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Swimming at low Reynolds number in fluids with odd, or Hall, viscosity

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We apply the geometric theory of swimming at low Reynolds number to the study of nearly circular swimmers in two-dimensional fluids with nonvanishing "odd," or Hall, viscosity. The odd viscosity gives an off-diagonal contribution to the fluid stress tensor, which results in a number of striking effects. In particular, we find that a swimmer whose area is changing will experience a torque proportional to the rate of change of the area, with the constant of proportionality given by the coefficient η^o of odd viscosity. After working out the general theory of swimming in fluids with odd viscosity for a class of simple swimmers, we give a number of example swimming strokes which clearly demonstrate the differences between swimming in a fluid with conventional viscosity and a fluid which also has an odd viscosity. We also include a discussion of the extension of the famous Scallop theorem of low Reynolds number swimming to the case where the fluid has a nonzero odd viscosity. A number of more technical results, including a proof of the torque-area relation for swimmers of more general shape, are explained in a set of Appendixes.

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I. INTRODUCTION

The theory of swimming in classical fluids at low Reynolds number [1,2] is remarkable because of the connections it makes between seemingly disparate fields [3]. For example, the motion of swimmers with cyclic swimming strokes is determined purely from classical fluid dynamics, but it can be recast into an elegant geometric formulation reminiscent of Berry's phase physics and gauge fields [3–5]. In fact, the motion of tiny organisms in fluids with high viscosity can be captured by a "gauge theory" of shapes. Since the initial work on the geometric formulation of swimming, there have been generalizations to swimmers in quantum fluids [6] and even to swimmers in fluids on curved spaces [7,8]. The theory has also been successfully applied in practice to describe the swimming of robots [9] and microbots [10,11].

In this article we focus on swimmers in two-dimensional (2D) fluids with broken time-reversal symmetry, for example, fluids in magnetic fields or rotating fluids. We are not interested in the specific source of time-reversal breaking, but instead just consider a classical fluid with a microscopic source of local angular momentum (on a much smaller scale than the size of the swimmer) that gives rise to a nonvanishing "odd" viscosity coefficient [12,13] in addition to the usual isotropic viscosity coefficients. The odd viscosity is an off-diagonal viscosity term that is dissipationless and produces forces perpendicular to the direction of the fluid flow. It can have a quantum mechanical origin in, for example, systems exhibiting the quantum Hall effect [12–22], or a classical origin in plasmas at finite-temperature [23]. In the quantum Hall setting the odd viscosity is usually known as the Hall viscosity. It is also sometimes referred to as Lorentz shear stress.

We will not focus on the microscopic origin of the odd viscosity coefficient, but only assume it to be nonvanishing in conjunction with the usual viscosity coefficients. From this assumption we will determine the motion of swimmers at low Reynolds number in the presence of odd viscosity. Specifically, we will consider the problem of swimmers with circular boundaries that move via deformations of their boundaries analogous to the nearly circular swimmers in Refs. [3,4]. We find a general result that connects the torque on a swimmer to the rate of area change of the swimmer with a proportionality constant given by the odd viscosity. We use our results to give examples of swimmer motion due to cyclic circular deformations and compare cases where the conventional and odd viscosities each dominate. Our paper is organized as follows: We first review the geometric formulation of swimming and the appearance of odd viscosity in 2D fluids with broken time-reversal symmetry. We then go on to derive the general consequences of the odd viscosity on swimmers and then give explicit examples of model swimming strokes that illustrate some differences between fluids with vanishing and nonvanishing odd viscosity. In the last section we consider reciprocal swimming strokes and show how the famous Scallop theorem of low Reynolds number swimming carries over to the case of fluids with odd viscosity. Finally, we have some Appendixes which collect derivations of the more technical results.

II. REVIEW OF GEOMETRIC FORMULATION OF THE SWIMMING PROBLEM

We begin by reviewing the geometric formulation of the problem of swimming at low Reynolds number developed by Shapere and Wilczek [3,4]. The instantaneous rigid motion (translation and rotation) of a swimmer is determined by the condition that the swimmer not be able to exert a net force or torque on itself, and the condition that the fluid velocity vanishes at infinity.

We should first explain why the problem of swimming at low Reynolds number can be formulated in a purely geometric way, independent of the mass of the swimmer or the speed of the swimming stroke (assuming the speed of the stroke is still small enough so that there is no appreciable momentum transfer to the fluid). Recall that the Reynolds number, which is associated with a viscous fluid and an object in motion in that fluid, is a ratio of the inertial and viscous forces on that object [we are not yet considering systems with odd viscosity so in this sentence the word "viscous" refers to the traditional dissipative (even) viscosity of the fluid]. If η^e is the even viscosity coefficient, V is a typical speed of the fluid flow, L is a characteristic dimension of the swimming object, and ρ is the density of the fluid, then the Reynolds number can be expressed as

$$\mathrm{Re} = \frac{\rho V L}{\eta^e}. \tag{2.1}$$

The low Reynolds number regime can be interpreted as the regime where the momentum density of the fluid, ρV, is negligible compared to the scale η^e/L.

At low Reynolds number the drag force on the swimmer is proportional to its velocity. This means that if the swimmer stops its stroke and just coasts through the fluid, its speed will decay exponentially until it comes to a stop. In the low Reynolds number regime this exponential decay is so fast that the motion of the swimmer at any given time can be considered to be completely independent of what the swimmer was doing at all previous times [2]. The motion of the swimmer at time t depends only on its shape and the velocity of its surface at time t. With these remarks in mind we can move on to discuss the geometric theory of swimming at low Reynolds number.

In two dimensions, for swimmers modeled as the interior of deformed circles, we can represent the swimming stroke (the motion of the boundary of the swimmer) by a time-dependent complex function $S_0(\sigma,t)$, $\sigma = e^{i\theta}$, whose real and imaginary parts give the x and y positions of the point on the swimmer described by the parameter $\theta \in [0,2\pi)$ at the time t. When we want to emphasize the dependence of $S_0(\sigma,t)$ on the real parameter θ instead of the complex parameter σ (as we do in Appendix B) we call it $S_0(\theta,t)$ instead.

The function $S_0(\sigma,t)$ lives in a space of "unlocated" shapes, which can be obtained from the space of "located" shapes by partitioning it into equivalence classes $[S_0(\sigma,t)]$ containing all shapes differing only by a rigid motion. The location and orientation of the swimmer in real space is specified by a rigid motion $\mathcal{R}(t)$ acting on a representative of the equivalence class $[S_0(\sigma,t)]$, the simplest choice being $S_0(\sigma,t)$ itself:

$$S(\sigma,t) = \mathcal{R}(t)S_0(\sigma,t). \tag{2.2}$$

To take an example, $S_0(\sigma,t)$ might be the representative of $[S_0(\sigma,t)]$ with its centroid at the origin and a distinguishing feature of the shape aligned with the x axis at time $t = 0$.

To be concrete, let us encode the translation and rotation represented by $\mathcal{R}(t)$ into a 3×3 matrix and let this matrix act on $S_0(\sigma,t)$ represented as a three-dimensional vector with third entry equal to 1,

$$\mathcal{R}(t)S_0(\theta,t) = \begin{pmatrix} \cos(\Theta) & \sin(\Theta) & X \\ -\sin(\Theta) & \cos(\Theta) & Y \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} \mathrm{Re}[S_0(\sigma,t)] \\ \mathrm{Im}[S_0(\sigma,t)] \\ 1 \end{pmatrix}, \tag{2.3}$$

where (X,Y) and Θ are the vector and angle representing the translation and rotation effected by $\mathcal{R}(t)$. The matrix $\mathcal{R}(t)$ is determined by integrating the equation

$$\frac{d\mathcal{R}(t)}{dt} = \mathcal{R}(t)\mathcal{A}(t), \tag{2.4}$$

where the matrix $\mathcal{A}(t)$ determines the infinitesimal rigid motion of the swimmer during a time dt in the sense that $\mathcal{A}(t)\ dt$ *is* the rigid motion of the swimmer during the interval dt. The matrix $\mathcal{A}(t)$ is completely determined by the requirements that the net force and torque on the swimmer vanish and that the fluid velocity goes to zero at infinity. To determine the swimming path we need to find $\mathcal{A}(t)$ for a given swimming stroke and then integrate Eq. (2.4).

Integrating this equation gives the solution for the rigid motion $\mathcal{R}(t)$,

$$\mathcal{R}(t) = \mathcal{R}(0)\bar{P}\exp\left[\int_0^t \mathcal{A}(t')dt'\right], \tag{2.5}$$

where $\bar{P}$ denotes a reverse path-ordering operation. Explicitly, we have

$$\bar{P}\exp\left[\int_0^t \mathcal{A}(t')dt'\right] = I + \int_0^t \mathcal{A}(t_1)dt_1 + \int_0^t \left(\int_0^{t_1} \mathcal{A}(t_2)\mathcal{A}(t_1)dt_2\right)dt_1 + \cdots \tag{2.6}$$

so the matrix $\mathcal{A}(t_i)$ with the latest time t_i appears furthest to the right in each integral, which is the reverse of the usual path ordering operation where the latest time goes furthest to the left in each integral. We show how this integration is carried out numerically in Appendix A.

To see how the idea of a gauge theory of shapes enters we first note that the choice of a representative from the equivalence class $[S_0(\sigma,t)]$ is analogous to a choice of gauge, and the matrix $\mathcal{A}(t)$ plays the role of a gauge potential. If we choose a different representative $\tilde{S}_0(\sigma,t)$, related to $S_0(\sigma,t)$ by a rigid motion $U(t)$ (we can choose a different representative at each time t),

$$\tilde{S}_0(\sigma,t) = U(t)S_0(\sigma,t), \tag{2.7}$$

then the requirement that the rigid motion of the swimmer in real space remain unchanged leads to the transformation law for $\mathcal{R}(t)$,

$$\mathcal{R}(t) \to \mathcal{R}'(t) = \mathcal{R}(t)U^{-1}(t). \tag{2.8}$$

The fact that the transformed gauge potential must satisfy the new differential equation

$$\frac{d\mathcal{R}'(t)}{dt} = \mathcal{R}'(t)\mathcal{A}'(t) \tag{2.9}$$

yields the familiar transformation law

$$\mathcal{A}(t) \to \mathcal{A}'(t) = U(t)\mathcal{A}(t)U^{-1}(t) + U(t)\frac{dU^{-1}(t)}{dt}, \tag{2.10}$$

which shows that $\mathcal{A}(t)$ does indeed transform like a gauge potential.

We can also represent $\mathcal{A}(t)$ in the form of a 3×3 matrix,

$$\mathcal{A}(t) = \begin{pmatrix} 0 & \omega & V_x \\ -\omega & 0 & V_y \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.11}$$

where (V_x, V_y) and ω are the instantaneous linear and angular velocity of the swimmer [so $\mathcal{A}(t)$ is in the Lie algebra of rigid motions]. Sometimes we will refer to the translational and rotational parts $\mathcal{A}_{\rm tr}$ and $\mathcal{A}_{\rm rot}$ of the gauge potential, defined by

$$\mathcal{A}_{\rm tr} = V_x + iV_y, \tag{2.12a}$$

$$\mathcal{A}_{\rm rot} = \omega. \tag{2.12b}$$

The components of $\mathcal{A}(t)$ can be completely determined by solving the equations of motion for Stokes flow of the viscous fluid surrounding the swimmer, subject to no-slip boundary conditions at the surface of the swimmer. Now that we have reviewed the geometric formulation of swimming we will introduce the concept of the odd viscosity in time-reversal breaking fluids.

III. ODD VISCOSITY

We now review the basic definition of odd viscosity and the derivation of the isotropic odd viscosity contribution to the fluid stress tensor in two dimensions. Throughout this section we follow the presentation of Ref. [13] where most of these details were first worked out.

The general linear relation between the fluid stress tensor T_{ij} and the rate of strain tensor $v_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j)$ (v_i are the components of the fluid velocity vector $\mathbf{v}$) is of the form

$$T_{ij} = \eta_{ijkl} v_{kl}. \tag{3.1}$$

The symmetry of the stress and rate of strain tensors imply the symmetry of the viscosity tensor η_{ijkl} under the exchanges $i \leftrightarrow j$ and $k \leftrightarrow l$, but in general η_{ijkl} can contain terms which are symmetric or antisymmetric under the exchange of the pair of indices $\{ij\}$ with the pair of indices $\{kl\}$. We can always split η_{ijkl} into parts which are even and odd under such an exchange by writing $\eta_{ijkl} = \eta^e_{ijkl} + \eta^o_{ijkl}$.

To extract the isotropic contribution to η^o_{ijkl} it is convenient to use a simple basis for representing a real, fourth-rank tensor that is symmetric under exchange of its first two and second two indices. One such basis is provided by the tensor products

$$\sigma^a \otimes \sigma^b,\ a,b \in \{0,1,3\} \tag{3.2}$$

of the Pauli matrices σ^1, σ^3 and the 2×2 identity matrix σ^0, where we have been careful to only use the symmetric matrices. We can expand the viscosity tensor as

$$\eta_{ijkl} = \sum_{a,b=0,1,3} \eta_{ab} \sigma^a_{ij} \sigma^b_{kl} \tag{3.3}$$

and then identify the odd part as

$$\eta^o_{ijkl} = \sum_{a \neq b} \eta^o_{ab} \left(\sigma^a_{ij}\sigma^b_{kl} - \sigma^b_{ij}\sigma^a_{kl}\right). \tag{3.4}$$

In two dimensions the generator of spatial rotations is $i\sigma^2$, where σ^2 is the second Pauli matrix. In an isotropic fluid the viscosity tensor must commute with $\sigma^2 \otimes \sigma^2$ to be rotationally invariant. Using the familiar commutation and anticommutation relations for the Pauli matrices, and the fact that all matrices commute with the identity σ^0, we find that in an isotropic fluid the odd part of the viscosity tensor must have the form

$$\eta^o_{ijkl} = \eta^o\left(\sigma^1_{ij}\sigma^3_{kl} - \sigma^3_{ij}\sigma^1_{kl}\right), \tag{3.5}$$

where the single constant η^o is the coefficient of odd viscosity. Finally we can use the explicit expressions

$$\sigma^1_{ij} = \delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1}, \tag{3.6a}$$

$$\sigma^3_{ij} = \delta_{i1}\delta_{j1} - \delta_{i2}\delta_{j2} \tag{3.6b}$$

for the elements of the Pauli matrices σ^1 and σ^3 to write down the form of the odd viscosity contribution to the stress tensor,

$$\begin{aligned} T^o_{ij} = \eta^o_{ijkl} v_{kl} &= -2\eta^o(\delta_{i1}\delta_{j1} - \delta_{i2}\delta_{j2})v_{12} \\ &\quad + \eta^o(\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1})(v_{11} - v_{22}), \end{aligned} \tag{3.7}$$

which was first obtained in Ref. [13]. For comparison we also display the much more familiar even viscosity part of the stress tensor (for an incompressible fluid)

$$T^e_{ij} = 2\eta^e v_{ij}, \tag{3.8}$$

where η^e is the coefficient of even viscosity.

We see that diagonal elements of T^o_{ij} are proportional to off-diagonal elements of v_{ij} and off-diagonal elements of T^o_{ij} are proportional to diagonal elements of v_{ij}. This atypical relation between the elements of T^o_{ij} and v_{ij} has a number of nonintuitive consequences. For example, a circular object rotating in a fluid with odd viscosity will feel a pressure, directed either radially inwards or outwards depending on the sense of the rotation (see [13] and Fig. 1). This is quite different from what would happen in a fluid with even viscosity only, where a rotating circle would feel a torque that opposes the rotation. The fact that the direction of the pressure force (radially inwards or outwards) on a circle rotating in an odd viscosity fluid depends on the sense of the rotation means that time-reversal symmetry is broken in systems with odd viscosity.

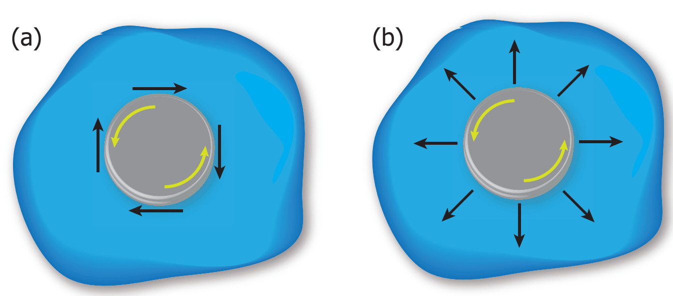

FIG. 1. (Color online) (a) In a fluid with even viscosity only, a rotating circle will feel a torque that opposes its rotation and is proportional to the coefficient of even viscosity η^e. (b) In a fluid with odd viscosity a rotating circle will also feel a pressure directed radially inwards or outwards (depending on the direction of the rotation) and proportional to the coefficient of odd viscosity η^o. The dependence of this pressure force on the direction of the rotation indicates that time-reversal symmetry is broken in systems with nonvanishing odd viscosity.

IV. EQUATIONS OF MOTION, FORCE, AND TORQUE

In classical fluids with both even and odd viscosity Avron has shown (see [13]) that the equations of motion for incompressible Stokes flow (viscous force-dominated flow) are

$$\nabla(p - \eta^o \xi) = \eta^e \nabla^2 \mathbf{v}, \tag{4.1a}$$
$$\nabla \cdot \mathbf{v} = 0, \tag{4.1b}$$

where p is the pressure, $\xi = (\nabla \times \mathbf{v}) \cdot \hat{\mathbf{z}}$ is the vorticity, and η^e and η^o are the coefficients of even and odd viscosity, respectively. We will refer to these equations as the "slow flow" equations, as that is what they are called in the usual case where only even viscosity is present. If $\eta^e \neq 0$, taking the curl of the first equation shows that the vorticity is a harmonic function, i.e., $\nabla^2 \xi = 0$. This means that the stream function ψ (which can be used here because the flow is incompressible), defined by $\mathbf{v} = \nabla \times (\psi \hat{\mathbf{z}})$, is a biharmonic function,

$$\nabla^2(\nabla^2 \psi) = 0. \tag{4.2}$$

In two dimensions we can package the velocity vector $\mathbf{v} = (v_1, v_2)$ into a complex variable $v = v_1 + i v_2$. The solution for v can then be expressed in the complex form (see [4])

$$v = \phi_1(z) - z\overline{\partial_z \phi_1(z)} + \overline{\phi_2(z)}, \tag{4.3}$$

where $z = x + iy = Re^{i\varphi}$, the bar denotes complex conjugation and $\partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$. The functions $\phi_1(z)$ and $\phi_2(z)$ are analytic functions (away from the point $z = 0$, which lies inside the swimmer) with the Laurent series expansions

$$\phi_1(z) = \sum_{k<0} a_k z^{k+1}, \tag{4.4a}$$

$$\phi_2(z) = \sum_{k<-1} b_k z^{k+1}. \tag{4.4b}$$

To solve for the coefficients a_k and b_k we impose no-slip boundary conditions at the surface of the swimmer. Solving for these coefficients can be very difficult for general swimming strokes, so we will focus our attention on a class of simple swimmers introduced in Ref. [4] whose shapes are conformal maps of the circle of degree $\mathcal{D} = 2$. In Appendix C we extend our results to swimmers that are conformal maps of the circle of degree $\mathcal{D} = 3$.

To calculate the force and torque on the swimmer we will need the stress tensor. We have seen in Sec. III that in the presence of odd viscosity the stress tensor gets an extra contribution. The full stress tensor is now

$$\begin{aligned} T_{ij} = &-p\delta_{ij} + 2\eta^e v_{ij} - 2\eta^o(\delta_{i1}\delta_{j1} - \delta_{i2}\delta_{j2})v_{12} \\ &+ \eta^o(\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1})(v_{11} - v_{22}). \end{aligned} \tag{4.5}$$

The components of the odd-viscosity part of the stress tensor are

$$T^o_{11} = -\eta^o(\partial_2 v_1 + \partial_1 v_2), \tag{4.6a}$$
$$T^o_{12} = \eta^o(\partial_1 v_1 - \partial_2 v_2), \tag{4.6b}$$
$$T^o_{21} = \eta^o(\partial_1 v_1 - \partial_2 v_2), \tag{4.6c}$$
$$T^o_{22} = \eta^o(\partial_2 v_1 + \partial_1 v_2). \tag{4.6d}$$

Since the fluid is incompressible, an application of the divergence theorem shows that the force and torque on the surface of the swimmer are the same as the force and torque on the fluid at infinity. Using this equivalence, the components of the force on the swimmer are

$$F_i = \lim_{R\to\infty} \int_0^{2\pi} (T_{ij} r_j) R d\varphi \tag{4.7}$$

and the torque on the swimmer is

$$N = \lim_{R\to\infty} \int_0^{2\pi} (\epsilon_{ij} r_i T_{jk} r_k) R^2 d\varphi. \tag{4.8}$$

In these formulas r_i are the components of the radial unit vector $\hat{\mathbf{r}} = \cos\varphi \hat{\mathbf{x}} + \sin\varphi \hat{\mathbf{y}}$ and the integral is taken over the circle at infinity.

Using these equations, and the components of the odd-viscosity part of the stress tensor, we can derive expressions for the odd-viscosity contribution to the force and torque on the swimmer. In complex form they are

$$F^o = \lim_{R\to\infty} -2\eta^o \oint_{\mathcal{C}} (\partial_{\bar{z}} v)\, d\bar{z} \tag{4.9}$$

and

$$N^o = \lim_{R\to\infty} -2\eta^o \,\mathrm{Re}\left\{ i \oint_{\mathcal{C}} z(\partial_z \bar{v})\, dz \right\}, \tag{4.10}$$

where $\mathcal{C}$ is a circular contour of radius R (to be taken to infinity), and we have switched to a complex notation for the force, $F = F_1 + iF_2$ (the torque, being a scalar in two dimensions, is real).

Plugging in the velocity expansion (4.3) into these formulas gives

$$F^o = 0, \tag{4.11a}$$
$$N^o = -4\pi\eta^o \mathrm{Re}[b_{-2}]. \tag{4.11b}$$

In the next subsection we will show that the physical interpretation of this result is that the odd-viscosity contribution to the torque is proportional to the flux of the fluid at infinity (see Sec. IV). Previously it has been shown [4] that the even-viscosity contribution to the force and torque on the swimmer is given by

$$F^e = 0, \tag{4.12a}$$
$$N^e = 4\pi\eta^e \mathrm{Im}[b_{-2}]. \tag{4.12b}$$

The swimmer feels no net force (a generic result for Stokes flows in two dimensions [24]) and the total torque is

$$N = 4\pi(\eta^e \mathrm{Im}[b_{-2}] - \eta^o \mathrm{Re}[b_{-2}]). \tag{4.13}$$

We can cancel the torque on the swimmer by having the swimmer rotate at a certain angular velocity ω. This uniquely determines the rotational part of the gauge potential. In dimensions $D > 2$ the translational part of the gauge potential can be determined by the condition that the net force on the swimmer vanish. In two dimensions, however, the net force vanishes identically [24] and so one must instead determine the translational part of the gauge potential by requiring that the fluid velocity vanish at infinity [25]. We discuss this condition in more detail in Sec. VI.

Physical interpretation of the torque formula

The physical content of the formula (4.13) for the net torque on the swimmer can be better understood by looking at the relation of the coefficient b_{-2} to the circulation and flux of the fluid at infinity, denoted by $\Gamma(\infty)$ and $\Phi(\infty)$, respectively. We can express the circulation and flux of the fluid at infinity in the form of line integrals of the velocity around a large circle of radius R, to be taken to infinity. We have

$$\Gamma(\infty) = \lim_{R\to\infty} \int_0^{2\pi} \mathbf{v}\cdot\hat{\boldsymbol{\varphi}} R d\varphi \tag{4.14}$$

and

$$\Phi(\infty) = \lim_{R\to\infty} \int_0^{2\pi} \mathbf{v}\cdot\hat{\mathbf{r}} R d\varphi. \tag{4.15}$$

Using the velocity expansion (4.3), we find

$$\Gamma(\infty) = -2\pi \,\mathrm{Im}[b_{-2}], \tag{4.16a}$$

$$\Phi(\infty) = 2\pi \,\mathrm{Re}[b_{-2}]. \tag{4.16b}$$

Using these expressions, the net torque on the swimmer can be rewritten in the form

$$N = -2\eta^e \Gamma(\infty) - 2\eta^o \Phi(\infty). \tag{4.17}$$

The condition of vanishing torque in the different cases can then be interpreted in terms of zero circulation at infinity for even viscosity only, zero flux at infinity for odd viscosity only, or a proportionality between the flux and circulation at infinity when both types of viscosity are present.

V. MODEL SWIMMING STROKES AND AREA FORMULA

Following Ref. [4], we will begin by considering nearly circular swimmers with swimming strokes of the form

$$S_0(\sigma,t) = \alpha_0(t)\sigma + \alpha_{-2}(t)\sigma^{-1} + \alpha_{-3}(t)\sigma^{-2}, \tag{5.1}$$

where the $\alpha_i(t)$'s are coefficients which determine the time evolution of the swimming stroke. This kind of stroke is just a conformal map of degree $\mathcal{D} = 2$ from the unit circle to the complex z plane. The absence of a term $\alpha_{-1}(t)$ "fixes the gauge" with respect to translations [4]. An important formula is the area of the swimmer at time t, which is given by

$$\begin{aligned} A(t) &= \frac{1}{2}\mathrm{Im}\left\{\oint \overline{S_0(\theta,t)}\, dS_0(\theta,t)\right\} \\ &= \frac{1}{2}\mathrm{Im}\left\{\int_0^{2\pi} \overline{S_0(\theta,t)}\, \frac{dS_0(\theta,t)}{d\theta} d\theta\right\}, \end{aligned} \tag{5.2}$$

which gives

$$A(t) = \pi(|\alpha_0|^2 - |\alpha_{-2}|^2 - 2|\alpha_{-3}|^2) \tag{5.3}$$

for the simple stroke (5.1). General swimmers represented by conformal maps of degree $\mathcal{D}$ have the form

$$S_0(\sigma,t) = \alpha_0(t)\sigma + \sum_{n=1}^{\mathcal{D}} \alpha_{-n}(t)\sigma^{-n} \tag{5.4}$$

and in Appendix C we extend the swimming motion formulas to swimmers with $\mathcal{D} = 3$.

VI. SOLUTION FOR TRANSLATIONAL AND ROTATIONAL MOTION OF SWIMMER

To determine the coefficients a_k and b_k in the velocity expansion (4.3) we need to conformally map the flow field back to the $\zeta = re^{i\theta}$ plane [4]. Recall that the shape of the swimmer $S_0(\sigma,t)$ is a conformal map in the other direction, from the unit circle $\sigma = e^{i\theta}$ in the ζ plane to the z plane. For general swimmers of the form (5.4) the conformal mappings between the ζ and z planes take the form [4]

$$z = S_0(\zeta) = \alpha_0(t)\zeta + \sum_{n=1}^{\mathcal{D}} \alpha_{-n}(t)\zeta^{-n}, \tag{6.1a}$$

$$\zeta = S_0^{-1}(z) = \frac{z}{\alpha_0} - \frac{\alpha_{-2}}{z} + \cdots . \tag{6.1b}$$

We now introduce a star $*$ symbol to denote the pull-back of a function in the z plane to the ζ plane obtained by substituting (6.1a) for z in that function. The pull-backs of $\phi_1(z)$ and $\phi_2(z)$ are denoted by

$$\phi_1^*(\zeta) = \sum_{k<0} a_k^* \zeta^{k+1}, \tag{6.2a}$$

$$\phi_2^*(\zeta) = \sum_{k<-1} b_k^* \zeta^{k+1}, \tag{6.2b}$$

where the a_k^* and b_k^* are a new set of coefficients related to the original a_k and b_k through the conformal mapping.

Next we pull back the velocity field onto the unit circle σ in the ζ plane so that we can apply the no-slip boundary conditions there and determine the pull-back coefficients a_k^* and b_k^* in terms of the $\alpha_i(t)$. On the unit circle σ the velocity expansion takes the form (suppressing the t dependence)

$$v^*(\sigma) = \phi_1^*(\sigma) - \frac{S(\sigma)}{\overline{\partial_\sigma S(\sigma)}}\overline{\partial_\sigma \phi_1^*(\sigma)} + \overline{\phi_2^*(\sigma)}. \tag{6.3}$$

The only coefficients we need to determine the translational and rotational motion of the swimmer are a_{-1} and b_{-2}. This is because a_{-1} gives the fluid flow at infinity, so it determines the translational motion of the swimmer, and b_{-2} is related to the torque on the swimmer, so it determines the rotational motion of the swimmer. Using the conformal mapping (6.1), the coefficients a_{-1} and b_{-2} can be expressed in terms of the pulled-back coefficients a_k^* and b_k^* as

$$a_{-1} = a_{-1}^*, \tag{6.4a}$$

$$b_{-2} = \alpha_0 b_{-2}^*. \tag{6.4b}$$

We can solve for the pulled-back coefficients a_k^* and b_k^* in terms of the parameters α_i using (6.3), and then use the pulled-back coefficients to solve for a_{-1} and b_{-2}. As in [4] we find

$$a_{-1} = -\bar{\alpha}_0^{-1}\alpha_{-3}\dot{\bar{\alpha}}_{-2}, \tag{6.5a}$$

$$b_{-2} = \bar{\alpha}_0\dot{\alpha}_0 - \alpha_{-2}\dot{\bar{\alpha}}_{-2} - 2\alpha_{-3}\dot{\bar{\alpha}}_{-3}. \tag{6.5b}$$

To determine the translational part of the gauge potential we note that the coefficient a_{-1} is a constant contribution to the velocity expansion, which means that the fluid velocity at infinity is uniform and nonzero. Following Sec. 7.5 of Ref. [25], we argue that a finite-size swimmer located near the origin should not be able to induce a nonzero fluid velocity at infinity, and so we make a Galilean transformation to a frame in which the fluid is at rest at infinity and the swimmer moves with a velocity

$$\mathcal{A}_{\mathrm{tr}} \equiv V_x + i V_y = -a_{-1}, \tag{6.6}$$

where $\mathcal{A}_{\mathrm{tr}}$ denotes the translational part of the gauge potential (2.11).

To determine the rotational part of the gauge potential we attempt to cancel the torque (4.13) on the swimmer by having the swimmer rotate at an appropriately chosen angular velocity ω. In the parametrization (5.1) of the swimming stroke, having the swimmer rotate at an angular velocity ω amounts to the replacement

$$\alpha_i \to \alpha_{i,\mathrm{rot}} = \alpha_i e^{i\omega t}. \tag{6.7}$$

Under this replacement we find

$$b_{-2} \to b_{-2,\mathrm{rot}} = b_{-2} + i\omega(|\alpha_0|^2 + |\alpha_{-2}|^2 + 2|\alpha_{-3}|^2), \tag{6.8}$$

so that the condition that the net torque on the swimmer vanish becomes

$$\eta^e \mathrm{Im}[b_{-2,\mathrm{rot}}] - \eta^o \mathrm{Re}[b_{-2,\mathrm{rot}}] = 0. \tag{6.9}$$

Solving this equation for ω yields the rotational part of the gauge potential,

$$\mathcal{A}_{\mathrm{rot}} \equiv \omega = \frac{-\mathrm{Im}[b_{-2}] + \frac{\eta^o}{\eta^e}\mathrm{Re}[b_{-2}]}{|\alpha_0|^2 + |\alpha_{-2}|^2 + 2|\alpha_{-3}|^2}. \tag{6.10}$$

This expression shows that in the presence of odd viscosity the rotational part of the gauge potential picks up a term proportional to $\mathrm{Re}[b_{-2}]$. For the simple swimming stroke (5.1), one can verify by explicit computation that

$$\mathrm{Re}[b_{-2}] = \frac{1}{2\pi}\frac{dA(t)}{dt}, \tag{6.11}$$

which shows that the odd viscosity contribution to the angular velocity of the swimmer is proportional to the rate of change of the area of the swimmer. This conclusion is not limited to swimming strokes which are conformal maps of degree $\mathcal{D} = 2$, but holds for *generic* swimmers bounded by a closed curve without any self-intersections, as we prove in Appendix B.

We can use this area relation and the relation $\Gamma(\infty) = -2\pi\,\mathrm{Im}[b_{-2}]$ for the circulation of the fluid at infinity to rewrite the angular velocity formula in a way which clearly shows the physical meaning of each term. We find

$$\omega = \frac{\Gamma(\infty) + \frac{\eta^o}{\eta^e}\frac{dA(t)}{dt}}{2\pi(|\alpha_0|^2 + |\alpha_{-2}|^2 + 2|\alpha_{-3}|^2)}. \tag{6.12}$$

As the ratio of η^o/η^e increases, the angular velocity (6.10) grows without bound. Therefore we conclude that in a fluid in which the odd viscosity terms completely dominate the stress tensor (i.e., $\eta^o/\eta^e \to \infty$), the condition that the swimmer experience zero net torque must be satisfied by taking $\frac{dA(t)}{dt} = 0$, otherwise the angular velocity of the swimmer would have to be infinite. So a swimmer in a fluid where odd viscosity effects are dominant must have constant area.

When discussing the limit $\eta^o/\eta^e \to \infty$ in this context, we must always assume that η^e is finite and large enough so that we can still neglect any inertial forces in the problem (and so we can still take advantage of the geometric formulation of the problem of swimming at low Reynolds number). This is why we have been careful to say "when the odd viscosity is dominant" and not "when $\eta^e = 0$."

VII. EXAMPLE SWIMMING STROKES

Here we present some simple examples of swimming strokes that clearly demonstrate the difference between swimming in a fluid with just even viscosity and swimming in a fluid with both even and odd viscosity.

1. Dipolar distortion

The first example is a swimmer which starts out as a circle but grows into an ellipse by elongating one of its axes through a dipolarlike distortion. We use the parametrization

$$\alpha_0 = 1 + \tfrac{t}{2}, \tag{7.1a}$$

$$\alpha_{-2} = \tfrac{t}{2}, \tag{7.1b}$$

$$\alpha_{-3} = 0 \tag{7.1c}$$

for this swimmer. The boundary of the swimmer is an ellipse with the lengths of the major and minor axes given by $a = 1 + t$, $b = 1$. With only the conventional even viscosity this stroke will not cause any motion other than an increase in the area. We can also see this from the reflection symmetry about the x axis, which is equivalent to the fact that all the coefficients are real. However, when there is also odd viscosity this swimmer will start to rotate because its area is growing and the torque has a term proportional to the odd viscosity and the rate of area change. The motion for different values of the odd viscosity can be seen in Fig. 2.

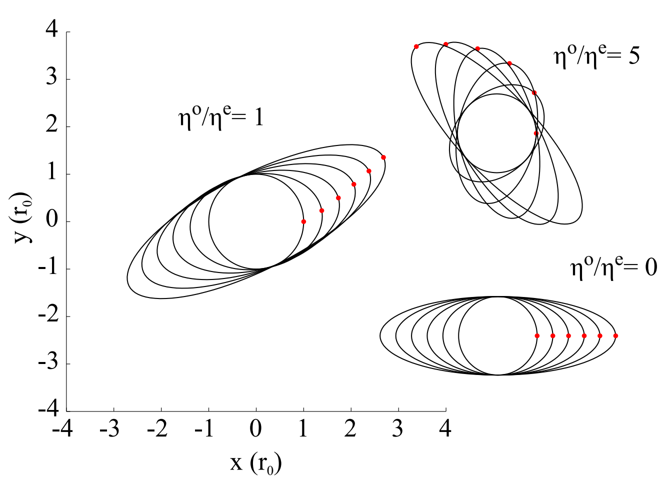

FIG. 2. (Color online) The elliptical distortion given by Eq. (7.1), shown in three different fluids with different ratios of odd to even viscosity. The time between each consecutive shape is 0.5 units of time. The dots (red) are a guide for the eye that indicates the same point on the boundary of the shape, so one can clearly see when the shape is rotating and when it is stationary. Distances are measured in units of r_0, which is the original radius of the swimmer before it starts expanding into an ellipse, i.e., $\alpha_0(t = 0) = r_0$.

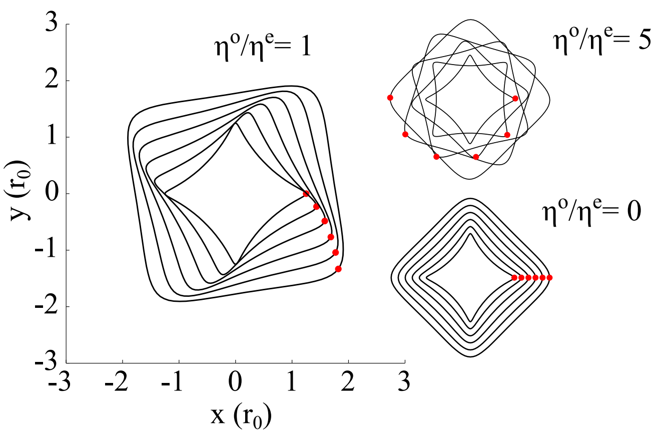

FIG. 3. (Color online) The quadrupolar distortion given by Eq. (7.2), shown in three different fluids with different ratios of odd to even viscosity. The time between each consecutive shape is 0.5 units of time. The dots (red) are a guide for the eye that indicates the same point on the boundary of the shape, so one can clearly see when the shape is rotating and when it is not. Distances are measured in units of r_0, defined by the relation $\alpha_0(t=0)=r_0$.

2. Quadrupolar distortion

To further test our results we chose a swimmer with a more complicated quadrupolar distortion which also has a uniform area growth. We used the parametrization

$$\alpha_0 = 1+t, \tag{7.2a}$$

$$\alpha_{-2} = 0, \tag{7.2b}$$

$$\alpha_{-3} = 0, \tag{7.2c}$$

$$\alpha_{-4} = \tfrac{1}{4}. \tag{7.2d}$$

This swimming parametrization represents a conformal map of degree $\mathcal{D}=3$. To see how to extend the analysis of the previous section to swimmers which are conformal maps of the circle of degree $\mathcal{D}=3$ (i.e., how to include α_{-4} terms), see Appendix C. In Fig. 3 we see very similar results to the dipolar case, e.g., the motion of the swimmer is just a rotation proportional to the growth of the area. This indicates, as we expected from the general result of Appendix B, that the odd viscosity does not distinguish between different types of shape distortions, and only couples to changes in the total area of the interior of the swimmer.

3. Wandering stroke

The third example is a swimmer parametrized with the cyclic stroke,

$$\alpha_0 = r_0, \tag{7.3a}$$

$$\alpha_{-2} = -i\xi_1 \sin(2\pi t), \tag{7.3b}$$

$$\alpha_{-3} = -i\xi_2 \cos(2\pi t), \tag{7.3c}$$

where r_0, ξ_1, and ξ_2 are all real parameters. We chose this particular stroke because in the case when only the even viscosity is present, the swimmer's centroid moves in a straight line through the fluid. Additionally, this stroke has a periodic

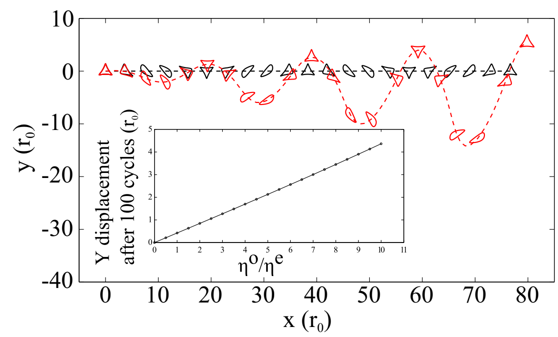

FIG. 4. (Color online) The swimming stroke of Eq. (7.3) with the parameter values $r_0=1$, $\xi_1=0.5$, and $\xi_2=0.4$, shown first with just even viscosity (horizontal trajectory in black) and then with both even and odd viscosity (oscillating trajectory in red) with $\eta^o/\eta^e=10$. The time between each consecutive shape in the figure is 6.1 cycles. When odd viscosity is also present, the swimmer wanders off of its straight trajectory because of rotations caused by changes in the area of the swimmer. The inset shows the y displacement of the swimmer after 100 cycles of this swimming stroke vs the ratio of the odd and even viscosity coefficients. Distances are measured in units of r_0, the parameter that appears in Eq. (7.3a).

time-dependent area,

$$A(t) = \pi\left[r_0^2 - \xi_1^2 + \left(\xi_1^2 - 2\xi_2^2\right)\cos^2(2\pi t)\right], \tag{7.4}$$

which implies that it will feel a cyclic stress from the odd viscosity term when present. In Fig. 4 one can clearly see the outcome as we show two trajectories, one with $\eta^o/\eta^e=0$ and one with $\eta^o/\eta^e=10$. In the case when η^o vanishes, the swimmer travels in a straight line, however in the second case the swimmer oscillates transverse to the straight-line path. In the inset we show that the amplitude of the transverse oscillation at a fixed time increases linearly with the slope η^o/η^e. As the swimmer continues it will wander further and further off of the straight-line course although on average it seems like it will still progress linearly at a similar rate to that of the swimmer in the fluid with vanishing odd viscosity.

4. Null-rotation stroke

The fourth example is a stroke which will nominally rotate when just even viscosity is present, but for which the variation of the area of the shape has been chosen carefully so that when odd viscosity is also present the shape will not rotate at all. In other words, the odd viscosity contribution to the angular velocity exactly cancels the even viscosity contribution for a given particular ratio η^o/η^e which, for the sake of this example, we pick to be unity.

A glance at Eq. (6.10) shows that in order to produce this cancellation, we need the stroke to satisfy

$$\mathrm{Im}[b_{-2}] = \mathrm{Re}[b_{-2}]. \tag{7.5}$$

For swimmers which are conformal maps of the circle of degree $\mathcal{D}=2$, the coefficient b_{-2} is given by Eq. (6.5b). We see from that equation that we can design such a stroke by taking $\alpha_0 = r_0 = \text{const}$ and

$$\alpha_j(t) = r_j(t)e^{i\theta_j(t)} \tag{7.6}$$

for $j = -2,-3$, where the functions $r_j(t)$ and $\theta_j(t)$ are functions which are determined in the following way. We would like to have

$$\alpha_j(t)\dot{\bar{\alpha}}_j(t) = (1+i)\dot{f}_j(t), \tag{7.7}$$

where the $f_j(t)$ are some real periodic functions of time (to give a periodic swimming stroke), which we are essentially free to choose. This choice will guarantee the cancellation of the even and odd viscosity contributions to the torque on the swimmer, since the real and imaginary parts of Eq. (7.7) are equal at all times. The reason for using the derivative of the functions $f_j(t)$ in the above formula is purely for convenience in the formulas that follow. Plugging the form (7.6) for the $\alpha_j(t)$ into this last equation and solving the two coupled ordinary differential equations for $r_j(t)$ and $\theta_j(t)$ gives the form of the stroke in terms of the functions $f_j(t)$,

$$r_j(t) = \sqrt{2(f_j(t) + C_{j,1})}, \tag{7.8a}$$

$$\theta_j(t) = -\tfrac{1}{2}\ln(f_j(t) + C_{j,1}) + C_{j,2}, \tag{7.8b}$$

where $C_{j,1}$ and $C_{j,2}$ are arbitrary constants (although they must be chosen carefully along with the functions f_j to keep the argument of the logarithm from ever equaling zero). Now any choice of the periodic functions $f_j(t)$ will give a cyclic swimming stroke that will not rotate in a fluid with our chosen ratio $\eta^o/\eta^e = 1$.

This shows in principle that it is possible to construct a stroke for which the even and odd viscosity contributions to the angular velocity exactly cancel each other. Swimmers using this type of stroke might be able to more efficiently navigate odd-viscosity fluids since the particular choice of stroke cancels the rotation effects due to the odd viscosity.

VIII. RECIPROCAL MOTIONS AND SCALLOP THEOREM WITH ODD VISCOSITY

An interesting aspect of swimming at low Reynolds number in an ordinary viscous fluid is the fact that a reciprocal swimming stroke leads to no net motion of the swimmer through the fluid. By a reciprocal swimming stroke we mean a swimming stroke which looks exactly the same whether time is run forwards or backwards. This fact has become known as the scallop theorem [2]. The opening and closing of a scallop's shell is the prototypical example of a reciprocal stroke. Examples of nonreciprocal strokes include corkscrew and undulatory motions. Reversing time in those situations will reverse the direction of rotation of the corkscrew motion, and it will reverse the direction of travel of the waves in the undulatory motion.

One can understand this result using the uniqueness theorem for the solutions of the slow flow equations, as proved in Ref. [25]. Running time backwards corresponds to negating the velocity of the fluid at the surface of the swimmer. In other words, the boundary condition for the time-reversed flow is obtained by changing the sign of $\mathbf{v}$ in the no-slip boundary conditions at the surface of the swimmer.

With only even viscosity, the unique solution to the slow flow equations with time-reversed boundary conditions is the time-reverse of the solution with the original boundary conditions (i.e., the velocity is negated everywhere). This means that whatever motion the scallop does as it opens its shell is immediately undone when it closes its shell. Therefore the scallop can make no net progress.

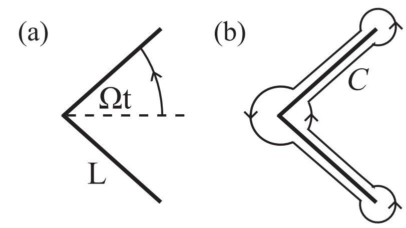

FIG. 5. (a) The simple model for the scallop swimmer. The legs have a length L and the scallop opens symmetrically about the x axis with angular velocity Ω. (b) The contour C we use to evaluate the flux of the scallop swimming stroke. This contour hugs the scallop tightly but circles around the wedge-shaped region in the center and also around the ends of the two arms.

In the geometric theory of swimming at low Reynolds number, one can make sense of this result by noting that a reciprocal swimming motion encloses no area in the space of unlocated shapes, therefore the reverse path-ordered integral of Eq. (2.5) is just the identity matrix.

Now we ask whether this result changes when we include odd viscosity. When the effects of odd viscosity are included there is the additional possibility that the swimmer can rotate itself with the reciprocal motion, and that the interplay between rotations and translations could lead to a net displacement after a full cycle of the swimming stroke, even though the stroke is reciprocal.

In fact, this is not the case, and the scallop theorem still holds in fluids with both even and odd viscosity. The uniqueness theorem argument is still valid in this case. One just needs to prove that the slow flow equations with odd viscosity terms still have unique solutions. In Appendix D we extend the usual uniqueness proof for the slow flow equations with just even viscosity (see Ref. [25]) to the case where odd viscosity terms are present. There we also give a separate uniqueness argument which applies even to swimmers whose boundary is not a smooth closed curve [for example the simple model of the scallop in Fig. 5(a) that we consider later in this section]. Therefore we can conclude that the scallop theorem is also true in viscous fluids with both even and odd viscosity.

No flux for the scallop swimming stroke

In the remainder of this section we give an argument showing that for a simple model of the scallop swimming stroke the flux of the fluid at infinity vanishes. This means that the scallop cannot rotate at all in a fluid with odd viscosity. Note that this is a stronger statement than the scallop theorem, which only states that a reciprocal swimming stroke cannot give any *net* motion (translation or rotation) through the fluid after one cycle.

We model the scallop as two infinitely thin arms of length L connected at the origin. Let Ωt be the angle between each arm and the positive x axis, so Ω is the angular velocity of the stroke at time t [see Fig. 5(a)]. The no-slip boundary conditions for the fluid on the scallop are then

$$\mathbf{v}(r,\pm\Omega t) = r\Omega[\mp\sin(\Omega t)\hat{\mathbf{x}} + \cos(\Omega t)\hat{\mathbf{y}}], \quad r < L. \tag{8.1}$$

Now we want to evaluate $\Phi(\infty)$, the flux of the fluid at infinity, due to the scallop swimming stroke. Since the fluid is incompressible, we can calculate this flux with any contour we want, instead of using the circular contour at infinity. We choose a contour C shown in Fig. 5(b). This contour hugs the scallop tightly but circles around the wedge-shaped region near the origin and around the ends of the arms.

The contribution to the flux from the straight parts of the contour vanishes, since on one side of each arm the velocity points towards the inwards normal and on the other side it points towards the outwards normal. So we just have to worry about the contribution to the flux from the small circular parts of the contour which surround the hinge of the scallop and the two ends. Therefore we need solutions to the viscous equations of motion which are valid in these small regions. We argue that we can use solutions to the equations for infinite geometries, since those should be approximately correct when we are very close to these small regions. Writing $\mathbf{v} = \nabla \times (\psi\hat{\mathbf{z}})$, where ψ is the stream function, the basic equation we need to solve is the biharmonic equation for ψ,

$$\nabla^2(\nabla^2\psi) = 0. \tag{8.2}$$

Let us suppose that $\Omega t < \pi/2$ and look at the flux from each small circular contour on either side of the wedge. We take the contour to have radius ϵ and the angle θ for integration over the contour ranges from $-\Omega t + \delta \leqslant \theta \leqslant \Omega t - \delta$ for the inner part of the wedge and from $\Omega t + \delta \leqslant \theta \leqslant 2\pi - \Omega t - \delta$ for the outer part of the wedge, where δ is a very small angle. For this infinite wedge geometry (i.e., $L \to \infty$), Eq. (8.2) has the solution (see Ref. [25])

$$\psi(r,\theta) = -\frac{1}{2}\Omega r^2\left(\frac{\sin(2\theta) - 2\theta\cos(2\Omega t)}{\sin(2\Omega t) - 2\Omega t\cos(2\Omega t)}\right). \tag{8.3}$$

This solution is valid for $2\Omega t \lesssim 257.45^\circ$, which is the angle where the denominator equals zero. Near that angle $\psi(r,\theta)$ shows more complicated scaling behavior and exhibits several different scaling regimes. Below, but very near, the critical angle $\psi(r,\theta)$ will instead scale with r as r^{p_2+2} and then deform to scale as $r^2\ln(r)$. For $2\Omega t$ greater than the critical angle $\psi(r,\theta)$ has three different regimes and will scale as r^{p_1+2}, r^2 and then r^{p_3+2}, where p_1, p_2, and p_3 are complex numbers with $\mathrm{Re}[p_i] > -1$, $i = 1,2,3$, though p_1 is actually real. The detailed solution to the infinite geometry wedge problem for all angles Ωt is discussed in Ref. [26], though we do not need much of the detail for what we are studying.

Now the radial and angular components of the fluid velocity are given in terms of the stream function by

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \quad v_\theta = -\frac{\partial\psi}{\partial r}. \tag{8.4}$$

The radial component is relevant for the computation of the fluxes

$$\Phi_{\text{wedge},1} = \int_{-\Omega t+\delta}^{\Omega t-\delta} v_r(\epsilon,\theta)\epsilon\, d\theta, \tag{8.5}$$

$$\Phi_{\text{wedge},2} = \int_{\Omega t+\delta}^{2\pi-\Omega t-\delta} v_r(\epsilon,\theta)\epsilon\, d\theta, \tag{8.6}$$

where ϵ is again the radius of the circular contour. We see that for all angles Ωt, the product $\epsilon v_r(\epsilon,\theta)$ scales as ϵ^α, $\mathrm{Re}[\alpha] > 1$, possibly multiplied by $\ln(\epsilon)$, which means that when we take the limit $\epsilon \to 0$ (the circle shrinks to zero radius), these contributions to the flux will vanish.

Next we have to look at the flow near the very ends of the arms. We argue that the flow here can be approximated by the flow near the end of a semi-infinite line or plate being dragged through the fluid with a velocity perpendicular to its length. For simplicity, we look at the solution of the slow flow equations where the semi-infinite plate occupies the negative x axis and is moving in the negative y direction with speed v_0. We again want to solve Eq. (8.2), but now subject to the boundary conditions

$$\mathbf{v}(r,\pi) = v_0\hat{\boldsymbol{\theta}}. \tag{8.7}$$

This time we find that the solution is

$$\psi(r,\theta) = r[A\cos(\theta) + B\sin(\theta) + C\theta\cos(\theta) + D\theta\sin(\theta)], \tag{8.8}$$

where A,B,C,D are constants that must be determined from the boundary conditions. Imposing the no-slip boundary conditions at $\theta = \pi$ gives

$$C = \frac{v_0 - A}{\pi}, \tag{8.9}$$

$$D = -\frac{B}{\pi} + \frac{A - v_0}{\pi^2}. \tag{8.10}$$

The important feature of this solution is that the fluid velocity actually scales as r^0. Now the semi-infinite plate problem is unrealistic, and this is reflected in the fact that in the solution the velocity has no r dependence at all. However, we expect this solution to still be valid very close to the tip of the plate, and that is where we will make use of it. We evaluate the flux of the fluid around a circular contour of radius ϵ with center at the origin (end of the plate) and $-\pi + \delta \leqslant \theta \leqslant \pi - \delta$. Since the fluid velocity scales there as r^0, the factor of ϵ we get from the line element $ds = \epsilon d\theta$ in the integration is the only factor of ϵ present, and so this contribution to the flux vanishes as we take $\epsilon \to 0$. Therefore we conclude that the contribution to the total flux from the two ends of the scallop is also zero, and so the total flux $\phi(\infty)$ of the scallop swimming stroke is zero. This means that in an odd viscosity fluid the scallop cannot rotate at all as it opens and closes its shell.

IX. CONCLUSION

We have applied the geometric theory of swimming at low Reynolds number developed by Wilczek and Shapere [4] to the case where the fluid has a nonvanishing odd, or Hall, viscosity. The main effect of the odd viscosity is to introduce an additional torque on the swimmer, proportional to the rate of change of the area of the swimmer, independent of the other shape changes occurring in the stroke pattern. This torque is the companion effect to the fact that a swimmer rotating in a fluid with odd viscosity feels an inwards or outwards pressure proportional to its angular velocity [13]. As we show in Appendix B this conclusion applies to generic swimming shapes and is not limited to swimmers whose boundaries are simple conformal maps of the unit circle.

As a consequence of this extra torque, a swimming stroke which would not cause the swimmer to rotate in a fluid with conventional viscosity can cause the swimmer to rotate in a fluid with odd viscosity if the area of the swimmer is changing. It is even possible to design a stroke which will rotate the swimmer in an even viscosity fluid but not in a fluid with both even and odd viscosity, for a certain value of the ratio η^o/η^e. It is possible that swimmers placed in fluids with an odd viscosity would have to adapt their strokes to efficiently move in a straight line. Additionally, it would be interesting to see if swimmers could use the interplay between the even and odd viscosity to perform more interesting or efficient motion patterns.

ACKNOWLEDGMENTS

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APPENDIX A: COMPUTATION OF THE PATH-ORDERED INTEGRAL

To calculate the matrix $\mathcal{R}(t)$, which gives the rigid motion of the swimmer after a finite time t, we need to evaluate the reverse path-ordered integral (2.6). In practice we do this by slicing time into many small steps (say N steps) of size Δt. We can write

$$\bar{P}e^{\int_0^t \mathcal{A}(t')dt'} = \bar{P}e^{\sum_{i=1}^N \int_{(i-1)\Delta t}^{i\Delta t} \mathcal{A}(t')dt'}. \tag{A1}$$

If the time steps are small enough then we can approximate this as

$$\bar{P}e^{\sum_{i=1}^N \int_{(i-1)\Delta t}^{i\Delta t} \mathcal{A}(t')dt'} \approx \prod_{i=1}^N \bar{P}e^{\int_{(i-1)\Delta t}^{i\Delta t} \mathcal{A}(t')dt'}, \tag{A2}$$

where on the right side we now have a product of reverse path-ordered integrals over many small time intervals of size Δt and we should put the earliest times on the right so that we are applying the rigid motions in these small intervals in chronological order. Since these time intervals are very small we can make a further approximation by expanding the path-ordered integral over the time interval Δt to first order and neglecting the higher order terms to find

$$\bar{P}e^{\int_{(i-1)\Delta t}^{i\Delta t} \mathcal{A}(t')dt'} \approx I + \int_{(i-1)\Delta t}^{i\Delta t} \mathcal{A}(t')dt'. \tag{A3}$$

Finally we can make one further approximation for the integral of the matrix $\mathcal{A}(t)$ over the small time interval Δt,

$$\int_{t_{i-1}}^{t_i} \mathcal{A}(t')dt' \approx \mathcal{A}(t_{i-1})\Delta t, \tag{A4}$$

where $t_i = i\Delta t$ (and $t_0 = 0$). Our final expression for the approximation of the full path-ordered integral is then

$$\bar{P}e^{\int_0^t \mathcal{A}(t')dt'} \approx \prod_{i=1}^N (I + \mathcal{A}(t_{i-1})\Delta t)\,, \tag{A5}$$

where again the matrices for the earliest times must be to the right so that the rigid motions are applied in the proper order.

We have also tried expanding the reverse path-ordered integrals over the time interval Δt to second order, but it seems that this makes almost no visible correction to the swimming trajectory when the swimming deformations are not too large and the step size Δt is small.

APPENDIX B: PROOF THAT Re[b_{-2}] = $\frac{1}{2\pi}\frac{dA(t)}{dt}$ FOR GENERAL SWIMMING STROKES

Our analysis of the simple swimmer (5.1) suggests a deeper connection between the area of the swimmer and the odd viscosity contribution to the torque on the swimmer. To explore this connection further we now show that Eq. (6.11) holds for any swimmer whose boundary is a smooth curve without self-intersections.

The boundary of the swimmer is just a smooth curve parametrized by θ which also depends on the time t. If we write the shape in terms of real components,

$$S_0(\theta,t) = x(\theta,t) + iy(\theta,t), \tag{B1}$$

and plug into the area formula (5.2) we find

$$A(t) = \frac{1}{2}\int_0^{2\pi} [x(\theta,t)y'(\theta,t) - y(\theta,t)x'(\theta,t)]d\theta, \tag{B2}$$

where the prime denotes a derivative with respect to θ. Next take a time derivative to get

$$\frac{dA(t)}{dt} = \frac{1}{2}\int_0^{2\pi} [\dot{x}(\theta,t)y'(\theta,t) + x(\theta,t)\dot{y}'(\theta,t) - \dot{y}(\theta,t)x'(\theta,t) - y(\theta,t)\dot{x}'(\theta,t)]d\theta. \tag{B3}$$

We can integrate by parts on the terms with mixed partial derivatives and use the fact that the boundary terms vanish since $x(\theta,t)$, $y(\theta,t)$, $\dot{x}(\theta,t)$, and $\dot{y}(\theta,t)$ are 2π-periodic in θ to get

$$\frac{dA(t)}{dt} = \int_0^{2\pi} [\dot{x}(\theta,t)y'(\theta,t) - \dot{y}(\theta,t)x'(\theta,t)]d\theta. \tag{B4}$$

Because of the no-slip boundary conditions the vector $(\dot{x}(\theta,t),\dot{y}(\theta,t))$ is just the fluid velocity $\mathbf{v}(\mathbf{r})$ evaluated on the surface of the swimmer,

$$\mathbf{v}(\mathbf{r})|_{\text{swimmer}} = \dot{x}(\theta,t)\hat{\mathbf{x}} + \dot{y}(\theta,t)\hat{\mathbf{y}}. \tag{B5}$$

Then we can write

$$\frac{dA(t)}{dt} = \oint_{\text{swimmer}} \mathbf{v}\cdot\hat{\mathbf{n}}\, ds = \Phi(\text{swimmer}), \tag{B6}$$

where $\hat{\mathbf{n}}\, ds = d\mathbf{r} \times \hat{\mathbf{z}}$ is a vector normal to the surface of the swimmer with magnitude $ds = |d\mathbf{r}|$. This integral is just the flux of the fluid at the surface of the swimmer. By the divergence theorem we have

$$\Phi(\infty) - \Phi(\text{swimmer}) = \int_{\text{fluid}} \nabla \cdot \mathbf{v}\, dx dy \tag{B7}$$

and since the fluid is incompressible, $\nabla \cdot \mathbf{v} = 0$, we get

$$\frac{dA(t)}{dt} = \Phi(\infty). \tag{B8}$$

A comparison with Eq. (4.15) for the flux of the fluid at infinity yields the final result

$$\mathrm{Re}[b_{-2}] = \frac{1}{2\pi}\frac{dA(t)}{dt}, \tag{B9}$$

proving that this relation is valid for general swimming shapes in incompressible fluids.

It is known that an object which rotates in a fluid with odd viscosity will feel a pressure directed radially inwards or outwards depending on the direction of the rotation [13]. The relation (6.11) is the companion to this statement. It says that an object which tries to expand or contract in a fluid with odd viscosity will feel a torque whose direction ($\pm\hat{\mathbf{z}}$) depends on whether the area of the object is growing or shrinking.

APPENDIX C: EXTENSION AND SOLUTION OF CONFORMAL MAPS OF DEGREE $\mathcal{D} = 3$

A swimmer whose boundary is a degree 3 ($\mathcal{D} = 3$) conformal map of the circle has the form

$$S_0(\sigma,t) = \alpha_0(t)\sigma + \alpha_{-2}(t)\sigma^{-1} + \alpha_{-3}(t)\sigma^{-2} + \alpha_{-4}\sigma^{-3}, \tag{C1}$$

with area

$$A(t) = \pi(|\alpha_0|^2 - |\alpha_{-2}|^2 - 2|\alpha_{-3}|^2 - 3|\alpha_{-4}|^2). \tag{C2}$$

To solve for a^*_{-1} and b^*_{-2} we need the coefficients a^*_{-2}, a^*_{-3} and a^*_{-4}. Equations for these coefficients can be obtained by plugging into the pulled-back velocity expansion (6.3). We find that

$$\alpha_{-4}\bar{a}^*_{-2} + \bar{\alpha}_0 a^*_{-2} = \bar{\alpha}_0 \dot{\alpha}_{-2}, \tag{C3a}$$

$$a^*_{-3} = \dot{\alpha}_{-3}, \tag{C3b}$$

$$a^*_{-4} = \dot{\alpha}_{-4}. \tag{C3c}$$

The equation for a^*_{-2} is really just a matrix equation for a two-component vector consisting of the real and imaginary parts of a^*_{-2}. The solution is

$$a^*_{-2} = \frac{|\alpha_0|^2\dot{\alpha}_{-2} - \alpha_0\alpha_{-4}\dot{\bar{\alpha}}_{-2}}{|\alpha_0|^2 - |\alpha_{-4}|^2}. \tag{C4}$$

In terms of this coefficient we find that

$$a_{-1} = -(\bar{\alpha}_0)^{-1}(\bar{a}^*_{-2}\alpha_{-3} + 2\dot{\alpha}_{-3}\alpha_{-4}) \tag{C5}$$

and

$$b_{-2} = \bar{\alpha}_0\dot{\alpha}_0 - \bar{\alpha}_{-2}\dot{\alpha}_{-2} - 2\alpha_{-3}\dot{\bar{\alpha}}_{-3} - 3\alpha_{-4}\dot{\bar{\alpha}}_{-4} + \bar{\alpha}_{-2}a^*_{-2} - \alpha_{-2}\bar{a}^*_{-2}. \tag{C6}$$

Note that the last two terms in b_{-2} are complex conjugates of each other and appear with the opposite sign so that they will cancel when we take the real part of b_{-2}. This means that the relation $\mathrm{Re}[b_{-2}] = \frac{1}{2\pi}\frac{dA(t)}{dt}$ still holds in this case, as we expect based on the general arguments presented in Appendix B.

To solve for the new form of the angular velocity necessary to cancel the torque on the swimmer, we again send $\alpha_i \to \alpha_{i,\text{rot}} = \alpha_i e^{i\omega t}$ and solve the equation

$$\eta^e \mathrm{Im}[b_{-2,\text{rot}}] - \eta^o \mathrm{Re}[b_{-2,\text{rot}}] = 0, \tag{C7}$$

where now

$$b_{-2,\text{rot}} = b_{-2} + i\omega J \tag{C8}$$

with

$$J = \frac{2}{|\alpha_0|^2 - |\alpha_{-4}|^2}(|\alpha_0|^2|\alpha_{-2}|^2 + \mathrm{Re}[\alpha_0\alpha_{-4}(\bar{\alpha}_{-2})^2]) + |\alpha_0|^2 - |\alpha_{-2}|^2 + 2|\alpha_{-3}|^2 + 3|\alpha_{-4}|^2. \tag{C9}$$

The new angular velocity needed to cancel the torque on the swimmer is then

$$\omega = \frac{1}{J}\left(-\mathrm{Im}[b_{-2}] + \frac{\eta^o}{\eta^e}\mathrm{Re}[b_{-2}]\right), \tag{C10}$$

so that the translational and rotational parts of the gauge potential are now given by

$$\mathcal{A}_{\text{tr}} = (\bar{\alpha}_0)^{-1}(\bar{a}^*_{-2}\alpha_{-3} + 2\dot{\alpha}_{-3}\alpha_{-4}), \tag{C11}$$

$$\mathcal{A}_{\text{rot}} = \frac{1}{J}\left(-\mathrm{Im}[b_{-2}] + \frac{\eta^o}{\eta^e}\mathrm{Re}[b_{-2}]\right). \tag{C12}$$

APPENDIX D: UNIQUENESS THEOREM FOR SLOW FLOW EQUATIONS WITH ODD VISCOSITY

The slow flow equations for fluids with odd viscosity are

$$\nabla(p - \eta^o\xi) = \eta^e\nabla^2\mathbf{v} \tag{D1}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{D2}$$

where $\xi = (\nabla \times \mathbf{v}) \cdot \hat{\mathbf{z}}$ is the vorticity. We first prove uniqueness of the solution in the case where the swimmer occupies a region S, bounded by a smooth closed curve ∂S. The fluid occupies the space in the plane outside of the swimmer, which we denote by $\mathbb{R}^2 \backslash S$. The proof here is very similar to the proof one uses in the even viscosity case when no-slip boundary conditions are imposed on a smooth closed curve (for that proof see Ref. [25]).

Suppose we have two solutions, $\mathbf{v}_1$ and $\mathbf{v}_2$, to the above equations that both satisfy the no-slip boundary conditions

$$\mathbf{v}|_{\partial S} = \mathbf{v}_0 \tag{D3}$$

on the surface of the swimmer. Construct the difference of the two velocity fields, $\mathbf{V} = \mathbf{v}_1 - \mathbf{v}_2$, and the differences of the pressures and vorticities $P = p_1 - p_2$, $\Xi = \xi_1 - \xi_2$. Since the slow-flow equations are linear, these quantities satisfy the equations

$$\nabla(P - \eta^o\Xi) = \eta^e\nabla^2\mathbf{V} \tag{D4}$$

$$\nabla \cdot \mathbf{V} = 0, \tag{D5}$$

but with the boundary condition

$$\mathbf{V}|_{\partial S} = \mathbf{0}. \tag{D6}$$

We want to show that this boundary condition forces $\mathbf{V} = \mathbf{0}$ everywhere inside the fluid. Take the dot product of Eq. (D4) with $\mathbf{V}$ and integrate both sides over the region containing the fluid,

$$\iint_{\mathbb{R}^2\backslash S} \mathbf{V} \cdot \nabla(P - \eta^o \Xi) dA = \eta^e \iint_{\mathbb{R}^2\backslash S} \mathbf{V} \cdot \nabla^2 \mathbf{V} dA. \tag{D7}$$

Since $\nabla \cdot \mathbf{V} = 0$, the integrand on the left-hand side can be written as

$$\mathbf{V} \cdot \nabla(P - \eta^o \Xi) = \nabla \cdot [(P - \eta^o \Xi)\mathbf{V}]. \tag{D8}$$

We can then use the divergence theorem on the left side, so that our relation becomes

$$\oint_{\partial S} (P - \eta^o \Xi)\mathbf{V} \cdot \hat{\mathbf{n}} ds = \eta^e \iint_{\mathbb{R}^2\backslash S} \mathbf{V} \cdot \nabla^2 \mathbf{V} dA, \tag{D9}$$

where $\hat{\mathbf{n}}$ is the unit normal vector to the curve ∂S. But $\mathbf{V} = \mathbf{0}$ on ∂S, so the integral on the left-hand side is zero, and we just get

$$\iint_{\mathbb{R}^2\backslash S} \mathbf{V} \cdot \nabla^2 \mathbf{V} dA = 0. \tag{D10}$$

From now on it will be more useful to write everything out in coordinates (but not using the summation convention). Let $\mathbf{V} = (V_1, V_2)$. The velocity is a function of position $\mathbf{x} = (x_1, x_2)$ inside the fluid. We have

$$\mathbf{V} \cdot \nabla^2 \mathbf{V} = \sum_i V_i \left(\sum_j \frac{\partial^2 V_i}{\partial x_j^2} \right). \tag{D11}$$

We can use the chain rule to rewrite this as

$$\mathbf{V} \cdot \nabla^2 \mathbf{V} = \sum_{i,j} \left[\frac{\partial}{\partial x_j} \left(V_i \frac{\partial V_i}{\partial x_j} \right) \right] - \sum_{i,j} \left(\frac{\partial V_i}{\partial x_j} \right)^2. \tag{D12}$$

If we define the vector $\mathbf{W}$ with components

$$W_j = \sum_i V_i \frac{\partial V_i}{\partial x_j} \tag{D13}$$

then we can write this compactly as

$$\mathbf{V} \cdot \nabla^2 \mathbf{V} = \nabla \cdot \mathbf{W} - \sum_{i,j} \left(\frac{\partial V_i}{\partial x_j} \right)^2. \tag{D14}$$

Now

$$\iint_{\mathbb{R}^2\backslash S} \nabla \cdot \mathbf{W} dA = \oint_{\partial S} \mathbf{W} \cdot \hat{\mathbf{n}} ds = 0, \tag{D15}$$

where we have again used the fact that $\mathbf{V}$ vanishes on ∂S. So we are left with

$$\iint_{\mathbb{R}^2\backslash S} \sum_{i,j} \left(\frac{\partial V_i}{\partial x_j} \right)^2 dA = 0. \tag{D16}$$

But the integrand in this expression is greater than or equal to zero, so we conclude that

$$\frac{\partial V_i}{\partial x_j} = 0 \;\forall\; i,j \tag{D17}$$

so $\mathbf{V}$ is a constant independent of position. But $\mathbf{V} = \mathbf{0}$ on the surface of the swimmer, therefore $\mathbf{V} = \mathbf{0}$ everywhere, and so $\mathbf{v}_1 = \mathbf{v}_2$. The solution is unique.

1. Possible Proof for Swimmers of More General Shapes

Now we consider swimmers of a more general shape. We can imagine two situations here. In the first situation the swimmer is bounded by a closed curve which might not be smooth, for example a swimmer with a "blocky" shape. In the second situation we have a very thin swimmer whose entire body consists of a one-dimensional curve, open on both ends, and not necessarily smooth. An example of this situation is our simple model of the scallop in Fig. 5(a) of Sec. VIII. In both cases we call the curve Γ, and we impose no-slip boundary conditions for the fluid on this curve,

$$\mathbf{v}|_{\Gamma} = \mathbf{v}_0. \tag{D18}$$

Define

$$v_{\max} = \max \{|\mathbf{v}(\mathbf{x})| : \mathbf{x} \in \Gamma\}. \tag{D19}$$

It is the speed of the fastest moving point on the swimmer.

Next, we recall the physical meaning of the slow flow equations. The full Navier-Stokes equations (for an incompressible fluid with odd viscosity) are

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) = \eta^e \nabla^2 \mathbf{v} - \nabla(p - \eta^o \xi), \tag{D20}$$

$$\nabla \cdot \mathbf{v} = 0. \tag{D21}$$

The slow flow equations are obtained from these by setting the convective derivative term to zero,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = 0. \tag{D22}$$

The physical meaning of this statement is that the net force on each fluid element, represented by the right-hand side of Eq. (D20), is equal to zero. The fact that the convective derivative is equal to zero means that the fluid velocity $\mathbf{v}$ is a constant along streamlines in the fluid. With an object moving in the fluid (with no-slip boundary conditions), the streamlines must begin on that object, and they either end on that object as well (if the flux of the fluid at infinity vanishes), or the streamlines go out to infinity (if the flux of the fluid at infinity does not vanish). Furthermore, every point in the fluid lies on exactly one streamline (there are no shocks in this situation).

These considerations imply that $v_{\max}$ is actually an upper bound for the speed of the fluid anywhere in the entire plane $\mathbb{R}^2$. So we can say that

$$|\mathbf{v}(\mathbf{x})| \leqslant v_{\max} \;\forall\; \mathbf{x}. \tag{D23}$$

We can now use this fact to prove the uniqueness of solutions to the slow flow equations for more general shapes, including our nonsmooth, possibly open-ended, curve Γ. Again, suppose we have two solutions $\mathbf{v}_1$ and $\mathbf{v}_2$, both satisfying the boundary condition $\mathbf{v}|_{\Gamma} = \mathbf{v}_0$. Then the difference $\mathbf{V} = \mathbf{v}_1 - \mathbf{v}_2$ again satisfies the slow flow equations, but with the boundary condition $\mathbf{V}|_{\Gamma} = \mathbf{0}$. For this boundary condition we have $v_{\max} = 0$. Then our bound Eq. (D23) implies that $|\mathbf{V}| \leqslant 0$ everywhere, so we can conclude that $\mathbf{V} = \mathbf{0}$ everywhere. This implies that $\mathbf{v}_1 = \mathbf{v}_2$, so the solution is unique.

Since we used the vanishing of the convective derivative in the derivation, the bound Eq. (D23) applies only to time-independent viscous flows, and almost certainly does not apply at all to any other kinds of fluid flows.

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