

Critical exponents in two dimensions and pseudo- ϵ expansion

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The critical behavior of two-dimensional n -vector $\lambda\phi^4$ field model is studied within the framework of pseudo- ϵ expansion approach. Pseudo- ϵ expansions for Wilson fixed-point location g^* and critical exponents originating from five-loop two-dimensional renormalization-group series are derived. Numerical estimates obtained within Padé and Padé-Borel resummation procedures as well as by direct summation are presented for $n = 1$, $n = 0$, and $n = -1$, i.e., for the models which are exactly solvable. The pseudo- ϵ expansions for g^* , critical exponents γ , and ν have small lower-order coefficients and slow increasing higher-order ones. As a result, direct summation of these series with optimal cutoff provides numerical estimates that are no worse than those given by the resummation approaches mentioned. This enables one to consider the pseudo- ϵ expansion technique itself as some specific resummation method.

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I. INTRODUCTION

Pseudo- ϵ expansion is known to be rather effective when used to estimate numerical values of universal quantities characterizing critical behavior of three-dimensional systems [1–4]. Moreover, even in two dimensions, where original renormalization-group (RG) series are shorter and more strongly divergent, the pseudo- ϵ expansion technique is able to give good or satisfactory results [1,5–7]. To obtain numerical estimates from pseudo- ϵ expansions one applies a resummation technique, since corresponding series have growing higher-order coefficients, i.e., look divergent. In contrast to RG expansions in fixed and $4 - \epsilon$ dimensions, pseudo- ϵ expansions do not need advanced resummation procedures based on Borel transformation. As a rule, use of simple Padé approximants turns out to be sufficient to obtain proper numerical estimates [3,5–7].

In this paper, we study the critical behavior of two-dimensional $O(n)$ -symmetric systems within the frame of the pseudo- ϵ expansion technique. The series for the Wilson fixed-point location g^* and critical exponents originating from the five-loop RG expansions will be derived for arbitrary order parameter dimensionality n . The pseudo- ϵ expansions obtained will be analyzed in detail for $n = 1$, $n = 0$, and $n = -1$, i.e., for the models with exactly known critical exponents [8–10]. The first of them (the Ising model) describes phase transitions in numerous physical systems including uniaxial ferromagnets and liquid mixtures, while the second corresponds to a long polymer in solution [11] (self-avoiding walks). Three models mentioned may be considered as test beds for clarification of the numerical effectiveness of various approximation schemes including RG perturbation theory and the method of pseudo- ϵ expansion. Numerical estimates for critical exponents will be extracted from the pseudo- ϵ expansions by means of Padé and Padé-Borel resummation techniques as well as by direct summation. The latter approach will be applied under the assumption that the best numerical results may be obtained by means of cutting divergent pseudo- ϵ expansions off by smallest terms, i.e., applying the procedure valid for asymptotic series.

II. PSEUDO- ϵ EXPANSIONS FOR GENERAL n

The critical behavior of two-dimensional systems with $O(n)$ -symmetric vector order parameters is described by Euclidean field theory with the Hamiltonian:

$$H = \int d^2x \left[\frac{1}{2} [m_0^2 \varphi_\alpha^2 + (\nabla \varphi_\alpha)^2] + \frac{\lambda}{24} (\varphi_\alpha^2)^2 \right], \quad (1)$$

where φ_α is a real n -vector field, bare mass squared m_0^2 being proportional to $T - T_c^{(0)}$, and $T_c^{(0)}$ is the phase transition temperature in the absence of order-parameter fluctuations. The β function and the critical exponents for the model (1) have been calculated within the massive theory [5,12], with the Green function, the four-point vertex, and the ϕ^2 insertion normalized in a conventional way:

$$G_R^{-1}(0, m, g_4) = m^2, \quad \left. \frac{\partial G_R^{-1}(p, m, g_4)}{\partial p^2} \right|_{p^2=0} = 1, \quad (2)$$

$$\Gamma_R(0, 0, 0, m, g) = m^2 g_4, \quad \Gamma_R^{1,2}(0, 0, m, g_4) = 1.$$

Starting from the five-loop RG expansion for the β function [12], we replace the linear term in this expansion with τg , calculate the Wilson fixed-point coordinate g^* as a series in τ , and arrive at the following expression:

$$\begin{aligned} g^* = & \tau + \frac{\tau^2}{(n+8)^2} (10.33501055n + 47.67505273) \\ & + \frac{\tau^3}{(n+8)^4} (-5.00027593n^3 + 24.4708201n^2 \\ & + 253.297221n + 350.808487) + \frac{\tau^4}{(n+8)^6} \\ & \times (0.088842906n^5 - 77.270445n^4 + 45.052398n^3 \\ & + 3408.2839n^2 + 14721.151n + 27649.346) \\ & - \frac{\tau^5}{(n+8)^8} (-0.00407946n^7 - 0.305739n^6 \\ & + 1464.58n^5 + 11521.4n^4 + 98803.3n^3 \\ & + 794945n^2 + 3.14662 \cdot 10^6 n + 4.73412 \cdot 10^6). \end{aligned} \quad (3)$$

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Substituting this expansion into the five-loop RG series for critical exponents γ and η [5,12] we obtain

$$\begin{aligned} \gamma^{-1} = & 1 - \frac{\tau(n+2)}{(n+8)} - \frac{\tau^2}{(n+8)^3}(6.95938160n^2 + 34.58878428n + 41.34004218) \\ & + \frac{\tau^3}{(n+8)^5}(0.338391156n^4 - 53.7045862n^3 - 181.874852n^2 + 471.838217n + 1236.12490) \\ & - \frac{\tau^4}{(n+8)^7}(-0.23015013n^6 + 21.848143n^5 + 1537.3578n^4 + 12405.258n^3 + 41577.259n^2 + 75410.316n + 59869.804) \\ & + \frac{\tau^5}{(n+8)^9}(0.115623n^8 + 17.8566n^7 + 83.1552n^6 + 14850.5n^5 - 84964.7n^4 + 318099n^3 + 3.76620 \cdot 10^6n^2 \\ & + 1.08837 \cdot 10^7n + 1.01285 \cdot 10^7), \end{aligned} \tag{4}$$

$$\begin{aligned} \eta = & \frac{\tau^2}{(n+8)^2}0.9170859698(n+2) + \frac{\tau^3}{(n+8)^4}(-0.0546089776n^3 + 17.9732248n^2 + 120.114155n + 167.898539) \\ & + \frac{\tau^4}{(n+8)^6}(-0.092684458n^5 - 8.2910597n^4 + 174.43187n^3 + 2120.0408n^2 + 7034.6638n + 7114.3103) \\ & + \frac{\tau^5}{(n+8)^8}(-0.0709196n^7 - 5.60392n^6 - 250.874n^5 + 1312.68n^4 + 36126.0n^3 + 201476n^2 + 470848n + 396119). \end{aligned} \tag{5}$$

Pseudo- ϵ expansions for other critical exponents can be deduced from (4) and (5) using well-known scaling relations. The series for the correlation length exponent ν , for example, results from the formula

$$\gamma = \nu(2 - \eta). \tag{6}$$

It is worthy to note that in two dimensions only models with $-2 < n < 2$ undergo transitions into an ordered phase, i.e., into the spatially uniform state with nonzero order parameter. From the physical point of view, the series obtained apply to this domain of n . On the other hand, two-dimensional

phase-transition models with $n \geq 2$ are widely explored [12–19] to evaluate numerical power of various lattice and field-theoretical approaches. It looks instructive in this context to study τ series (3)–(5) for $n \geq 2$ as well. The first step in this direction has been recently taken [20].

III. CRITICAL EXPONENTS FOR $n = 1, n = 0,$ AND $n = -1$

It is of major interest to analyze numerical results given by the obtained expansions for the values of n under which

TABLE I. Numerical values of critical exponents for $n = 1, n = 0,$ and $n = -1$ found by direct summation (DS) of the pseudo- ϵ expansions (see the text) and of corresponding inverse series (DS⁻¹), by Padé resummation of the series for γ and ν , and by Padé-Borel resummation of the pseudo- ϵ expansions and of their inverses using Padé approximants [2/3] and [3/2]. Padé estimates presented are averaged over those given by [2/3] and [3/2] approximants. Exact values of critical exponents and the estimates obtained from original five-loop renormalization-group series [12] are also presented for comparison.

Critical exponents (CE) for various n								
CE	Exact	DS	DS ⁻¹	Padé	PB _[2/3]	(PB ⁻¹) _[2/3]	(PB ⁻¹) _[3/2]	Five-loop RG
$n = 1$								
γ	1.75	1.7145	1.7304	1.775	1.6105	1.7746		1.790
ν	1	0.9067	0.9204	0.959	0.8136	0.9652		0.966
η	0.25	0.1372						0.146
$n = 0$								
γ	1.34375 (43/32)	1.4115	1.4740	1.435	1.3804	1.4285	1.4429	1.449
ν	0.75	0.7250	0.7399	0.753	0.7069	0.7514		0.774
η	0.20833 (5/24)	0.1204						0.128
$n = -1$								
γ	1.15625 (37/32)	1.1917	1.1952	1.192	1.1641	1.1843		1.184
ν	0.625	0.6021	0.6183	0.606	0.5945	0.6054	0.6076	0.617
η	0.15	0.0793						0.082

TABLE II. Padé table originating from pseudo- ϵ expansion (14) for critical exponent ν at $n = 0$. The exact value of this critical exponent is equal to 0.75.

L/M	0	1	2	3	4	5
0	0.500	0.625	0.704	0.725	0.753	0.731
1	0.667	0.838	0.733	0.639	0.741	
2	0.763	0.744	0.763	0.752		
3	0.740	0.755	0.754			
4	0.781	0.754				
5	0.706					

the model (1) is exactly solvable. That is why further we concentrate on the cases $n = 1$, $n = 0$, and $n = -1$. Pseudo- ϵ expansions for critical exponents we'll deal with are as follows: $n = 1$,

$$\gamma = 1 + \tau/3 + 0.224812357\tau^2 + 0.087897190\tau^3 + 0.086443008\tau^4 - 0.0180209\tau^5, \quad (7)$$

$$\gamma^{-1} = 1 - \tau/3 - 0.113701246\tau^2 + 0.024940678\tau^3 - 0.039896059\tau^4 + 0.0645210\tau^5, \quad (8)$$

$$\nu = 1/2 + \tau/6 + 0.120897626\tau^2 + 0.0584361287\tau^3 + 0.056891652\tau^4 + 0.00379868\tau^5, \quad (9)$$

$$\nu^{-1} = 2 - 2\tau/3 - 0.261368281\tau^2 + 0.0145750797\tau^3 - 0.091312521\tau^4 + 0.118121\tau^5, \quad (10)$$

$$\eta = 0.0339661470\tau^2 + 0.0466287623\tau^3 + 0.030925471\tau^4 + 0.0256843\tau^5, \quad (11)$$

$n = 0$,

$$\gamma = 1 + \tau/4 + 0.143242270\tau^2 + 0.018272597\tau^3 + 0.035251118\tau^4 - 0.0634415\tau^5, \quad (12)$$

$$\gamma^{-1} = 1 - \tau/4 - 0.080742270\tau^2 + 0.037723538\tau^3 - 0.028548147\tau^4 + 0.0754631\tau^5, \quad (13)$$

$$\nu = 1/2 + \tau/8 + 0.0787857831\tau^2 + 0.0211750671\tau^3 + 0.028101050\tau^4 - 0.0222040\tau^5, \quad (14)$$

TABLE III. Padé triangle for pseudo- ϵ expansion (19) of exponent ν at $n = -1$. The exact ν value equals 0.625.

L/M	0	1	2	3	4	5
0	0.500	0.571	0.606	0.602	0.614	0.587
1	0.583	0.640	0.603	0.605	0.606	
2	0.619	0.606	0.610	0.606		
3	0.600	0.609	0.607			
4	0.618	0.605				
5	0.577					

TABLE IV. Padé table for pseudo- ϵ expansion (12) of exponent γ at $n = 0$. The exact exponent value is 1.34375.

L/M	0	1	2	3	4	5
0	1.000	1.25	1.393	1.412	1.447	1.383
1	1.333	1.585	1.414	1.374	1.424	
2	1.494	1.439	1.449	1.429		
3	1.414	1.448	1.441			
4	1.474	1.430				
5	1.326					

$$\nu^{-1} = 2 - \tau/2 - 0.190143132\tau^2 + 0.0416212976\tau^3 - 0.071673308\tau^4 + 0.136330\tau^5, \quad (15)$$

$$\eta = 0.0286589366\tau^2 + 0.0409908542\tau^3 + 0.027138940\tau^4 + 0.0236106\tau^5, \quad (16)$$

$n = -1$,

$$\gamma = 1 + \tau/7 + 0.060380873\tau^2 - 0.023532210\tau^3 + 0.012034268\tau^4 - 0.0638772\tau^5, \quad (17)$$

$$\gamma^{-1} = 1 - \tau/7 - 0.039972710\tau^2 + 0.037868436\tau^3 - 0.018392201\tau^4 + 0.0649966\tau^5, \quad (18)$$

$$\nu = 1/2 + \tau/14 + 0.0348693698\tau^2 - 0.00424514372\tau^3 + 0.011608435\tau^4 - 0.0268913\tau^5, \quad (19)$$

$$\nu^{-1} = 2 - 2\tau/7 - 0.0986611527\tau^2 + 0.0510003794\tau^3 - 0.049264800\tau^4 + 0.116842\tau^5, \quad (20)$$

$$\eta = 0.0187160402\tau^2 + 0.0274103364\tau^3 + 0.017144702\tau^4 + 0.0159901\tau^5. \quad (21)$$

The expansions for “big” critical exponents γ , ν and for their inverses are seen to possess coefficients which begin to grow from certain terms indicating that these series are divergent. Moreover, they are not alternative, i.e., their coefficients have irregular signs. At the same time, lower-order

TABLE V. Padé-Borel table for pseudo- ϵ expansion of ν^{-1} at $n = 0$. The exact exponent value equals 0.75. Some estimates are absent because corresponding Padé approximants turn out to be spoiled by “dangerous” (positive axis) poles.

L/M	0	1	2	3	4	5
0	0.5	0.6058	0.6555	0.6762	0.6888	0.6954
1	0.6667		0.7170		0.7145	
2	0.7634	0.7449		0.7514		
3	0.7399	0.7538				
4	0.7814	0.7549				
5	0.7061					

TABLE VI. Same as Table V for $n = -1$. The exact value of ν is equal to 0.625.

L/M	0	1	2	3	4	5
0	0.5	0.5640	0.5903	0.5961	0.6012	
1	0.5833		0.5990		0.6009	
2	0.6190	0.6072		0.6054		
3	0.6000	0.6086	0.6076			
4	0.6183	0.6059				
5	0.5766					

coefficients in expansions (7)–(10), (12)–(15), and (17)–(20) decrease, and decrease more rapidly than their counterparts in the original RG series. This enables one to consider them as suitable for some resummation and getting proper numerical estimates.

The structure of pseudo- ϵ expansions for “small” exponent η is quite different. These series have positive coefficients of the same order of magnitude what makes questionable an applicability of any procedure employed nowadays for resummation of diverging RG series.

To demonstrate a power of various resummation techniques and to clear up to what extent they are necessary in the case considered we present below numerical results given by several relevant procedures. Namely, we evaluate critical exponents γ and ν for $n = 1$, $n = 0$, and $n = -1$ by means of the Padé resummation, by Padé-Borel resummation of the pseudo- ϵ expansions for exponents themselves and for their inverses, and by direct summation of the series (7)–(10), (12)–(15), and (17)–(20). Direct summation is performed under the assumption that one can get the best numerical estimates cutting off divergent pseudo- ϵ expansions by smallest terms, i.e., adopting the procedure true for asymptotic series.

The results thus obtained are collected in Table I. Along with pseudo- ϵ expansion estimates the exact values of critical exponents and the numbers originating from resummed five-loop RG series [12] are presented for comparison. Numerical values of the Fisher exponent given by direct summation of series (11), (16), and (21) are also included to give an idea about the accuracy of the pseudo- ϵ expansion technique in the case of small critical exponent.

Before discussing content of Table I we present some details concerning the critical exponent values obtained. In principle, the Padé resummation procedure is known to be rather effective when applied to pseudo- ϵ expansions for critical exponents

TABLE VII. Padé-Borel triangle for pseudo- ϵ expansion of γ^{-1} at $n = 0$. The exact exponent value is 1.34375. Absent estimates are due to Padé approximant dangerous poles.

L/M	0	1	2	3	4	5
0	1	1.3907	1.3055	1.3411	1.3622	1.3721
1	1.3333		1.3946		1.3907	
2	1.4942	1.4424		1.4285		
3	1.4145	1.4458	1.4429			
4	1.4740	1.4320				
5	1.3264					

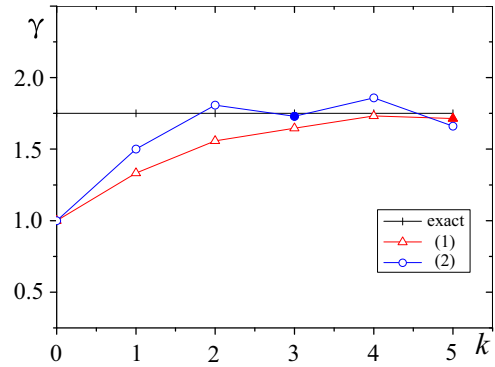


FIG. 1. (Color online) Values of critical exponent γ for $n = 1$ as functions of the order in τ k obtained by direct summation of pseudo- ϵ expansions (7) (curve 1, triangles) and (8) (curve 2, rounds). Horizontal line depicts the exact value. Filled triangle and round mark the points of optimal cutoff, i.e., the orders at which coefficients of the series finish to decrease.

and other universal quantities [3–6]. It demonstrates, as a rule, good convergence if one works within high enough orders in τ . In two dimensions, however, the numbers given by Padé resummed expansions may converge to the values differing considerably from their exact counterparts. Padé triangles presented below illustrate this situation. The first one (Table II) shows the most favorable situation—exponent ν at $n = 0$ —when numerical estimates regularly converge to the true value $\nu = 0.75$. The second example (Table III) demonstrates that good convergence may not result in quite a good numerical estimate: the asymptotic value $\nu = 0.606$ differs appreciably from the exact one $\nu = 0.625$ for $n = -1$. At last, from Table IV (the exponent γ , $n = 0$) one can see that fair convergence does not guarantee satisfactory numerical results—the estimates in this table concentrate near 1.435, i.e., far from the exact value 1.34375.

A similar situation takes place when we address the Padé-Borel resummation technique. This procedure may result in either good numerical results or unsatisfactory ones depending on the critical exponent evaluated and on the value of n . Tables V–VII illustrate this statement. Padé-Borel resummation of the pseudo- ϵ expansion of the inverse exponent ν for $n = 0$ gives quite good numerical estimates

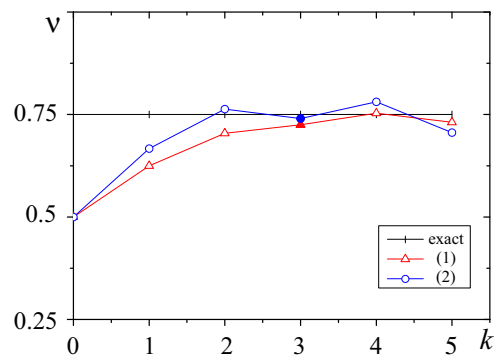


FIG. 2. (Color online) Same as Fig. 1, but for the exponent ν at $n = 0$. Triangles correspond to series (14) and rounds to series (15).

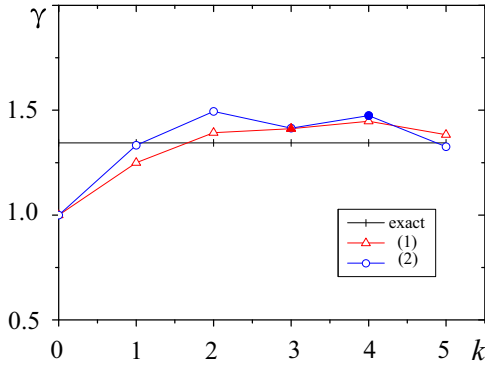


FIG. 3. (Color online) Critical exponent γ at $n = 0$ as function of k (the order in τ) obtained by direct summation of series (12) (triangles) and (13) (rounds).

(Table V), while estimates of ν for $n = -1$ and γ for $n = 0$ via inverse expansions (Tables VI and VII) “miss” the exact values. Moreover, Padé-Borel triangles for exponents γ and ν themselves at $n = 1$ and some others turn out to be half-empty because many Padé approximants are spoiled by “dangerous” (positive axis) poles.

IV. TO RESUM OR NOT TO RESUM?

Let us return to Table I. As is seen, numerical estimates provided by Padé and Padé-Borel resummation techniques may (i) be considerably scattered and (ii) differ from the exact values no less than numbers given by direct summation of pseudo- ϵ expansions and of corresponding inverse series. On the other hand, direct summation of these expansions generates an iteration procedure which rapidly converges to asymptotic values that are as close to the exact ones as those obtained within various resummation methods. Figures 1–4, where partial sums of series (7)–(10) and (12)–(15) are depicted as functions of k , k being the order in τ , illustrate the situation. Filled rounds and triangles mark the points of optimal cutoff, i.e., the order from which the coefficients of pseudo- ϵ expansions start to grow. Figures 1 and 2 show the favorable cases when approximate values almost coincide with exact ones. Figures 3 and 4, to the contrary, show most unfavorable regimes when the difference between approximate and exact values turns out to reach 0.1. An analogous level of accuracy is observed when small critical exponent η is estimated. Indeed, the direct summation of the pseudo- ϵ expansion (see Table I) and application of the resummation techniques result in numbers grouping around the exact values within the range of order of 0.1.

So, the resummation of pseudo- ϵ expansions for two-dimensional models practically does not improve numerical estimates of critical exponents. Moreover, the direct summation leads to approximate values which are as accurate as those resulting from the original five-loop RG series

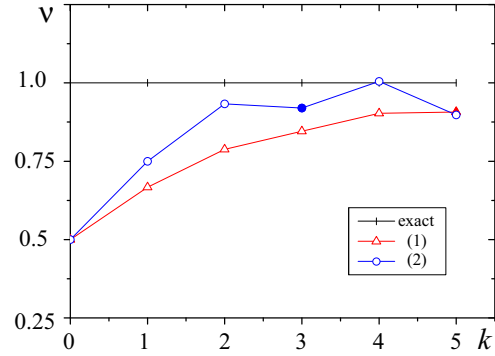


FIG. 4. (Color online) Same as Fig. 1, but for the exponent ν at $n = 1$. Triangles correspond to series (9) and rounds to series (10).

(see Table I). This enables us to conclude that, estimating critical exponents in two dimensions within the pseudo- ϵ expansion approach, one can use the simplest possible way to process the series—direct summation with optimal cutoff [21].

In this sense the pseudo- ϵ expansion itself may be considered as some special resummation method. There are two reasons for such a point of view. First, this approach transforms strongly divergent field-theoretical RG expansions into power series with much smaller lower-order coefficients and much slower increasing higher-order ones. Second, the physical value of expansion parameter τ is equal to 1, while the Wilson fixed-point coordinate g^* playing an analogous role within the field-theoretical RG approach is almost two times bigger in two dimensions ($g^* = 1.84$ – 1.86 for $n = 1, 0, -1$ [12]). This difference looks essential, especially keeping in mind the importance of higher-order terms.

V. CONCLUSION

To summarize, we have calculated pseudo- ϵ expansions for dimensionless effective coupling constant g^* and critical exponents of two-dimensional Euclidean n -vector field theory up to τ^5 order. Numerical estimates of critical exponents for models with $n = 1, 0, -1$ exactly solvable at criticality have been found using Padé and Padé-Borel resummation techniques as well as by direct summation with optimal cutoff. Comparison of the results obtained with each other and with their exact counterparts has shown that direct summation of pseudo- ϵ expansions provides, in general, numerical estimates that are no worse than those given by resummation approaches mentioned. This implies that the pseudo- ϵ expansion approach itself may be thought of as some specific resummation technique.

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