Many-body interaction in fast soliton collisions

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We study n-pulse interaction in fast collisions of N solitons of the cubic nonlinear Schrödinger (NLS) equation in the presence of generic weak nonlinear loss. We develop a generalized reduced model that yields the contribution of the n-pulse interaction to the amplitude shift for collisions in the presence of weak $(2m+1)$-order loss, for any n and m. We first employ the reduced model and numerical solution of the perturbed NLS equation to analyze soliton collisions in the presence of septic loss ($m = 3$). Our calculations show that the three-pulse interaction gives the dominant contribution to the collision-induced amplitude shift already in a full-overlap four-soliton collision, and that the amplitude shift strongly depends on the initial soliton positions. We then extend these results for a generic weak nonlinear loss of the form $G(|\psi|^2)\psi$, where ψ is the physical field and G is a Taylor polynomial of degree m_c. Considering $m_c = 3$, as an example, we show that three-pulse interaction gives the dominant contribution to the amplitude shift in a six-soliton collision, despite the presence of low-order loss. Our study quantitatively demonstrates that n-pulse interaction with high n values plays a key role in fast collisions of NLS solitons in the presence of generic nonlinear loss. Moreover, the scalings of n-pulse interaction effects with n and m and the strong dependence on initial soliton positions lead to complex collision dynamics, which is very different from that observed in fast NLS soliton collisions in the presence of cubic loss.

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I. INTRODUCTION

The problem of predicting the dynamic evolution of N physical interacting objects or quantities, commonly known as the N-body problem, is an important subject of research in science and engineering. The study of this problem plays a key role in many fields, including celestial mechanics [1,2], nuclear physics, solid-state physics, and molecular physics [3]. In many cases, the dynamics of the N objects is governed by a force which is a sum over two-body forces. This is the situation in celestial mechanics [1,2] and in other systems [3], and it has been discussed extensively in the literature. A different but equally interesting dynamic scenario emerges when the N-body dynamics is determined by a force involving n-body interaction with $n \geqslant 3$ [4]. Indeed, n-body forces with $n \geqslant 3$ have been employed in a variety of problems including van der Waals interaction between atoms [5], interaction between nucleons in atomic nuclei [6–9], and in cold atomic gases in optical lattices [10–12]. A fundamental question in these studies concerns the physical mechanisms responsible for the emergence of n-body interaction with a given n value. A second important question revolves around the dependence of the interaction strength on n and on the other physical parameters. In the current study we investigate a different class of N-body problems, in which n-body forces play a dominant role. More specifically, we study the role of n-body interaction in fast collisions between N solitons of the cubic nonlinear Schrödinger (NLS) equation in the presence of generic weak nonlinear loss. In this case the solitons experience significant collision-induced amplitude shifts, and important questions arise regarding the role of n-pulse interaction in the process, and the dependence of the amplitude shift and the n-pulse interaction on the physical parameters.

The NLS equation is one of the most widely used nonlinear wave models in the physical sciences. It was successfully employed to describe a large variety of physical systems, including water waves [13,14], Bose-Einstein condensates [15,16], pulse propagation in optical waveguides [17,18], and nonlinear waves in plasma [19–21]. The most common solutions of the NLS equation are the fundamental solitons. The dynamics of fundamental solitons in these systems can be affected by loss, which is often nonlinear [22]. Nonlinear loss arises in optical waveguides due to gain or loss saturation or multiphoton absorption [23]. In fact, M-photon absorption with $3 \leqslant M \leqslant 5$ has been the subject of intensive theoretical and experimental research in recent years due to a wide variety of potential applications, including lasing, optical limiting, laser scanning microscopy, material processing, and optical data storage [24–32]. More specifically, strong four-photon and five-photon absorption were recently observed in a variety of experimental setups [25,28,31,32], while optical soliton generation and propagation in the presence of two-photon and three-photon absorption was experimentally demonstrated in several recent works [33–37]. It should be emphasized that nonlinear loss is also quite common in other physical systems that can support soliton pulses, including Bose-Einstein condensates [38,39] and systems described by the complex Ginzburg-Landau equation [40]. It is therefore important to study the impact of nonlinear loss on the propagation and dynamics of NLS solitons.

The main effect of weak nonlinear loss on the propagation of a single NLS soliton is a continuous decrease in the soliton's energy. This single-pulse amplitude shift is qualitatively similar to the one due to linear loss, and can be calculated in a straightforward manner by employing the standard adiabatic perturbation theory. Nonlinear loss also strongly affects the collisions of NLS solitons, by causing an additional decrease of soliton amplitudes. The character of this collision-induced amplitude shift was recently studied in Refs. [41,42] for fast soliton collisions in the presence of cubic and quintic loss [43]. The results of these studies indicate that the amplitude dynamics in soliton collisions in the presence of generic nonlinear loss might be quite complicated due to n-pulse interaction effects. More specifically, in Ref. [41] it was shown that the total collision-induced amplitude shift in a fast three-soliton collision in the presence of *cubic* loss is given by a sum over amplitude shifts due to two-pulse interaction, i.e., the contribution to the amplitude shift from three-pulse interaction is negligible. In contrast, In Ref. [42] it was found that three-pulse interaction enhances the amplitude shift in a fast three-soliton collision in the presence of *quintic* loss by a factor of 1.38.

The results of Ref. [42] indicate that n-pulse interaction with $n \geqslant 3$ might play an important role in fast NLS soliton collisions in the presence of generic or high-order nonlinear loss. However, the study in Ref. [42] was rather limited, in the sense that only two- and three-soliton collisions were studied and the effects of n-pulse interaction with $n > 3$ were not considered. In addition, the scalings of the amplitude shifts with the parameter m, characterizing the order of the loss, were not systematically analyzed and dependences on initial soliton positions and phase differences were not treated. Thus, a systematic analytic or numerical study of the role of n-pulse interaction in fast soliton collisions in the presence of generic weak nonlinear loss is still missing. In the current study we address this important problem. For this purpose, we first develop a general reduced model for amplitude dynamics, which allows us to calculate the contribution of n-pulse interaction to the amplitude shift for collisions in the presence of weak $(2m+1)$-order loss, for any n and m. We then use the reduced model and numerical solution of the perturbed NLS equation to analyze soliton collisions in the presence of septic loss ($m = 3$). Our calculations show that three-pulse interaction gives the dominant contribution to the collision-induced amplitude shift already in a full-overlap four-soliton collision, while both three-pulse and four-pulse interaction are important in a six-soliton collision. Furthermore, we find that the amplitude shift is insensitive to the initial intersoliton phase differences, but strongly depends on the initial soliton positions, with a pronounced maximum in the case of full-overlap collisions. We then generalize these results for generic weak nonlinear loss of the form $G(|\psi|^2)\psi$, where ψ is the physical field and G is a Taylor polynomial of degree m_c. We consider $m_c = 3$, as an example. That is, we take into account the effects of linear, cubic, quintic, and septic loss on the collision. We show that in this case three-pulse interaction gives the dominant contribution to the amplitude shift in a six-soliton collision, despite the presence of linear and cubic loss. Our study presents a generalized reduced model for amplitude dynamics in fast collisions of NLS solitons in the presence of weak nonlinear loss, which allows us to systematically characterize the scalings of the collision-induced amplitude shifts. Analysis with the reduced model along with numerical solution of the perturbed NLS equation show that n-body interaction plays a key role in the collisions. Moreover, the scalings of n-pulse interaction effects with n and m and the strong dependence on initial positions lead to complex collision dynamics. This dynamics is very different from that encountered in fast N-soliton collisions in the presence of weak cubic loss, where the total collision-induced amplitude shift is a sum over amplitude shifts due to two-pulse interaction [41].

The rest of the paper is organized as follows. In Sec. II, we obtain the generalized reduced model for amplitude dynamics in a fast N-soliton collision in the presence of weak nonlinear loss. We then employ the model to calculate the total collision-induced amplitude shift and the contribution from n-soliton interaction. In Sec. III, we analyze in detail the predictions of the reduced model for the amplitude shifts in four-soliton and six-soliton collisions. In addition, we compare the analytic predictions with results of numerical simulations with the perturbed NLS equation. In Sec. IV, we present our conclusions. The Appendix is devoted to the derivation of the equation for the collision-induced change in the soliton's envelope due to n-pulse interaction in a fast N-soliton collision.

II. AMPLITUDE DYNAMICS IN N-SOLITON COLLISIONS

Consider propagation of soliton pulses of the cubic NLS equation in the presence of generic weak nonlinear loss $L(\psi)$, where ψ is the physical field. In the context of propagation of light through optical waveguides, for example, ψ is proportional to the envelope of the electric field. Assume that $L(\psi)$ can be approximated by $G(|\psi|^2)\psi$, where G is a Taylor polynomial of degree m_c. Thus, we can write

$$L(\psi) \simeq G(|\psi|^2)\psi = \sum_{m=0}^{m_c} \epsilon_{2m+1}|\psi|^{2m}\psi, \tag{1}$$

where $0 \leqslant \epsilon_{2m+1} \ll 1$ for $m \geqslant 0$. We refer to the mth summand on the right-hand side of Eq. (1) as $(2m+1)$-order loss and note that it is often associated with $(m+1)$-photon absorption [23]. Under the aforementioned assumption on the loss, the dynamics of the pulses is governed by

$$i\partial_z\psi + \partial_t^2\psi + 2|\psi|^2\psi = -i\sum_{m=0}^{m_c} \epsilon_{2m+1}|\psi|^{2m}\psi. \tag{2}$$

Here we adopt the notation used in nonlinear optics, in which z is the propagation distance and t is time. The fundamental soliton solution of the unperturbed NLS equation with central frequency β_j is

$$\psi_j(t,z) = \eta_j \frac{\exp(i\chi_j)}{\cosh(x_j)}, \tag{3}$$

where $x_j = \eta_j(t - y_j - 2\beta_j z)$, $\chi_j = \alpha_j + \beta_j(t - y_j) + (\eta_j^2 - \beta_j^2)z$, and η_j, y_j, and α_j are the soliton amplitude, position, and phase, respectively.

The effects of the nonlinear loss on single-pulse propagation can be calculated by employing the standard adiabatic perturbation theory [17]. This perturbative calculation yields the following expression for the rate of change of the soliton amplitude:

$$\frac{d\eta_j(z)}{dz} = -\sum_{m=0}^{m_c} \epsilon_{2m+1}a_{2m+1}\eta_j^{2m+1}(z), \tag{4}$$

where $a_{2m+1} = (2^{m+1}m!)/[(2m+1)!!]$. The z dependence of the soliton amplitude is obtained by integration of Eq. (4).

Let us discuss the calculation of the effects of weak nonlinear loss on a fast collision between N NLS solitons. The solitons are identified by the index j, where $1 \leqslant j \leqslant N$. Since we deal with a fast collision, $|\beta_j - \beta_k| \gg 1$ for any $j \neq k$. The only other assumption of our calculation is that $0 \leqslant \epsilon_{2m+1} \ll 1$ for $m \geqslant 0$. Under these assumptions, we can employ a generalization of the perturbation technique, developed in Ref. [44], and successfully applied for studying fast two-soliton and three-soliton collisions in different setups [41,42,44–50]. Note that the generalized technique in the current paper is more complicated than the one used in Refs. [41,42,44–50]. We therefore provide a brief outline of the main steps in the generalized calculation. (1) We first consider the effects of $(2m+1)$-order loss, and calculate the contribution of n-soliton interaction with $n \leqslant m+1$ to the collision-induced amplitude shift, for a given n-soliton combination [51]. (2) We then add the contributions coming from all possible n-soliton combinations. This sum is the total contribution of n-pulse interaction to the amplitude shift in a fast collision in the presence of $(2m+1)$-order loss. (3) Summing the amplitude shifts calculated in (2) over all relevant m values, $1 \leqslant m \leqslant m_c$, we obtain the total contribution of n-pulse interaction to the amplitude shift in a collision in the presence of *generic nonlinear loss*. (4) The total collision-induced amplitude shift is obtained by summing the amplitude shifts in (3) over all possible n values, $2 \leqslant n \leqslant m+1$.

Following this procedure, we first calculate the collision-induced change in the amplitude of the jth soliton due to $(2m+1)$-order loss. The dynamics is determined by the following perturbed NLS equation:

$$i\partial_z\psi + \partial_t^2\psi + 2|\psi|^2\psi = -i\epsilon_{2m+1}|\psi|^{2m}\psi. \tag{5}$$

We start by considering the amplitude shift of the jth soliton due to n-pulse interaction with solitons with indices $l_1, l_2, \dots, l_{n-1}$, where $1 \leqslant l_{j'} \leqslant N$ and $l_{j'} \neq j$ for $1 \leqslant j' \leqslant n-1$. Employing a generalization of the perturbation method developed in Ref. [44], we look for an n-pulse solution of Eq. (5) in the form $\psi_n = \psi_j + \phi_j + \sum_{j'=1}^{n-1}[\psi_{l_{j'}} + \phi_{l_{j'}}] + \cdots$, where ψ_k is the kth single-soliton solution of Eq. (5) with $0 < \epsilon_{2m+1} \ll 1$, ϕ_k describes collision-induced effects for the kth soliton, and the ellipsis represents higher-order terms. We then substitute ψ_n along with $\psi_j(t,z) = \Psi_j(x_j)\exp(i\chi_j)$, $\phi_j(t,z) = \Phi_j(x_j)\exp(i\chi_j)$, $\psi_{l_{j'}}(t,z) = \Psi_{l_{j'}}(x_{l_{j'}})\exp(i\chi_{l_{j'}})$, and $\phi_{l_{j'}}(t,z) = \Phi_{l_{j'}}(x_{l_{j'}})\exp(i\chi_{l_{j'}})$, for $j' = 1, \dots, n-1$, into Eq. (5). Since the frequency difference for each soliton pair is large, we can employ the resonant approximation, and neglect terms with rapid oscillations with respect to z. Under this approximation, Eq. (5) decomposes into a system of equations for the evolution of Φ_j and the $\Phi_{l_{j'}}$. (See, for example, Refs. [41,42], for a discussion of the cases $n=2$ and $n=3$ for $m=1$ and $m=2$.) The system of equations is solved by expanding Φ_j and each of the $\Phi_{l_{j'}}$ in a perturbation series with respect to ϵ_{2m+1} and $1/|\beta_{l_{j'}} - \beta_j|$. We focus attention on Φ_j and comment that the equations for the $\Phi_{l_{j'}}$ are obtained in a similar manner. The only collision-induced effect in order $1/|\beta_{l_{j'}} - \beta_j|$ is a phase shift $\Delta\alpha_j = 4\sum_{j'=1}^{n-1}\eta_{l_{j'}}/|\beta_{l_{j'}} - \beta_j|$, which already exists in the unperturbed collision [52]. Thus, we find that the main effect of $(2m+1)$-order loss on the collision is of order $\epsilon_{2m+1}/|\beta_{l_{j'}} - \beta_j|$. We denote the corresponding term in the expansion of Φ_j by $\Phi_{j2}^{(1m)}$, where the first subscript stands for the soliton index, the second subscript indicates the combined order with respect to both ϵ_{2m+1} and $1/|\beta_{l_{j'}} - \beta_j|$, and the superscripts represent the order in ϵ_{2m+1} and the order of the nonlinear loss, respectively. Furthermore, the contribution to $\Phi_{j2}^{(1m)}$ from n-soliton interaction with the $l_1, l_2, \dots, l_{n-1}$ solitons is denoted by $\Phi_{j2(l_1,\dots,l_{n-1})}^{(1mn)}$. In the Appendix, we show that the latter contribution satisfies

$$\partial_z\Phi_{j2(l_1,\dots,l_{n-1})}^{(1mn)} = -\epsilon_{2m+1}\sum_{k_{l_1}=1}^{m-(n-2)}\ \sum_{k_{l_2}=1}^{m-k_{l_1}-(n-3)}\cdots\sum_{k_{l_{n-1}}=1}^{m-s_{n-2}}\frac{m!(m+1)!}{(k_{l_1}!\cdots k_{l_{n-1}}!)^2}$$
$$\times[(m+1-s_{n-1})!(m-s_{n-1})!]^{-1}|\Psi_{l_1}|^{2k_{l_1}}\cdots|\Psi_{l_{n-1}}|^{2k_{l_{n-1}}}|\Psi_j|^{2m-2s_{n-1}}\Psi_j, \tag{6}$$

where $s_n = \sum_{j'=1}^{n} k_{l_{j'}}$. Note that all terms in the sum on the right-hand side of Eq. (6) contain the products $|\Psi_{l_1}|^{2k_{l_1}}\cdots|\Psi_{l_{n-1}}|^{2k_{l_{n-1}}}|\Psi_j|^{2k_j}\Psi_j$, where $k_{l_1} + \cdots + k_{l_{n-1}} + k_j = m$, and $1 \leqslant k_{l_{j'}} \leqslant m-(n-2)$ for $1 \leqslant j' \leqslant n-1$. Therefore, the largest value of n that can induce nonvanishing effects is obtained by setting $k_j = 0$ and $k_{l_{j'}} = 1$ for $1 \leqslant j' \leqslant n-1$. This yields $n_{\max} = m+1$ for the maximum value of n.

Next, we obtain the equation for the rate of change of the jth soliton's amplitude due to n-pulse interaction with the $l_1, l_2, \dots, l_{n-1}$ solitons. For this purpose, we first expand both sides of Eq. (6) with respect to the eigenmodes of the linear operator $\hat{L}$ describing small perturbations about the fundamental NLS soliton [41,42,44–46]. We then project the two expansions onto the eigenmode $f_0(x_j) = \mathrm{sech}(x_j)(1,-1)^T$ and integrate over x_j. This calculation yields the following equation for the rate of change of the amplitude:

$$\frac{d\eta_{j(l_1,\dots,l_{n-1})}^{(mn)}}{dz} = -\epsilon_{2m+1}\sum_{k_{l_1}=1}^{m-(n-2)}\cdots\sum_{k_{l_{n-1}}=1}^{m-s_{n-2}}\frac{m!(m+1)!\eta_{l_1}^{2k_{l_1}}\cdots\eta_{l_{n-1}}^{2k_{l_{n-1}}}\eta_j^{2m-2s_{n-1}+1}}{(k_{l_1}!\cdots k_{l_{n-1}}!)^2(m+1-s_{n-1})!(m-s_{n-1})!}$$
$$\times\int_{-\infty}^{\infty}dx_j[\cosh(x_{l_1})]^{-2k_{l_1}}\cdots[\cosh(x_{l_{n-1}})]^{-2k_{l_{n-1}}}[\cosh(x_j)]^{-(2m-2s_{n-1}+2)}. \tag{7}$$

We now proceed to the second calculation step, in which we obtain the total rate of change in the jth soliton's amplitude due to n-pulse interaction in a fast N-soliton collision in the presence of $(2m+1)$-order loss. For this purpose, we sum

Eq. (7) over all n-soliton combinations $(j,l_1,\dots,l_{n-1})$, where $1 \leqslant l_{j'} \leqslant N$, $l_{j'} \neq j$, and $1 \leqslant j' \leqslant n-1$. Thus, the total rate of change of the amplitude due to n-pulse interaction is

$$\frac{d\eta_j^{(nm)}}{dz} = \sum_{l_1=1}^{N}\sum_{l_2=l_1+1}^{N}\cdots\sum_{l_{n-1}=l_{n-2}+1}^{N}\Pi_{j'=1}^{n-1}\left(1-\delta_{jl_{j'}}\right) \times \frac{d\eta^{(mn)}_{j(l_1,\dots,l_{n-1})}}{dz}, \tag{8}$$

where δ_{jk} is the Kronecker delta function. The total rate of change in the jth soliton's amplitude in an N-soliton collision in the presence of *the generic nonlinear loss* due to n-soliton interaction is calculated by summing both sides of Eq. (8) over m for $n-1 \leqslant m \leqslant m_c$. This yields

$$\frac{d\eta_j^{(n)}}{dz} = \sum_{m=n-1}^{m_c}\frac{d\eta_j^{(mn)}}{dz}. \tag{9}$$

To obtain the total rate of change of the amplitude in the collision, we sum Eq. (9) over n for $2 \leqslant n \leqslant m_c+1$, and also take into account the effects of single-pulse propagation, as described by Eq. (4). We arrive at the following equation:

$$\frac{d\eta_j}{dz} = \sum_{n=2}^{m_c+1}\frac{d\eta_j^{(n)}}{dz} - \sum_{m=0}^{m_c}\epsilon_{2m+1}a_{2m+1}\eta_j^{2m+1}, \tag{10}$$

for $j=1,\dots,N$. Equations (7)–(10) provide the generalized reduced model for amplitude dynamics in fast collisions of N NLS solitons. The model can be employed to obtain the contribution of n-pulse interaction to the collision-induced amplitude shifts for any values of n, m, and m_c. Furthermore, it can be used for both full-overlap collisions, in which the envelopes of all N solitons overlap at a certain distance z_c, and for more general collisions, in which the solitons' envelopes do not fully overlap. In this sense the reduced model given by Eqs. (7)–(10) is a major generalization of the reduced models in Refs. [41,42,44–50], which were limited to full-overlap collisions and to n-pulse interaction with $n=2$ [41,42,44–48,50] or $2 \leqslant n \leqslant 3$ [49].

Useful insight into the effects of n-pulse interaction on the collisions can be gained by studying full-overlap collisions. More specifically, we would like to calculate the total collision-induced amplitude shift $\Delta\eta_j$ in these collisions, and compare it with the contributions of n-pulse interaction to the amplitude shift $\Delta\eta_j^{(n)}$, for $n=2,\dots,m_c+1$. For this purpose, we consider first a full-overlap N-soliton collision in the presence of $(2m+1)$-order loss. The rate of change in the jth soliton's amplitude due to n-pulse interaction with solitons with indices $l_1,l_2,\dots,l_{n-1}$, where $1 \leqslant l_{j'} \leqslant N$ and $l_{j'} \neq j$ for $1 \leqslant j' \leqslant n-1$, is given by Eq. (7). In a fast full-overlap collision in the presence of weak $(2m+1)$-order loss, the main contribution to the amplitude shift comes from the close vicinity of the collision point z_c. Therefore, an approximate expression for the contribution of n-pulse interaction to the amplitude shift can be obtained by integrating Eq. (7) over z from $-\infty$ to ∞, while taking the amplitude values on the right-hand side of the equation as constants [53]: $\eta_k = \eta_k(z_c^-)$. Employing these steps, we arrive at

$$\Delta\eta^{(mn)}_{j(l_1,\dots,l_{n-1})} = -\epsilon_{2m+1}\sum_{k_{l_1}=1}^{m-(n-2)}\cdots\sum_{k_{l_{n-1}}=1}^{m-s_{n-2}}\frac{m!(m+1)!\eta_{l_1}^{2k_{l_1}}\cdots\eta_{l_{n-1}}^{2k_{l_{n-1}}}\eta_j^{2m-2s_{n-1}+1}}{\left(k_{l_1}!\cdots k_{l_{n-1}}!\right)^2(m+1-s_{n-1})!(m-s_{n-1})!} \times \int_{-\infty}^{\infty}dx_j[\cosh(x_j)]^{-(2m-2s_{n-1}+2)}\int_{-\infty}^{\infty}dz\left[\cosh\left(x_{l_1}\right)\right]^{-2k_{l_1}}\cdots\left[\cosh\left(x_{l_{n-1}}\right)\right]^{-2k_{l_{n-1}}}. \tag{11}$$

The total contribution of n-pulse interaction to the amplitude shift in a fast full-overlap N-soliton collision in the presence of $(2m+1)$-order loss is obtained by summing Eq. (11) over all n-soliton combinations $(j,l_1,\dots,l_{n-1})$:

$$\Delta\eta_j^{(mn)} = \sum_{l_1=1}^{N}\sum_{l_2=l_1+1}^{N}\cdots\sum_{l_{n-1}=l_{n-2}+1}^{N}\Pi_{j'=1}^{n-1}\left(1-\delta_{jl_{j'}}\right)\Delta\eta^{(mn)}_{j(l_1,\dots,l_{n-1})}. \tag{12}$$

Summation of Eq. (12) over m yields the total contribution of n-pulse interaction to the amplitude shift in a full-overlap collision in the presence of the *generic nonlinear loss*:

$$\Delta\eta_j^{(n)} = \sum_{m=n-1}^{m_c}\Delta\eta_j^{(mn)}. \tag{13}$$

Thus, the approximate expression for the total amplitude shift in a fast full-overlap collision is

$$\Delta\eta_j = \sum_{n=2}^{m_c+1}\Delta\eta_j^{(n)}. \tag{14}$$

Note that since Eqs. (7)–(14) are independent of the soliton phases, the total collision-induced amplitude shift and the contribution of n-soliton interaction are expected to be phase insensitive.

III. ANALYSIS AND NUMERICAL SIMULATIONS

The generalized reduced models given by Eqs. (7)–(14) enable a systematic study of n-pulse interaction effects in fast N-soliton collisions. We are especially interested in finding whether n-pulse interaction with $n \geqslant 3$ can give the dominant contribution to the amplitude shift and in analyzing the sensitivity of the amplitude shift to the initial soliton parameters. For this purpose, we analyze the scaling with n of the contribution of n-pulse interaction to the collision-induced amplitude shift. This is done for both collisions in the presence of weak $(2m+1)$-order loss and for collisions in the presence of generic weak nonlinear loss. Furthermore, we investigate the dependence of the total amplitude shift on the initial soliton positions and phases. We note that the reduced models are based on a perturbative approximation, which neglects high-order effects due to radiation emission and collision-induced frequency shifts. For this reason, it is important to check the predictions of the reduced models by comparison with results obtained with the more complete NLS model. In the current section we take this important task by numerically solving the perturbed NLS equations (2) and (5).

We start the analysis by considering the effects of fast full-overlap N-soliton collisions in the presence of $(2m+1)$-order loss, where the dynamics is described by Eq. (5). We first focus attention on collisions in the presence of septic loss ($m=3$), since analysis of this case is sufficient for demonstrating the importance of n-soliton interaction with $n \geqslant 3$. For concreteness, we consider four-soliton and six-soliton collisions with soliton frequencies, $\beta_1 = 0$, $\beta_2 = -\Delta\beta$, $\beta_3 = \Delta\beta$, $\beta_4 = 2\Delta\beta$ for $N=4$, and $\beta_1 = 0$, $\beta_2 = -2\Delta\beta$, $\beta_3 = -\Delta\beta$, $\beta_4 = \Delta\beta$, $\beta_5 = 2\Delta\beta$, $\beta_6 = 3\Delta\beta$ for $N=6$, where $3 \leqslant \Delta\beta \leqslant 40$. To ensure full-overlap collisions with this choice of the β_j, the initial positions are taken as $y_1(0)=0$, $y_2(0)=20$, $y_3(0)=-20$, $y_4(0)=-40$ for $N=4$, and $y_1(0)=0$, $y_2(0)=40$, $y_3(0)=20$, $y_4(0)=-20$, $y_5(0)=-40$, $y_6(0)=-60$ for $N=6$. The initial amplitudes and phases are $\eta_j(0)=1$ and $\alpha_j(0)=0$ for $1 \leqslant j \leqslant N$, respectively. This choice of soliton parameters corresponds, for example, to the one used in optical waveguide links employing wavelength division multiplexing [54]. It should be emphasized, however, that similar behavior is observed for other setups of full-overlap N-soliton collisions, e.g., in setups where the group velocity difference and temporal separation between the j and $j+1$ solitons vary with j. Notice that with the above choice of the initial positions, the solitons are well separated before the collision. In addition, the final propagation distance z_f is taken to be large enough, so that the solitons are well separated after the collision. The value of the septic loss coefficient is taken as $\epsilon_7 = 0.002$.

Figure 1 shows the $\Delta\beta$ dependence of the total collision-induced amplitude shift in four-pulse and six-pulse collisions, for the $j=1$ ($\beta_j = 0$) soliton. Both the prediction of Eqs. (11)–(14) and the result obtained by numerical solution of Eq. (5) are presented. The figure also shows the analytic prediction for the contributions of two-, three-, and four-soliton interaction to the amplitude shift, $\Delta\eta_1^{(2)}$, $\Delta\eta_1^{(3)}$, and $\Delta\eta_1^{(4)}$, respectively. The agreement between the analytic prediction and the numerical simulations is very good for $\Delta\beta \geqslant 15$, where the perturbation description is expected to hold. Moreover, our calculations show that the dominant contribution to the total amplitude shift in a four-soliton collision comes from three-soliton interaction. The contribution from four-soliton interaction increases from 15.9% in a four-soliton collision to 39.4% in a six-soliton collision. Consequently, in a six-soliton collision the effects of three-pulse and four-pulse interaction are both important, while those of two-pulse interaction are relatively small (about 9.6%).

An important prediction of the reduced models presented in Sec. II is the independence of the total collision-induced amplitude shifts and the contributions from n-pulse interaction on the initial soliton phases. In order to check this prediction, we carry out numerical simulations with Eq. (5) for the full-overlap four-soliton and six-soliton collisions in the presence of septic loss, discussed in the previous two paragraphs, with $\epsilon_7 = 0.002$ and $\Delta\beta = 30$. The initial values of soliton

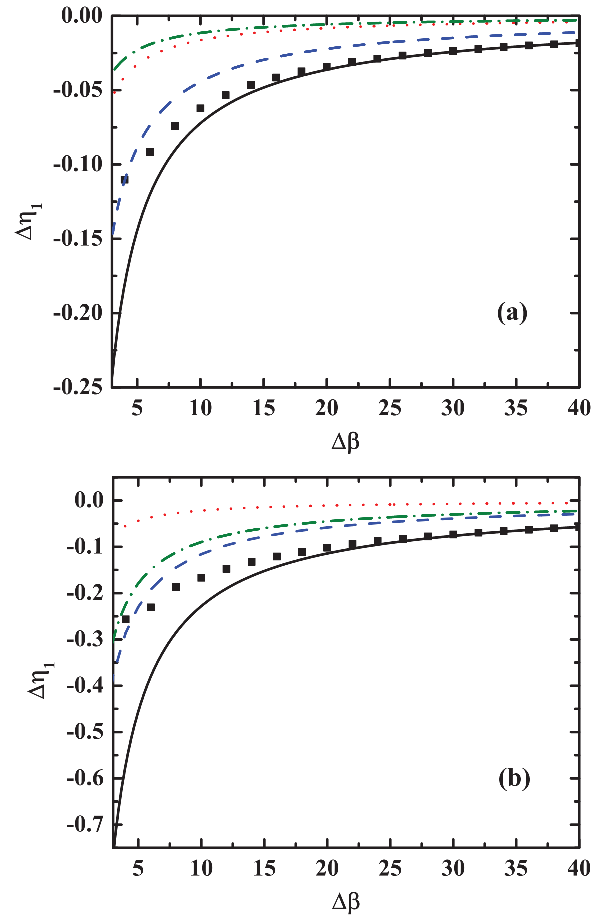

FIG. 1. (Color online) The total collision-induced amplitude shift of the $j=1$ soliton $\Delta\eta_1$ vs frequency difference $\Delta\beta$ in a full-overlap four-soliton collision (a) and in a full-overlap six-soliton collision (b) in the presence of septic loss with coefficient $\epsilon_7 = 0.002$. The solid black line is the analytic prediction of Eqs. (11)–(14) and the squares represent the result of numerical simulations with Eq. (5). The dotted red, dashed blue, and dash-dotted green lines correspond to the contributions of two-, three-, and four-soliton interactions to the amplitude shift, $\Delta\eta_1^{(2)}$, $\Delta\eta_1^{(3)}$, and $\Delta\eta_1^{(4)}$, respectively.

positions and amplitudes are the same as the ones considered in the previous two paragraphs. The initial phases are $\alpha_j(0)=0$ for $j=1,2,4$ and $0 \leqslant \alpha_3(0) \leqslant 2\pi$ for $N=4$, and $\alpha_j(0)=0$ for $j=1,2,3,5,6$ and $0 \leqslant \alpha_4(0) \leqslant 2\pi$ for $N=6$. That is, the initial phase of the soliton with frequency $\beta = 30$, which is denoted by $\alpha_3(0)$ in a four-soliton collision and by $\alpha_4(0)$ in a six-soliton collision, is varied, while the initial phases of the other solitons are not changed. The dependence of the collision-induced amplitude shift of the $j=1$ soliton on the initial position of the $\beta = 30$ soliton is shown in Fig. 2. The agreement between the predictions of the reduced model and numerical simulations with Eq. (5) is excellent for four-soliton collisions and good for six-soliton collisions. In the latter case, the values of $|\Delta\eta_1|$ obtained by simulations with the NLS equation are smaller than the ones predicted by Eqs. (11)–(14). Based on the results presented in Figs. 1 and 2 and similar results obtained for fast full-overlap collisions with other choices of the physical parameters, we conclude that phase-insensitive n-pulse interactions with high n values,

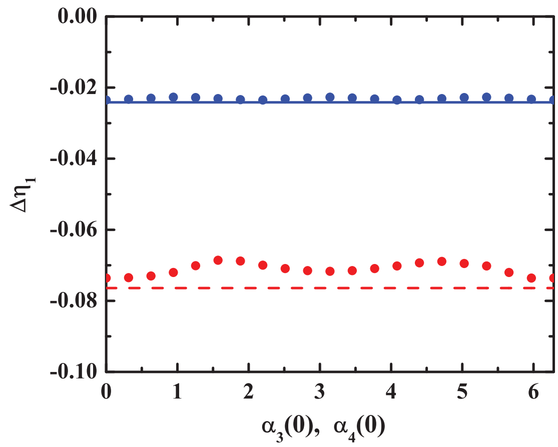

FIG. 2. (Color online) The collision-induced amplitude shift of the $j = 1$ soliton $\Delta\eta_1$ vs the initial phase of the soliton with frequency $\beta = 30$ in full-overlap N-soliton collisions in the presence of septic loss with $\epsilon_7 = 0.002$. The blue (upper) and red (lower) circles represent the results of numerical simulations with Eq. (5) for four-soliton and six-soliton collisions, respectively. The solid blue and dashed red lines correspond to the analytic predictions of Eqs. (11)–(14) for four-soliton and six-soliton collisions. The initial phase of the $\beta = 30$ soliton is denoted by $\alpha_3(0)$ in the four-soliton collision and by $\alpha_4(0)$ in the six-soliton collision.

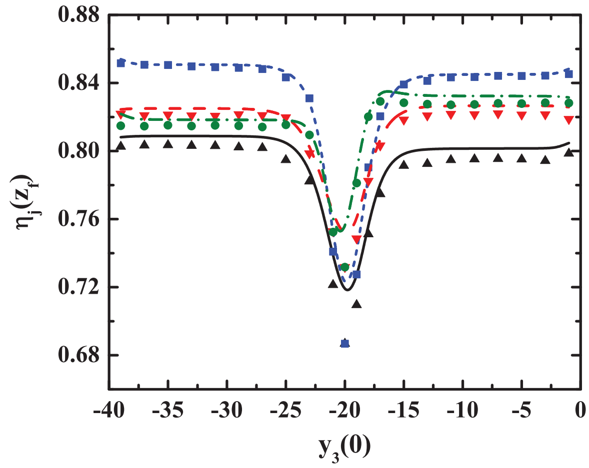

FIG. 3. (Color online) The final soliton amplitudes $\eta_j(z_f)$ vs the initial position of the $j = 3$ soliton $y_3(0)$ in a four-soliton collision in the presence of septic loss with $\epsilon_7 = 0.02$. The solid black curve, dashed red curve, short-dashed blue curve, and dash-dotted green curve represent the analytic predictions of Eqs. (7)–(10) for $\eta_j(z_f)$ with $j = 1,2,3,4$, respectively. The black up triangles, red down triangles, blue squares, and green circles correspond to the results obtained by numerical solution of Eq. (5) for $\eta_j(z_f)$ with $j = 1,2,3,4$, respectively.

satisfying $2 < n \leqslant m + 1$, play a crucial role in fast full-overlap N-soliton collisions in the presence of $(2m + 1)$-order loss.

We now turn to analyze more generic fast N-soliton collisions, in which the solitons' envelopes do not completely overlap. Based on Eq. (7), the contribution of n-pulse interaction to the total amplitude shift should strongly depend on the degree of soliton overlap during the collision, for $n \geqslant 3$, $m \geqslant 2$, and $N \geqslant 3$. Consequently, the total collision-induced amplitude shift might strongly depend on the initial soliton positions in this case. We therefore focus our attention on this dependence. We consider, as an example, a four-soliton collision in the presence of septic loss with $\epsilon_7 = 0.02$, where the soliton frequencies are $\beta_1 = 0$, $\beta_2 = -10$, $\beta_3 = 10$, and $\beta_4 = 20$. The initial amplitudes and phases are $\eta_j(0) = 1$ and $\alpha_j(0) = 0$ for $1 \leqslant j \leqslant 4$. The initial positions are $y_1(0) = 0$, $y_2(0) = 20$, $y_4(0) = -40$, and $-39 \leqslant y_3(0) \leqslant -1$. That is, the initial position of the $j = 3$ soliton is varied, while the initial positions of the other solitons are not changed. Notice that in this setup, the four-soliton collision is *not* a full-overlap collision, except at $y_3(0) = -20$. As a result, Eqs. (11)–(14), which were used in earlier studies of fast soliton collisions, do not apply and the more general reduced model, consisting of Eqs. (7)–(10), should be employed. We therefore solve Eqs. (7)–(10) with the aforementioned initial parameter values for $0 \leqslant z \leqslant z_f$, where $z_f = 6$, and plot the final amplitudes $\eta_j(z_f)$ vs $y_3(0)$. The curves are shown in Fig. 3 along with the curves obtained by numerical solution of Eq. (5). The agreement between the analytic prediction and the simulations result is good. As can be seen, each $\eta_j(z_f)$ vs $y_3(0)$ curve has a pronounced minimum at $y_3(0) = -20$, i.e., at the initial position value of the $j = 3$ soliton corresponding to a full-overlap collision. Thus, a strong dependence of the collision-induced amplitude shift on the initial soliton positions is observed already in a four-soliton collision in the presence of septic loss. This means that the collision-induced amplitude dynamics in fast N-soliton collisions in the presence of weak generic loss can be quite complex due to the dominance of contributions from n-pulse interaction with high n values. This behavior is sharply different from the one encountered in fast N-soliton collisions in the presence of weak cubic loss. In the latter case, the total collision-induced amplitude shift is a sum over contributions from two-pulse interaction, and the collision can be accurately viewed as consisting of a collection of pointwise two-soliton collisions [41].

The analysis of the effects of $(2m + 1)$-order loss on N-soliton collisions is very valuable, since it explains the importance of n-pulse interaction and uncovers the scaling laws for this interaction. However, in most systems one has to take into account the impact of the low-order loss terms, whose presence can enhance the effects of two-pulse interaction. It is therefore important to take into account all the relevant loss terms when analyzing collision-induced dynamics in the presence of generic loss. We now turn to address this aspect of the problem, by considering the effects of generic weak nonlinear loss of the form (1) on fast N-soliton collisions. For concreteness, we assume $m_c = 3$ and loss coefficients $\epsilon_1 = 0.002$, $\epsilon_3 = 0.004$, $\epsilon_5 = 0.006$, and $\epsilon_7 = 0.001$. We also assume full-overlap collisions, but emphasize that the analysis can be extended to treat the general case by the same method described in the preceding paragraph. We consider four-soliton and six-soliton collisions with the same pulse parameters used for full-overlap collisions in the presence of septic loss. Figure 4 shows the $\Delta\beta$ dependence of the total collision-induced amplitude shift in four-soliton and six-soliton collisions for the $j = 1$ soliton, as obtained by

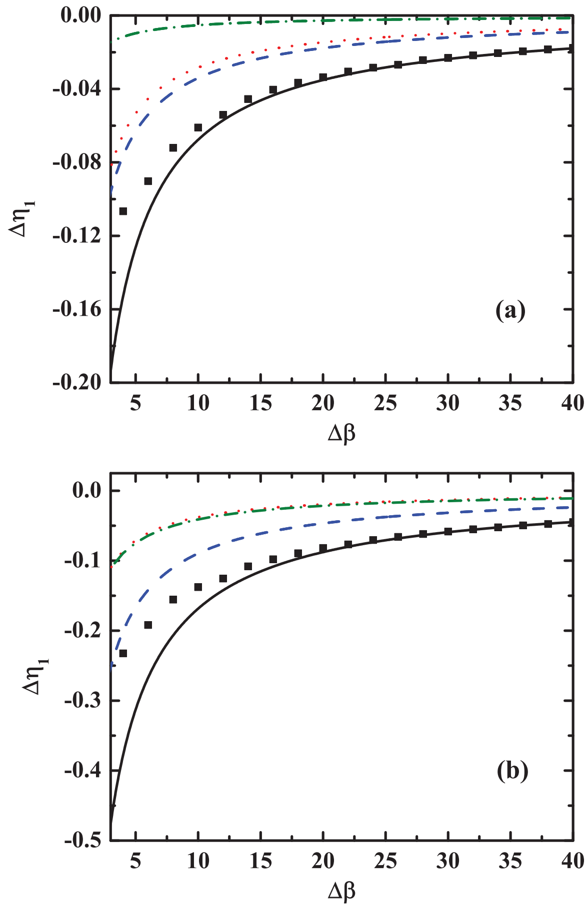

FIG. 4. (Color online) The total collision-induced amplitude shift of the $j = 1$ soliton $\Delta\eta_1$ vs frequency difference $\Delta\beta$ in a full-overlap four-soliton collision (a) and in a full-overlap six-soliton collision (b) in the presence of generic nonlinear loss of the form (1) with $m_c = 3$ and loss coefficients $\epsilon_1 = 0.002$, $\epsilon_3 = 0.004$, $\epsilon_5 = 0.006$, and $\epsilon_7 = 0.001$. The solid black line is the analytic prediction of Eqs. (11)–(14) and the squares correspond to the result of numerical simulations with Eq. (2). The dotted red, dashed blue, and dash-dotted green lines represent the contributions of two-, three-, and four-soliton interactions to the amplitude shift, $\Delta\eta_1^{(2)}$, $\Delta\eta_1^{(3)}$, and $\Delta\eta_1^{(4)}$, respectively.

Eqs. (11)–(14). The result obtained by numerical solution of Eq. (2) and the analytic predictions for the contributions of two-, three-, and four-soliton interactions, $\Delta\eta_1^{(2)}$, $\Delta\eta_1^{(3)}$, and $\Delta\eta_1^{(4)}$, are also shown. We observe that in four-soliton collisions, $\Delta\eta_1^{(2)}$ is comparable to $\Delta\eta_1^{(3)}$, while $\Delta\eta_1^{(4)}$ is much smaller. That is, the inclusion of the low-order loss terms does lead to an enhancement of the fractional contribution of two-pulse interaction to the amplitude shift. In contrast, in six-soliton collisions, $\Delta\eta_1^{(3)}$ (53.2%) is significantly larger than $\Delta\eta_1^{(2)}$ (22.2%), while $\Delta\eta_1^{(4)}$ (24.6%) is comparable to $\Delta\eta_1^{(2)}$. Based on the latter observation, we conclude that when the low-order loss coefficients ϵ_1 and ϵ_3 are comparable in magnitude to the higher-order loss coefficients, the contributions to the amplitude shift from n-pulse interaction with $n \geqslant 3$ can be much larger than that coming from two-pulse interaction.

IV. CONCLUSIONS

In summary, we studied n-pulse interaction in fast collisions of N solitons of the cubic NLS equation in the presence of generic weak nonlinear loss, which can be approximated by the series (1). Due to the presence of nonlinear loss, the solitons experience collision-induced amplitude shifts that are strongly enhanced by n-pulse interaction. We first developed a general reduced model that allowed us to calculate the contribution of n-pulse interaction to the amplitude shift in fast N-soliton collisions in the presence of $(2m + 1)$-order loss, for any n and m. We then used the reduced model and numerical simulations with the perturbed NLS equation to analyze four-soliton and six-soliton collisions in the presence of septic loss ($m = 3$). Our calculations showed that three-pulse interaction gives the dominant contribution to the collision-induced amplitude shift already in a full-overlap four-soliton collision, while in a full-overlap six-soliton collision, both three-pulse and four-pulse interactions are important. Furthermore, we found that the collision-induced amplitude shift has a strong dependence on the initial soliton positions, with a pronounced maximum in the case of a full-overlap collision. We then generalized these results by considering N-soliton collisions in the presence of generic weak nonlinear loss of the form (1) with $m_c = 3$. Our analytic calculations and numerical simulations showed that three-pulse interaction gives the dominant contribution to the amplitude shift in a full-overlap six-soliton collision, despite the presence of linear and cubic loss. All the collision-induced effects were found to be insensitive to the soliton phases for fast collisions. Based on these observations, we conclude that phase-insensitive n-pulse interaction with high n values plays a key role in fast collisions of NLS solitons in the presence of generic weak nonlinear loss. The complex scalings of n-pulse interaction effects with n and m and the strong dependence on initial soliton positions lead to complex collision dynamics. This dynamics is very different from that observed in fast collisions of N NLS solitons in the presence of weak cubic loss, where the total collision-induced amplitude shift is a sum over amplitude shifts due to two-pulse interaction [41].

We conclude by remarking that the analysis carried out in the current paper might have important practical implications. Indeed, a fast N-pulse collision in the presence of generic weak nonlinear loss can be used as an effective mechanism for localized energy transfer from the electromagnetic field to the nonlinear medium. In this process, the dissipative inter-pulse interaction during the collision significantly enhances energy transfer to the medium. Furthermore, the large group velocity difference between the colliding pulses guarantees the localized character of the process. In view of this one might expect that in applications where effective and localized energy transfer between the electromagnetic field and the nonlinear medium is required, a fast N-pulse collision with $N \geqslant 3$ would be a better option compared with a two-pulse collision or singe-pulse propagation.

ACKNOWLEDGMENT

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APPENDIX: DERIVATION OF EQ. (6)

In this Appendix, we derive Eq. (6) for the collision-induced change in the envelope of a soliton due to n-pulse interaction in a fast N-soliton collision in the presence of weak $(2m+1)$-order loss. More specifically, we consider the change in the envelope of the jth soliton induced by n-pulse interaction with solitons with indexes $l_1, l_2, \dots, l_{n-1}$, where $1 \leqslant l_{j'} \leqslant N$ and $l_{j'} \neq j$ for $1 \leqslant j' \leqslant n-1$. The derivation is based on a generalization of the perturbation procedure developed in Ref. [44]. Following this procedure, we look for a solution of Eq. (5) in the form $\psi_n = \psi_j + \phi_j + \sum_{j'=1}^{n-1}[\psi_{l_{j'}} + \phi_{l_{j'}}] + \cdots$, where ψ_k is the kth single-soliton solution of Eq. (5) with $0 < \epsilon_{2m+1} \ll 1$, ϕ_k describes collision-induced effects for the kth soliton, and the ellipsis represents higher-order terms. We then substitute ψ_n along with $\psi_j(t,z) = \Psi_j(x_j)\exp(i\chi_j)$, $\phi_j(t,z) = \Phi_j(x_j)\exp(i\chi_j)$, $\psi_{l_{j'}}(t,z) = \Psi_{l_{j'}}(x_{l_{j'}})\exp(i\chi_{l_{j'}})$, and $\phi_{l_{j'}}(t,z) = \Phi_{l_{j'}}(x_{l_{j'}})\exp(i\chi_{l_{j'}})$ for $j' = 1, \dots, n-1$, into Eq. (5). Next, we use the resonant approximation, and neglect terms with rapid oscillations with respect to z. We find that the main effect of $(2m+1)$-order loss on the envelope of the jth soliton is of order $\epsilon_{2m+1}/|\beta_{l_{j'}} - \beta_j|$. We denote this collision-induced change in the envelope by $\Phi_{j2}^{(1m)}$, and the contribution to this change from n-soliton interaction with the $l_1, l_2, \dots, l_{n-1}$ solitons by $\Phi_{j2(l_1,\dots,l_{n-1})}^{(1mn)}$. Within the resonant approximation, the phase factor of terms contributing to changes in the jth soliton's envelope must be equal to χ_j. Consequently, these terms must be proportional to: $|\Psi_{l_1}|^{2k_{l_1}} \cdots |\Psi_{l_{n-1}}|^{2k_{l_{n-1}}} |\Psi_j|^{2k_j}\Psi_j$, where $k_{l_1} + \cdots + k_{l_{n-1}} + k_j = m$, and $1 \leqslant k_{l_{j'}} \leqslant m-(n-2)$ for $1 \leqslant j' \leqslant n-1$. Summing over all possible contributions of this form, we obtain the following evolution equation for $\Phi_{j2(l_1,\dots,l_{n-1})}^{(1mn)}$:

$$\partial_z \Phi_{j2(l_1,\dots,l_{n-1})}^{(1mn)} = -\epsilon_{2m+1} \sum_{k_{l_1}=1}^{m-(n-2)} \sum_{k_{l_2}=1}^{m-k_{l_1}-(n-3)} \cdots \sum_{k_{l_{n-1}}=1}^{m-s_{n-2}} b_{\mathbf{k}} \left|\Psi_{l_1}\right|^{2k_{l_1}} \cdots \left|\Psi_{l_{n-1}}\right|^{2k_{l_{n-1}}} |\Psi_j|^{2m-2s_{n-1}}\Psi_j, \tag{A1}$$

where $s_n = \sum_{j'=1}^{n} k_{l_{j'}}$, $b_{\mathbf{k}}$ are constants, and $\mathbf{k} = (k_{l_1}, k_{l_2}, \dots, k_{l_{n-1}})$.

To calculate the expansion coefficients $b_{\mathbf{k}}$, we first note that

$$|\Psi|^{2m}\Psi = \left(\Psi_j + \sum_{j'=1}^{n-1}\Psi_{l_{j'}}\right)^{m+1} \left(\Psi_j^* + \sum_{j'=1}^{n-1}\Psi_{l_{j'}}^*\right)^{m}. \tag{A2}$$

Employing the multinomial expansion formula for the two terms on the right-hand side of Eq. (A2), we obtain

$$\left(\Psi_j + \sum_{j'=1}^{n-1}\Psi_{l_{j'}}\right)^{m+1} = \sum_{k_{l_1}=0}^{m+1} \cdots \sum_{k_{l_{n-1}}=0}^{m+1} \frac{(m+1)!}{(k_{l_1}! \cdots k_{l_{n-1}}!)(m+1-s_{n-1})!} \Psi_{l_1}^{k_{l_1}} \cdots \Psi_{l_{n-1}}^{k_{l_{n-1}}} \Psi_j^{m+1-s_{n-1}} \tag{A3}$$

and

$$\left(\Psi_j^* + \sum_{j'=1}^{n-1}\Psi_{l_{j'}}^*\right)^{m} = \sum_{k_{l_1}=0}^{m} \cdots \sum_{k_{l_{n-1}}=0}^{m} \frac{m!}{(k_{l_1}! \cdots k_{l_{n-1}}!)(m-s_{n-1})!} \Psi_{l_1}^{*k_{l_1}} \cdots \Psi_{l_{n-1}}^{*k_{l_{n-1}}} \Psi_j^{*m-s_{n-1}}. \tag{A4}$$

Combining Eqs. (A2)–(A4), we find that the expansion coefficients $b_{\mathbf{k}}$ are given by

$$b_{\mathbf{k}} = \frac{m!(m+1)!}{(k_{l_1}! \cdots k_{l_{n-1}}!)^2 (m+1-s_{n-1})!(m-s_{n-1})!}. \tag{A5}$$

Substituting this relation into Eq. (A1), we arrive at Eq. (6).

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