(2A_1A^3k^3 \\
&- A^2k^4m^2 + 2A_1^4 + 6c_8'^2 - A^2k^4 + 6A_1^3Ak + 6A_1^2A^2k^2 - Ak^3A_1 - Ak^3A_1m^2 \\
&+ Ak^3A_1n)]S_n^2 - (Ak + A_1)^2C_1 + 4c_8'C(A_1 + Ak) - 3c_8'^2 - 6A_1^2A^2k^2 + A^2k^4n \\
&- 4A_1A^3k^3 - 4A_1^3Ak - A_1^4 + AA_1nk^3 - A^4k^4 = 0.
\end{aligned} \tag{62}$$

Vanishing all the coefficients of different powers of S_n leads to a unique nontrivial solution for constants

$$c_8'^2 = \frac{k^2A_1^2}{n}(n + nm^2 - 3m^2) - A_1^4 \pm \frac{kA_1\big[nA_1^2(2n - 3m^2 + 2nm^2 - n^2) + k^2m^2(n-1)(m^2 - n)\big]}{n\sqrt{n(1-n)(m^2 - n)}},$$

$$C_1 = \frac{3k^4m^2\big[k^2m^2(n-1)(m^2-1) + nA_1^2(2n(1+m^2) - n^2 - 3m^2)\big]}{n^2\big[n\big(A_1^4 + c_8'^2\big) - k^2A_1^2(n + nm^2 - 3m^2)\big]} - \frac{3}{A_1^2}\big(A_1^4 - c_8'^2\big) - \frac{2k^2}{n}(n + nm^2 - 3m^2),$$

$$C = \frac{c_8'}{2A_1} + \frac{A_1}{6nc_8'}\left[k^2(n + nm^2 - 3m^2) + n\left(2C_1 + 3A_1^2\right)\right],$$
$$A = \frac{2A_1}{3m^2k}(n + nm^2 - 3m^2) + \frac{nA_1}{3m^2k^3}\left(C_1 + 3A_1^2\right) - \frac{nc_8'^2}{A_1m^2k^3}, \tag{63}$$

while the remained five parameters m, n, k, A_1, and c (the velocity of the periodic wave included in η) are free.

In this case, the physical quantity I and J expressed by Eqs. (39) and (41) become

$$I = b\left[C - \frac{c'_8\left(1 - nS_n^2\right)}{Ak + A_1\left(1 - nS_n^2\right)}\right]^2 + b\left\{\frac{\tanh[A_1x - (A_1c - c'_8)t + AE_\pi(S_n,n,m)]}{1 - nS_n^2} - \frac{Ank^2S_nC_nD_n}{\left(1 - nS_n^2\right)\left[Ak + A_1\left(1 - nS_n^2\right)\right]}\right\}^2,$$
$$\equiv K + J \tag{64}$$

and

$$J = \begin{cases} \frac{b}{4}\left[C - \frac{c'_8(1-nS_n^2)}{Ak+A_1(1-nS_n^2)}\right]^2 + b\left\{\frac{1}{1-nS_n^2} - \frac{Ank^2S_nC_nD_n}{(1-nS_n^2)[Ak+A_1(1-nS_n^2)]}\right\}^2 - I_0, & x - v_st > 0, \\ \frac{b}{4}\left[C - \frac{c'_8(1-nS_n^2)}{Ak+A_1(1-nS_n^2)}\right]^2 + b\left\{\frac{-1}{1-nS_n^2} - \frac{Ank^2S_nC_nD_n}{(1-nS_n^2)[Ak+A_1(1-nS_n^2)]}\right\}^2 - I_0, & x - v_st < 0, \end{cases} \tag{65}$$

with the soliton velocity

$$v_s = c - \frac{c'_8}{A_1}, \tag{66}$$

while the quantity K defined in Eq. (40) is just $I - J$.

It is remarkable that this solution can describe interactions between solitons and cnoidal waves. Figure 1 displays the soliton-cnoidal wave interaction structure of Eq. (39) with Eq. (61) by fixing

$$m = 0.999, \quad n = 0.99, \quad k = -1, \quad A_1 = 1, \quad b = 1, \quad I_0 = 1, \tag{67}$$

and then

$$A = 0.00899, \quad C = 0.202, \quad C_1 = -0.938, \quad c_8' = 0.110, \quad c = 0.202. \tag{68}$$

Figure 1(a) exhibits the wave interaction structure of I determined by Eq. (39) for the defocusing NLS equation (with $b = 1$) at $t = 0$, and Fig. 1(d) is a density plot of Eq. (39) showing its time evolution. Figures 1(b) and 1(c) reveal the structures of the related quantities J and K. It is observed from Fig. 1(b) that apart from the soliton center, the solution rapidly tends to a cnoidal periodic wave expressed by Eq. (65) [i.e., Eq. (41)]. It is clear from Fig. 1(c) that after removing the periodic wave from I, the left is just a dark soliton. Figure 1(d) shows that the interaction between soliton and cnoidal wave (everyone peak of the cnoidal wave) is elastic except for a phase shift. The straight line plotted in Fig. 1(d) is determined by

$$x - v_st = 0, \tag{69}$$

with v_s given by Eq. (66).

It is noted that similar soliton-cnoidal wave interaction solutions can be obtained for other types of nonlinear systems [15,16].

Case (ii). $B = m$, $\nu = m^{-1}$. In this case, Eq. (60) is changed to

$$F(\eta) = \frac{A_1}{c_8'}\eta + \frac{A}{c_8'}\,E_\pi(m\,S_n,\,n,\,m^{-1}). \tag{70}$$

Substituting Eq. (61) into Eq. (33) yields further constraints of the constants

$$A = -\frac{n}{3mk^3A_1}\left(C_1A_1^2 - 4c_8'CA_1 + A_1^4 + 3c_8'^2\right),$$
$$C = \frac{A_1k^2}{6nc_8'}(n - 3 + nm^2) + \frac{c_8'}{2A_1}\frac{A_1}{6c_8'}\left(2C_1 + 3A_1^2\right),$$
$$C_1 = \frac{3k^2\left\{\left[n(n-1)k^2 + n^2A_1^2(n-2)\right]c_8'^2 - k^4A_1^2(n^2 - 3n + 2) - nk^2A_1^4(n-2)^2 + n^2(n-2)A_1^6\right\}}{n^2A_1^2\left[A_1^2k^2(n - 3 + nm^2) - n\left(A_1^4 + c_8'^2\right)\right]}$$
$$+ \frac{3c_8'^2}{A_1^2} + \frac{3(n-1)k^4}{n^2A_1^2} + (1 - 2m^2)k^2 - 3A_1^2, \tag{71}$$
$$c_8'^2 = \frac{A_1^2k^2}{n}(n - 3 + nm^2) - A_1^4 \pm \frac{kA_1\left[k^2(n-1)(nm^2 - 1) + nA_1^2(3 - 2nm^2 - 2n + n^2m^2)\right]}{n\sqrt{n(n-1)(nm^2 - 1)}}.$$

We will not go further on this type of solution because singularities of I occur for all possible parameter selections.

Case (iii). $n = 0$, $\nu = 1$. In this case, the elliptic π function in Eq. (60) reduces to the inverse hyperbolic tangent function; i.e., Eq. (60) becomes

$$F(\eta) = \frac{A_1}{c_8'}\eta + \frac{A}{c_8'}\,\text{arctanh}(B\,S_n). \tag{72}$$

Consequently, the quantities I and J expressed by Eqs. (39) and (41) become

$$I = b\left(\left\{\frac{S_{1n}}{S_{2n}}\tanh[c'_8(t+F)] - \frac{ABk^2S_n}{2S_{1n}S_{2n}}\left[S_{1n}\left(D_n^2+m^2C_n^2\right)+2B^2C_n^2D_n^2\right]\right\}^2 + \left(C + \frac{c'_8S_{1n}}{S_{2n}}\right)^2\right), \tag{73}$$

and

$$J = \begin{cases} b\left(\left\{\frac{S_{1n}}{S_{2n}} - \frac{ABk^2S_n}{2S_{1n}S_{2n}}\left[S_{1n}\left(D_n^2+m^2C_n^2\right)+2B^2C_n^2D_n^2\right]\right\}^2 + \left(C+\frac{c'_8S_{1n}}{S_{2n}}\right)^2\right) - I_0, & x - v_st > 0,\\ b\left(\left\{\frac{S_{1n}}{S_{2n}} + \frac{ABk^2S_n}{2S_{1n}S_{2n}}\left[S_{1n}\left(D_n^2+m^2C_n^2\right)+2B^2C_n^2D_n^2\right]\right\}^2 + \left(C+\frac{c'_8S_{1n}}{S_{2n}}\right)^2\right) - I_0, & x - v_st < 0, \end{cases} \tag{74}$$

with the same soliton velocity Eq. (66) and

$$S_{1n} \equiv B^2S_n^2 - 1, \quad S_{2n} \equiv ABkC_nD_n - A_1S_{1n}.$$

Substituting Eq. (72) into Eq. (33) leads to three possible subcases.

Case (iiia).

$$\begin{aligned} B^2 &= 1, \quad A^2 = \pm\frac{1}{4},\\ C_1 &= \frac{1}{4}(1+m^2)k^2 - 6A_1^2,\\ C &= \frac{A_1}{4c'_8}\left[k^2(1+m^2) - 8A_1^2\right],\\ c'^2_8 &= \frac{1}{16}\left(k^2-4A_1^2\right)\left(4A_1^2 - k^2m^2\right), \end{aligned} \tag{75}$$

where m, k, c'_8, and c are four free parameters.

In this subcase, the soliton-cnoidal periodic wave interaction solutions p, q, I expressed by Eqs. (37), (38), and (39) with Eq. (72) are analytic only for $m > 1$. Figure 2 displays a special structure of this type of interaction solution with the parameter selections

$$\begin{aligned} &B = b = k = 2A = 1, \quad m = \frac{5}{4}, \quad c = \frac{1}{5}, \quad c'_8 = \frac{7\sqrt{11}}{400},\\ &A_1 = \frac{3}{5}, \quad C_1 = -\frac{2431}{1600}, \quad C = -\frac{381\sqrt{11}}{1540}, \quad I_0 = \frac{3}{2}. \end{aligned} \tag{76}$$

Figure 2(a) explicitly displays that a dark (gray) soliton is dressed by a periodic wave at $t = 0$. Figure 2(b) shows that the solution rapidly approaches the cnoidal wave apart from the soliton center. Figure 2(c) reveals that only a dark soliton K is left after the periodic wave J is ruled out from the exact solution I. Figure 2(d) demonstrates the interaction behavior between soliton and every peak of the periodic wave.

Case (iiib).

$$\begin{aligned} B^2 &= m, \quad A^2 = 1,\\ C_1 &= \frac{1}{4}(1+6m+m^2)k^2 - 6A_1^2,\\ C &= \frac{A_1}{4c'_8}\left[k^2(1+6m+m^2) - 8A_1^2\right],\\ c'^2_8 &= \frac{1}{4}\left(mk^2 - A_1^2\right)\left[4A_1^2 - k^2(1+m)^2\right]. \end{aligned} \tag{77}$$

Different from the previous subcase (*Case iiia*), the soliton-periodic interaction waves here are analytic only for $m < 1$. Figure 3 is a special plot of this interaction solution with the parameter selections

$$\begin{aligned} &m = \frac{9}{10}, \quad B = \frac{3}{\sqrt{10}}, \quad A = 1, \quad k = 1, \quad c = 0, \quad I_0 = 3.7,\\ &c'_8 = \frac{3\sqrt{126\,811}}{1\,000\,000}, \quad A_1 = \frac{949}{1000},\\ &C = \frac{615\,901\sqrt{126\,811}}{190\,216\,500}, \quad C_1 = -\frac{1\,800\,553}{500\,000}. \end{aligned} \tag{78}$$

It is seen from Fig. 3(a) that a gray soliton is dressed by a periodic wave. Figure 3(b) displays that the solution exponentially approaches the cnoidal wave when $x \to \pm\infty$. Figure 3(c) manifests the fact that only a dark soliton K is revived after the periodic wave J is taken away from the exact solution I. Figure 3(d) is a three-dimensional plot of the soliton-cnoidal wave interaction solution.

Case (iiic).

$$\begin{aligned} B^2 &= m^2, \quad A^2 = \frac{1}{4},\\ C_1 &= \frac{1}{4}(1+m^2)k^2 - 6A_1^2,\\ C &= \frac{A_1}{4c'_8}\left[k^2(1+m^2) - 8A_1^2\right],\\ c'^2_8 &= \frac{1}{16}\left(k^2-4A_1^2\right)\left(4A_1^2 - k^2m^2\right). \end{aligned} \tag{79}$$

Similar to the second subcase, the soliton-periodic interaction solution in this subcase is analytic also for $m < 1$. Figure 4 is a special plot of this special interaction solution with

$$\begin{aligned} &m = A = B = I_0 = \frac{1}{2}, \quad k = b = 1, \quad c = 0, \quad A_1 = \frac{3}{8},\\ &c'_8 = \frac{\sqrt{35}}{64}, \quad C = \frac{3\sqrt{35}}{140}, \quad C_1 = -\frac{17}{32}. \end{aligned} \tag{80}$$

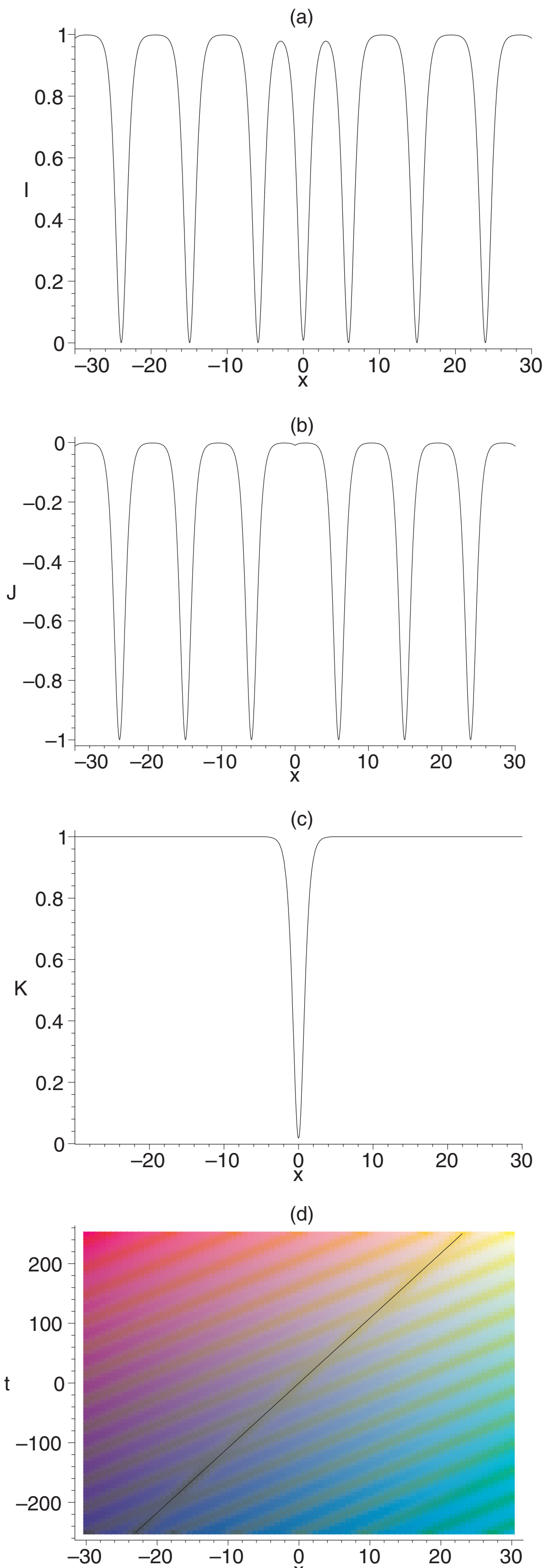

FIG. 1. (Color online) The first type of special soliton-cnoidal wave interaction solution for the NLS system given by Eq. (39) with the parameter selections Eq. (63): (a–c) the special structure of I, J, and K, respectively, at $t = 0$; (d) the density plot of I for time evolution.

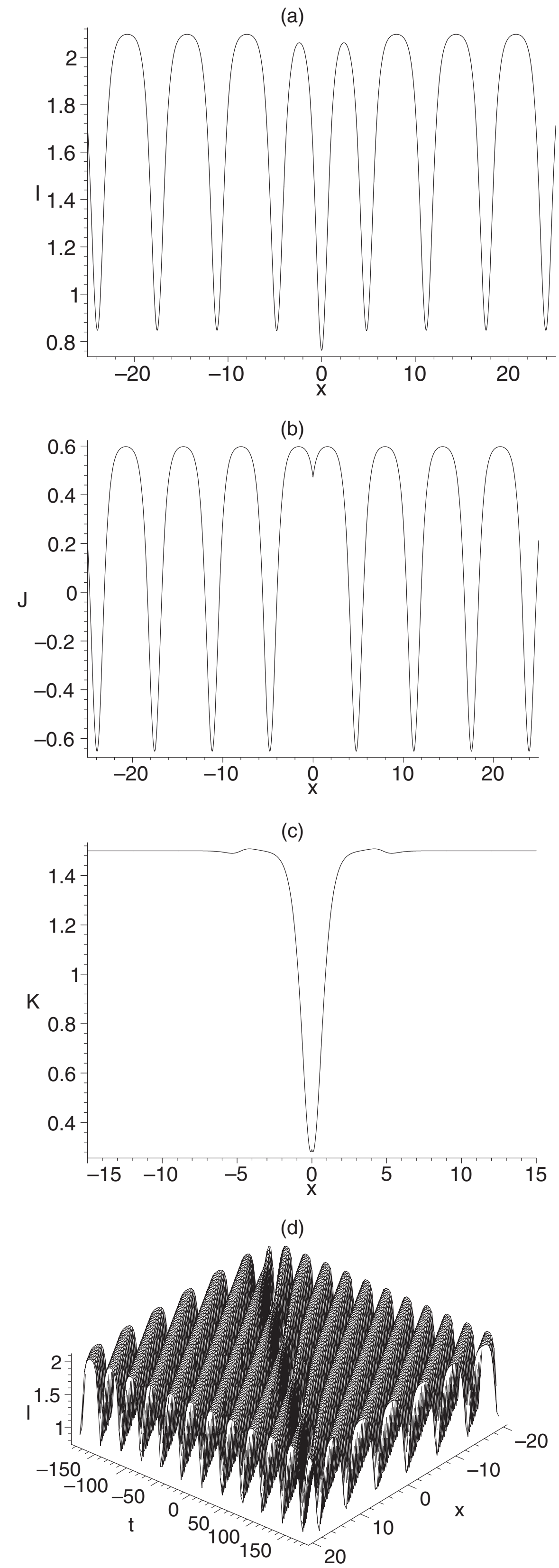

FIG. 2. The second type of special soliton-cnoidal wave interaction solution for the NLS system given by Eq. (39) with the parameter selections Eqs. (75) and (76): (a–c) a dark soliton dressed by a cnoidal periodic wave at $t = 0$ for I, J, and K, respectively; (d) the three-dimensional plot with time evolution and phase shifts for every peak of the periodic wave.

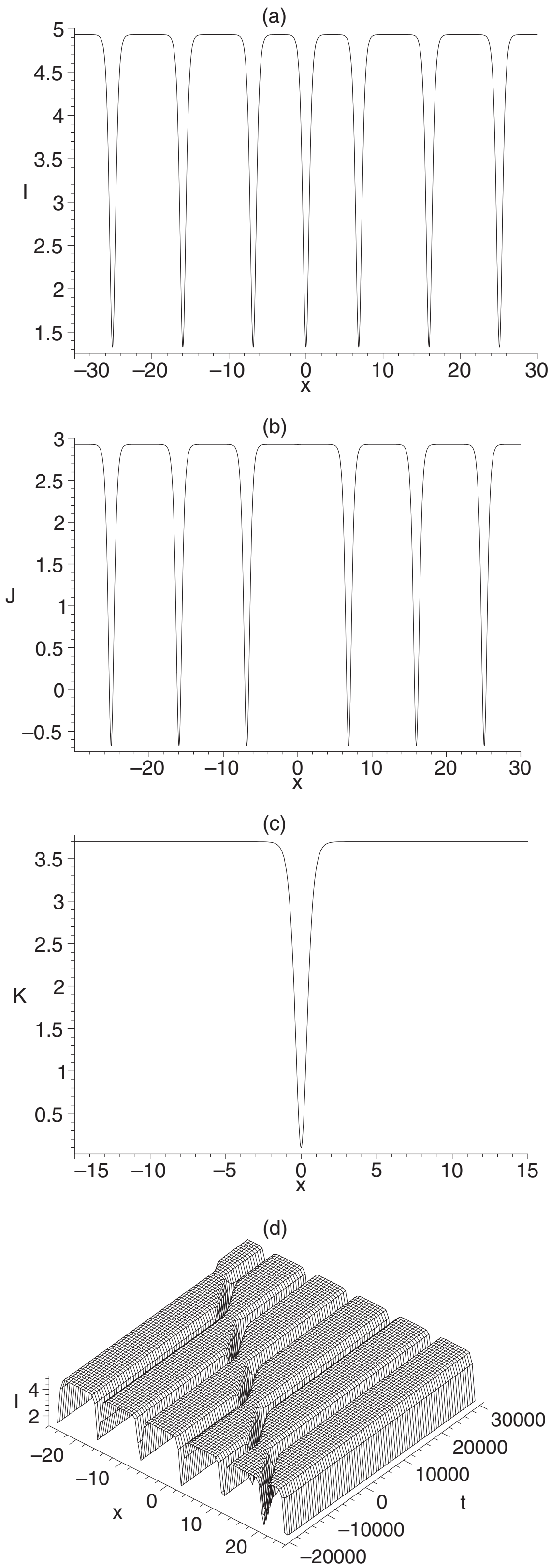

FIG. 3. The third special soliton-cnoidal wave interaction solution for the NLS system given by Eq. (39) with the parameter selections Eq. (77) and especially Eq. (78): the quantities I, J, and $K = I - J$ are displayed in (a–c), respectively, at $t = 0$; (d) the three-dimensional plot of the soliton-cnoidal wave with time evolution and explicit phase shifts for all peaks of the periodic wave.

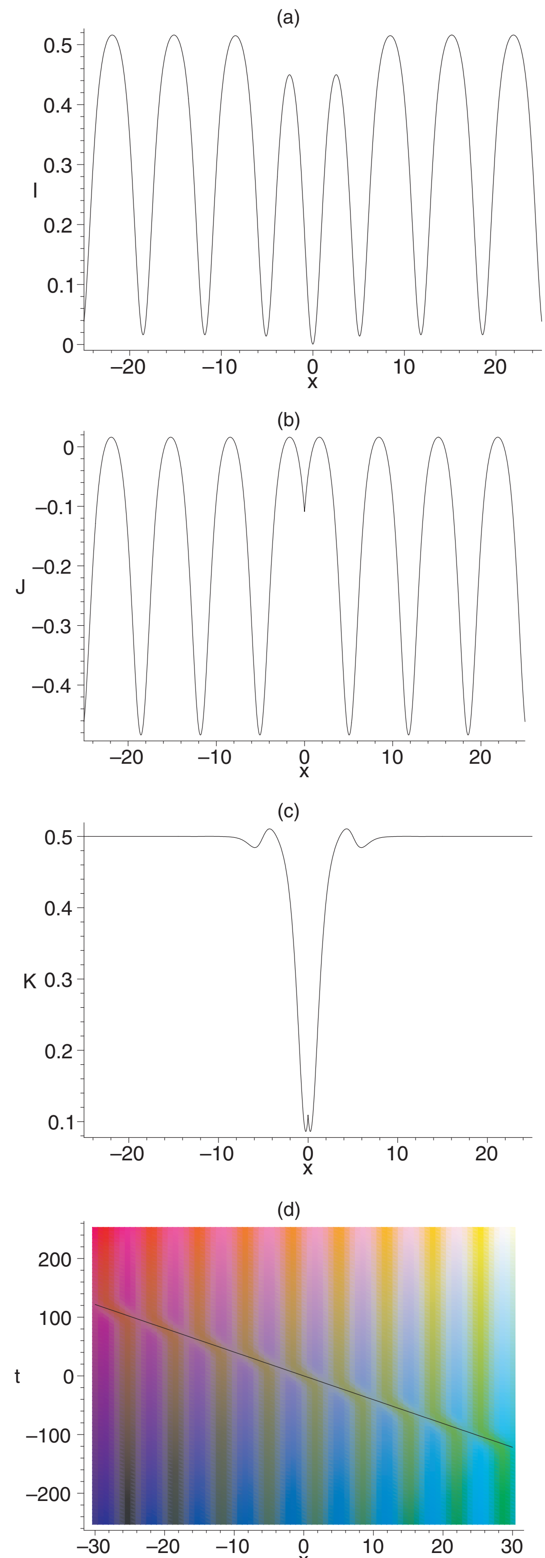

FIG. 4. (Color online) The fourth special soliton-cnoidal wave interaction solution for the NLS system given by Eq. (39) with the parameter selections Eqs. (79) and (80): (a–c) exhibit the detailed structure of the gray soliton dressed by a static cnoidal periodic wave at $t = 0$ for the quantities I, J, and K, respectively; (d) the density plot for the gray soliton with the interaction by a static cnoidal periodic wave. The black straight line denotes $x - v_s t = 0$, displaying the correctness of the soliton velocity Eq. (66).

Figure 4(a) shows the wave interaction structure of I at $t = 0$, while Fig. 4(d) is a density plot of Eq. (39) in addition to a straight line plot of $x - v_s t = 0$. Figures 4(b) and 4(c) exhibit the structure of the related cnoidal wave J and the soliton K, respectively.

VI. SUMMARY AND DISCUSSIONS

Though the square eigenfunction symmetry Eq. (12) is nonlocal for the original AKNS system, it can be localized for the extended system of Eqs. (2), (10), (11), and (23). The localization method is valid for other integrable systems. The extended AKNS system is studied by symmetry reductions with localized symmetries that are nonlocal for the original system. Especially, the soliton-cnoidal wave interaction solutions are explicitly expressed by the Jacobi elliptic functions and the third type of incomplete elliptic integral. The reduction solutions exhibit the interactions between solitons and other NLS waves such as the Painlevé IV waves determined by Eq. (45), rational waves given in Eq. (47), and periodic waves. For simplicity and analyticity, only the soliton-cnoidal wave interaction solutions are discussed in detail.

The soliton-cnoidal wave interaction solutions display some interesting physical phenomena. It is demonstrated that the interactions between the soliton and the background waves are elastic with phase shifts. From Figs. 1(a)–1(c), 2(a)–2(c), 3(a)–3(c), and 4(a)–4(c), one can find that at a fixed time the soliton-cnoidal interaction solution looks like a soliton dressed by a periodic wave. This kind of phenomena may be found in real physical systems, such as the Fermionic quantum plasma [30] and the unmagnetized plasma system where ion-acoustic and electric-plasma waves are simultaneously excited. The soliton structures dressed by periodic waves have also been observed in both experiments [31] and numerical simulations [30,32]. It is noted that the present special type of solutions is established by means of the symmetry reduction method; however, it is also obtainable by other methods, such as the CRE/CTE (consistent Riccati expansion and consistent tanh expansion) method and the truncated Painlevé expansion approach [33]. In addition, Fig. 4(d) indicates another potential application of the soliton-cnoidal interaction solution is to offer a new possible mechanism to produce the controllable routing switches in optical information and communications.

It is remarkable that the method and what we obtained here are valid to various integrable models. The details on the method for other nonlinear systems, other types of interacting wave solutions, other methods to solve interaction solutions between different types of nonlinear excitations, other possible new physical applications, and so on, will be reported in our future research work.

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