Generalized Sherrington-Kirkpatrick glass without reflection symmetry

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We investigate generalized Sherrington–Kirkpatrick glassy systems without reflection symmetry. In the neighborhood of the transition temperature, we, in general, uncover the structure of the glass state building the full-replica-symmetry-breaking solution. A physical example of the explicitly constructed solution is given.

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I. INTRODUCTION

The Sherrington-Kirkpatrick (SK) model—the Ising model with random exchange interactions—has proved to be a unique laboratory for understanding physics of spin glasses [1]. Its exact mean-field solution has become the cornerstone of modern glass physics. There are many generalizations of the SK model that allow understanding other glassy systems, more complicated than spin glasses [2–17]. One of the simplest ways to build the generalized SK model (GSK) is to replace the Ising-spin operator at each lattice site $\hat{S}_z$ by another diagonal operator $\hat{U}$ that also, of course, satisfies the relation Tr $\hat{U} = 0$ as well as the Ising spin does. Then one may naively suggest that this new glass forming model inherits, at least on a qualitative level, most of the features of the SK model. However, this is not so [12,18–24]. It follows that the physics of the generalized SK model strongly depends on the hidden symmetry of the $\hat{U}$ operator, particularly if there is "reflection" symmetry or not. Here we build an analytical solution for the glass state in the generalized SK models where the $\hat{U}$ operators do not have reflection symmetry.

Formally, the presence of the reflection symmetry means that $\mathrm{Tr}[\hat{U}^{(2k+1)}] = 0, k = 0,1,2,\ldots$. The Ising-spin operator $\hat{S}_z$ obviously satisfies the reflection-symmetry condition. It was shown recently [25] that all GSK models with reflection symmetry qualitatively behave as the SK model.

$\mathrm{Tr}[\hat{U}^{(2k+1)}] \neq 0$, $k = 1,2,\ldots$ in the GSK model without the reflection symmetry. A typical example is, e.g., the quadrupole SK where $\hat{U}$ is the quadrupole angular moment operator. More examples can be found in Ref. [12].

The nonsymmetric GSK model has a different glass structure than the reflection-symmetric one, see Fig. 1. Only recently has the full-replica-symmetry-breaking solution of the nonsymmetric GSK glass been built, but for the very special case when $\hat{U} = \hat{U}_0 + \hat{U}_1$, where $\hat{U}_1$ is much smaller (in some sense) than $\hat{U}_0$; here $\hat{U}_0$ is the reflection-symmetric diagonal operator, whereas, $\hat{U}_1$ is a non-reflection-symmetric (diagonal) perturbation, see Ref. [12]. In the present paper, we, in general, uncover the structure of the glass state in the non-reflection-symmetric problem without any simplifying assumption about the smallness of the nonsymmetric part of $\hat{U}$. Our investigation is focused on the region near the glass-transition temperature where we follow how the glass freezes and unfreezes when we cross the glass-transition temperature.

The concept of glass transition as a RSB has proven to be very successful [26–30]. The language of the overlap functions in the replica-symmetry-breaking formalism has become one of the standards for explaining the physical nature of glass forming events. For the SK model [1], the glass-transition problem in terms of RSB is explicitly solvable. Below we build the exact solution for the generalized SK model without reflection symmetry using the replica-symmetry-breaking formalism.

One of the most interesting features of spin-glass models is the connection between the replica method and the dynamics [5,29–34]. The results we obtain below are in line with the results of Ref. [33] when the one-step RSB (1RSB) branch can be continued to the full RSB (FRSB) branch of solutions of the Ising p-spin model. In this connection, generalizations of the SK model that include multispin (more than two) interactions are worth mentioning. Then it was shown that the violation of the FRSB scheme in generalized SK models is also correlated with the symmetry properties of the Hamiltonian [4,19,35–39]. There is a conjecture that, in the absence of the reflection symmetry, it is not possible to construct a continuous nondecreasing glass order parameter function $q(x)$ and so the FRSB solution does not occur instantly at the point of RS solution instability unlike that in the SK model, see, e.g, Refs. [40,41], where the Potts model is considered.

II. GLASS FORMING GSK MODEL

Our Hamiltonian is a straightforward generalization of the SK model [1]: $\hat{H} = -\frac{1}{2}\sum_{i\neq j} J_{ij}\hat{U}_i\hat{U}_j$, where i,j label the lattice cites. The exchange interactions have Gaussian distribution $P(J_{ij}) = \frac{\sqrt{N}}{\sqrt{2\pi}J}\exp[-(J_{ij})^2N/(2J^2)]$, where $J = \tilde{J}/\sqrt{N}$ and N is the number of lattice sites. Using the replica trick, we can write down, in a standard way, the disorder averaged free energy, order parameters, and Almeida-Thoulles replicon modes $\lambda_{n\mathrm{RSB}}$ that indicate the nth step of RSB while $\lambda_{n\mathrm{RSB}} = 0$, see, e.g., Ref. [12] and references therein.

The bifurcation condition $\lambda_{(\mathrm{RS})\,\mathrm{repl}} = 0$ defines the point T_c where the RS solution becomes unstable. Considering 1RSB, two-step RSB (2RSB), three-step RSB,…,nRSB,

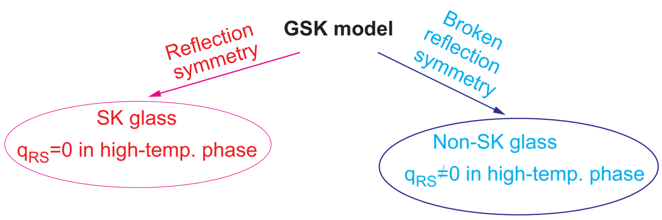

FIG. 1. (Color online) GSK models with and without reflection symmetry produce principally different glass states. Although symmetric GSK glass is well understood, the nonsymmetric one—is not [12]. One of the drastic differences between the two classes of GSK models is the behavior of the replica-symmetric (RS) *glass* order parameter $q_{\rm RS}$ in a parastate (at temperatures above the glass transition). Here we uncover the nature of nonsymmetric GSK glass using the RS-breaking (RSB) language.

respectively, we find that the equation $\lambda_{n\rm RSB}=0$ always has the solution which determines the point T_c and coincides with the solution of the equation $\lambda_{\rm (RS)\,repl}=0$ [24,42,43]. We emphasize that it is the nonzero RS solution that bifurcates.

To write ΔF, the difference between the free energy and its replica-symmetric value, we use a functional of Parisi FRSB glass order parameter $q(x)$, and so, to construct FRSB, we use the standard formalized algebra rules [27,44]. The properties of this algebra were formulated by Parisi for Ising-spin glasses. In our case, the expansion of the generalized expression for the free energy includes some terms of nonstandard form. Those terms are not formally described by the Parisi rules but can easily be reduced to the standard form. To do this, we compare the corresponding expressions, consistently producing 1RSB, 2RSB,...symmetry breaking. In what follows, we use the complete equation for the free energy, in this case, up to fourth order in the deviations $\delta q^{\alpha\beta}$ from $q_{\rm RS}$ where α,β number replica (see Eq. (16) of Ref. [12]). Finally, up to the terms of the third order in $q(x)$, we get near T_c [fourth-order terms in $q(x)$ will be considered below],

$$\frac{\Delta F}{Nk_BT}=-\frac{t^2}{4}\lambda_{\rm (RS)\,repl}\langle q^2\rangle-\frac{t^4}{2}L\langle q\rangle^2-t^6\Big\{-B_2\langle q\rangle^3-B_2'\langle q\rangle^3+B_3\left[\int_0^1 zq^3(z)dz+3\int_0^1 q(z)dz\int_0^z q^2(y)dy\right]+B_3'[\langle q\rangle\langle q^2\rangle]+B_4[-\langle q^3\rangle]\Big\}+\cdots, \tag{1}$$

where $t=\tilde J/kT$, $\langle q^n\rangle=\int_0^1 q^n(z)dz$ for $n=0,1,\ldots,$ and FRSB glass order parameter $q_{\rm FRSB}=q_{\rm RS}+q(x)$. The coefficients in Eq. (1) are the averages of linear combinations of the products of operators $\hat U$ averaged on the RS solution at the point T_c. We do not repeat the explicit total expression for ΔF here because it is rather lengthy and one can find it in Refs. [12,43]. In particular,

$$L=\big\langle\hat U_1^2\hat U_2\hat U_3\big\rangle_{\rm RS}-\langle\hat U_1\hat U_2\hat U_3\hat U_4\rangle_{\rm RS}\geqslant 0, \tag{2}$$

where averaging is performed on the RS solution at the transition point. It follows from the Cauchy-Schwarz inequality that the expression Eq. (2) is non-negative. The expression for L is not equal to zero only when $q_{\rm RS}\neq 0$.

The equation for the order parameter follows from the stationarity condition $\frac{\delta}{\delta q(x)}\Delta F=0$ applied to the free energy functional Eq. (1). Since $\lambda_{\rm (RS)\,repl}=0$ at T_c, we obtain

$$-\frac{t_c^2}{2}\frac{d[\lambda_{\rm (RS)\,repl}]}{dt}\bigg|_{t_c}\Delta t\,q(x)-t_c^4L\langle q\rangle-t_c^6\bigg[3(-B_2-B_2')\langle q\rangle^2+3B_3\left(xq^2(x)+2q(x)\int_x^1 q(z)dz+\int_0^x q^2(z)dz\right)+B_3'(\langle q^2\rangle+2q(x)\langle q\rangle)-3B_4q^2(x)\bigg]+\cdots=0. \tag{3}$$

The nontrivial solutions of Eq. (3) are fulfilled only if

$$\langle q\rangle=0+o(\Delta t)^2. \tag{4}$$

This is, in fact, the branching condition. It follows due to $L\neq 0$ in the expression Eq. (3). As can be seen from Eq. (2), this is a direct consequence of the fact that $q_{\rm RS}\neq 0$ if operators $\hat U$ do not possess the reflection symmetry. On the other hand, we know that the 1RSB solution near the branch point satisfies the branching equation $\langle q\rangle_{\rm 1RSB}=0$, see Refs. [10,42,43]. This branching condition fails for $\hat U$ with reflection symmetry. For the 2RSB solution, we receive a similar expression $\langle q\rangle_{\rm 2RSB}=0$. Within our expansion in Δt, the results for 1RSB and 2RSB near T_c coincide. This eventually leads to Eq. (4) for FRSB.

After the substitution in Eq. (3), $x=0$ and $x=1$, and taking into account Eq. (4), we obtain the following equation which will be needed later:

$$-\frac{t_c^2}{2}\frac{d[\lambda_{\rm (RS)\,repl}]}{dt}\bigg|_{t_c}\Delta t[q(0)-q(1)]+t_c^6 3[B_4[q^2(0)-q^2(1)]+B_3[q^2(1)+\langle q^2\rangle]]+\cdots=0. \tag{5}$$

Equation (3) can be further simplified using the differential operator $\hat O=\frac{1}{q'}\frac{d}{dx}\frac{1}{q'}\frac{d}{dx}$, where $q'=\frac{dq(x)}{dx}$. Then $t^6\{B_4-B_3x\}+\cdots=0$.

Now we get the key result—the significant one, depending on the x part of the solution $q(x)$, is concentrated in the neighborhood of

$$\tilde x=B_4/B_3. \tag{6}$$

We should repeat again that only for $\hat U$ with Tr $\hat U^{(2k+1)}\neq 0$, $k>0$ do we get: $\tilde x=B_4/B_3$. For operators with Tr $\hat U^{(2k+1)}=0$, we get $B_4=0$ and $\tilde x=0$, which correspond to the well-known result for SK and similar models with reflection symmetry.

Keeping the result (6) in mind, we can now build FRSB. To describe the FRSB function $q(x)$ of the variable x, we have to include the fourth-order terms in the expansion of ΔF in the consideration. This is, in general, performed in Ref. [12]. Here we must reproduce this result to explain the origin of the

FRSB solution that we build below,

$$t^6\{B_4 - B_3 x\} + t^8\Bigg\{[-2D_{33} + 4xD_{47}]\Bigg[-xq(x) - \int_x^1 dy\, q(y)\Bigg] + [-4D_2 + 2xD_{33}]q(x) + [D_{31} - D_{46}x]\langle q\rangle\Bigg\} = 0, \quad (7)$$

where $\langle q\rangle = \int_0^1 dy\, q(y)$ and, for the D coefficients, see Ref. [12].

The resulting equation (7) can be solved in a standard way as follows (we solve this equation formally in a similar way as in Ref. [11] where the expansion of the SK replica free energy functional around the nonzero RS solution, truncated to the fourth order, leads to a FRSB solution with a continuous nonlinear order parameter): We divide the equation by $[-2D_{33} + 4xD_{47}]$, differentiate with respect to x, take into account Eq. (4), and finally get

$$q(x) = \Gamma\frac{(x-s)}{\sqrt{(x-s)^2+\Delta}} - a, \quad (8)$$

where

$$s = \frac{D_{33}}{2D_{47}}, \quad (9)$$

$$\Delta = -s^2 + \frac{D_2}{D_{47}}, \quad (10)$$

$$a = \frac{1}{2t_c^2}\frac{(B_3D_{33} - 2B_4D_{47})}{\left(-D_{33}^2 + 4D_2D_{47}\right)}, \quad (11)$$

and x is near $\tilde{x}$. The values of Γ and x_c (the boundary value of x) should be found from the initial conditions.

It should be noted that our conclusions are consistent with Ref. [33] where static and dynamics of a class of mean-field spin-glass models were considered. It was shown earlier that it may exist that a temperature at which the stable, at higher temperatures, 1RSB branch becomes unstable at lower temperatures and it can be continued to a FRSB branch. An analytical study of the fourth-order expansion of the free energy in the context of some truncated model reveals that the FRSB branch of solutions is characterized by the two plateau structure and the continuous region. The numerical solutions of the FRSB equations for the Ising p-spin model with $p = 3$ have been obtained where $q(x)$ depends on x in a nonlinear manner. This is essentially a generalization of the result obtained originally by Gross *et al.* in the context of the Potts model [40].

Next, we proceed by successive steps. From Eq. (6) follows that $\tilde{x} \equiv \tilde{x}_{1\mathrm{RSB}}$: i.e., the value at which the 1RSB solution changes abruptly from $q_{1\mathrm{RSB}}(0)$ to $q_{1\mathrm{RSB}}(1)$, see Refs. [10,42,43]. We obtained that, within our expansion in Δt, the results for 1RSB and 2RSB near T_c coincide. Therefore, we start from the 1RSB solution, which is already a good approximation [27,45]. We recall here that the 1RSB solution behaves much differently when operators $\hat{U}$ have reflection symmetry and when they do not have such symmetry (a strict detailed derivation is given in Ref. [43] for $p = 2$). At first,

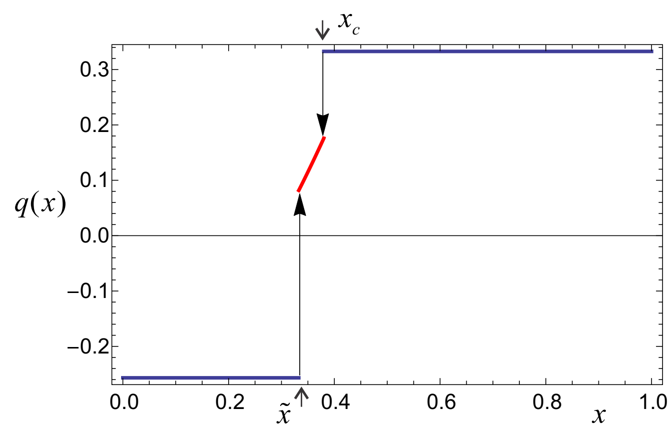

FIG. 2. (Color online) Order parameter $q(x)$ is defined by the expression (8) where $\hat{U} = \hat{S} + 0.5\hat{Q}$ and $(T - T_c) = -0.3$. [$\hat{S}$ is the z component of the spin (for $\mathbf{S} = 1$), and $\hat{Q}$ is the axial quadrupole moment $\hat{Q} = 3\hat{S}^2 - 2$.] Horizontal sections are $q_{1\mathrm{RSB}}(0)$ and $q_{1\mathrm{RSB}}(1)$, respectively. Function $q_{\mathrm{FRSB}} = q_{\mathrm{RS}} + q(x)$, where $q_{\mathrm{RS}}|_{T=T_c} = 1.154$.

from Eqs. (4)–(6), we obtain the values of $\tilde{x}$, $q_{1\mathrm{RSB}}(0)$, and $q_{1\mathrm{RSB}}(1)$. General equations (4)–(6) can easily be rewritten, assuming any nRSB wherein $\tilde{x} = \tilde{x}_{n\mathrm{RSB}}$. Furthermore, we use Eq. (4) as

$$q_{1\mathrm{RSB}}(0)\tilde{x} + q_{1\mathrm{RSB}}(1)(1 - x_c) + \int_{\tilde{x}}^{x_c} dy\, q(y) = 0, \quad (12)$$

and Eq. (7) for $x = \tilde{x}$,

$$[-D_{33} + 2\tilde{x}D_{47}][-\tilde{x}q(\tilde{x}) + q_{1\mathrm{RSB}}(0)\tilde{x}] + [-2D_2 + \tilde{x}D_{33}]q(\tilde{x}) = 0. \quad (13)$$

Finally, from Eqs. (12) and (13), we find Γ and x_c.

III. DISCUSSION

As an example, we consider operators $\hat{U} = \hat{S} + \eta\hat{Q}$ where η is a tuning parameter, not small. Here $\hat{S}$ is the z component of the spin (for $\mathbf{S} = 1$) taking values $(0,1,-1)$ and $\hat{Q}$ is the axial quadrupole moment $\hat{Q} = 3\hat{S}^2 - 2$ and it takes values $(-2,1,1)$ (see, e.g., Ref. [25]). Algebra of the operators $\hat{Q}$, $\hat{S}$, and E is closed: $\hat{Q}^2 = 2 - \hat{Q}$, $3\hat{S}^2 = 2 + \hat{Q}$, and $\hat{Q}\hat{S} = \hat{S}\hat{Q} = \hat{S}$. The operator $\hat{S}$ has the reflection symmetry, whereas, $\hat{Q}$ does not. FRSB is valid for the reflection-symmetric operator $\hat{S}$ [25]. The operator $\sqrt{3}S = V$ is a second component of the quadrupole momentum operator considered in the problem of anisotropic quadrupole glass.

For $\eta = 0.5$ (see Fig. 2), we obtain $T_c = 1.809$, $q_{\mathrm{RS}}|_{T_c} = 1.154$, $\tilde{x} = 0.333$, and $x_c = 0.392$ for $(T - T_c) = -0.3$. For $(T - T_c) = -0.1$, we have $x_c = 0.35$. For $(T - T_c) = -0.2$, we obtain $x_c = 0.37$. In the case of $\hat{U} = \hat{Q}$, we obtain $T_c = 1.37$ and $\tilde{x} = 0.43$, $x_c = 0.446$ for $(T - T_c) = -0.2$. Since $q_{\mathrm{RS}} \neq 0$, we have, for FRSB, glass order parameter $q_{\mathrm{FRSB}} = q_{\mathrm{RS}} + q(x)$. We remind here that, in the case of the standard truncated SK model, the well-known Parisi FRSB solution is $q_{\mathrm{FRSB}} = q(x) = x/2$ for $0 \leqslant x \leqslant (-2)(T - T_c)$ and $q_{\mathrm{FRSB}} = -(T - T_c)$ for $x > (-2)(T - T_c)$, where $T_c = 1$, see, e.g., Ref. [27].

So, in terms of qualitative physical arguments, we can define, in a conventional manner, the distribution function $P(q)$ as the order parameter, which gives the probability of finding a pair of glass states having the overlap q. In terms of the FRSB scheme, the distribution function $P(q)$ is defined by the Parisi function $q(x)$: $P(q) = dx(q)/dq$. So, the continuous spectrum of the overlaps appears in the whole interval $\tilde{x}(T) < x < x_c(T)$. When the nonsymmetric part of $\hat{U}$ is small, our solution for $q(x)$ is linear [12], such as the SK-model solution in the presence of a small external field [27].

The proximity of the boundary parameter x_c to $\tilde{x}$ says that the 1RSB solution is generally a quite good physical approximation for the problem we are considering. In this regard, our solution is close to that obtained in Ref. [39] and in a series of subsequent papers where the phase diagram was presented of the spherical $2 + p$ spin-glass model. The main outcome is the presence of a new phase with an order parameter made of a continuous part much similar to the FRSB order parameter and a discontinuous jump resembling the 1RSB case.

To say for sure whether the presence of such a jump is an intrinsic feature of our model or if we next need to use subsequent successive steps, it is necessary to consider the further approximation for free energy up to next order in (ΔT).

IV. CONCLUSIONS

To summarize, we have considered a model with pair interaction where the absence of reflection symmetry is caused by the characteristics of the operators $\hat{U}$ themselves. The principal prescription for obtaining a full replica-symmetry-breaking solution is derived in the general case in the neighborhood of the transition temperature. An illustrating example is considered, demonstrating the explicit build solution.

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