Characterization of kinetic coarsening in a random-field Ising model

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We report a study of nonequilibrium relaxation in a two-dimensional random field Ising model at a nonzero temperature. We attempt to observe the coarsening from a different perspective with a particular focus on three dynamical quantities that characterize the kinetic coarsening. We provide a simple generalized scaling relation of coarsening supported by numerical results. The excellent data collapse of the dynamical quantities justifies our proposition. The scaling relation corroborates the recent observation that the average linear domain size satisfies different scaling behavior in different time regimes.

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Study of the effect of disorder on nondisordered magnetic systems has been a subject of intense interest for the last several years [1–10]. When a magnetic system is quenched from a high temperature to a low temperature, it locally orders with the formation of domains separated by domain walls. The average linear size of the domains $R(t)$ grows with time. This linear size can also be understood as a nonequilibrium correlation length of the system. The growth of the characteristic length scale $R(t)$ with time is known as the coarsening of the system. Although coarsening in nondisordered systems is well understood [11], progress in understanding the same in disordered systems has been rather slow. Unavailability of reliable theoretical tools makes it difficult to study the dynamics of disordered systems out of equilibrium. Moreover, the dynamics of disordered systems is typically so slow that we cannot access the truly asymptotic time regime in numerical simulations. Despite all these, last several years have witnessed an appreciable development in the study of disordered systems. These include coarsening of disordered magnets [12–14], polymers in random media [15–17], or vortex lines in disordered type II superconductors [18–20]. The fundamental quantity of interest in the coarsening is the growing length scale $R(t)$ and almost all studies of coarsening is primarily concerned with the determination of this $R(t)$. However, the growth law governing the coarsening of disordered systems is at the center of some controversies. Some numerical simulations on disordered ferromagnets [21–23] yielded an algebraic growth $R(t) \sim t^{1/z}$, with a nonuniversal dynamical exponent z that depends on the temperature and on the nature of disorder. Huse and Henley [24] suggested a logarithmic increase of $R(t) \sim (\ln t)^{1/\psi}$, with the barrier exponent $\psi > 0$. Later a series of papers on the dynamics of elastic lines in a random potential [16,17,25] claims a dynamic crossover from a pre-asymptotic algebraic regime to a asymptotic slow logarithmic regime. Recent studies on other disordered systems [26,27] supports this claim, too.

In this work, we investigate the coarsening dynamics of a disordered system, namely the random field Ising model (RFIM), focusing our attention on three morphological quantities which are functions of the strength of the random fields (η_0) and the temperature (T). These are the total length of the interfaces ($\Pi(\eta_0,t)$), i.e., the total number of boundary spins of all the domains, the total number of domains ($\Lambda(\eta_0,t)$) in the system and the length of the interface of the domain with largest mass ($\Omega(\eta_0,t)$), i.e., the number of boundary spins of the domain containing maximum number of spins. In this work, we provide a empirical scaling relation of the coarsening. The scaling relation is found to be nicely obeyed by the three morphological quantities and is capable of explaining coarsening in disordered magnets in the conventional way, i.e., the behavior of the average linear domain size for the entire time regime can be reproduced from the proposed scaling relation and in this sense it is general.

The Hamiltonian of the RFIM is given by

$$H = -J \sum_{\langle i,j \rangle} s_i s_j + \sum_i \eta_i s_i + H_{\text{ext}} \sum_i s_i, \qquad (1)$$

where $s_i = \pm 1$ is the spin variable at site i, J is the strength of the exchange interaction (conventionally set to unity), and η_i is the quenched random fields taken from an uniform distribution with varying strength η_0. The external field H_{ext} has been set to zero to observe the unbiased dynamics of the system. We consider a $L \times L$ square lattice (here $L = 256$) with periodic boundary conditions along both directions. We start our simulations from a completely random spin configurations, characteristic of a high temperature ($T = \infty$) phase and then suddenly quench the system to a temperature $T = 0.50$, well below the critical temperature of a nondisordered system (Ising model) and then observe the time evolution of the system. The single spin flip Metropolis algorithm [28] is used to simulate the system. Here an unit of time (i.e., a time step) refers to one Monte Carlo (MC) step and one MC step is taken to be completed (i.e., $t = 1$) when the number of attempted single spin moves equals the total number of spins in the system. The number of domains with their sizes are determined by the Hoshen-Kopelman algorithm [29]. All the quantities are averaged over 50 independent simulations to get a precise estimate. The quantities are normalized with respect to the

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total number of spins (L^2) of the system. The temperature is taken sufficiently low to reduce the thermal fluctuations.

The relevance of domain wall roughening due to temperature in comparison to that due to random fields is explained by Binder [30]. For low temperature, the length scale [$\sim\exp(2J/K_BT)$] over which the thermal fluctuations is relevant is much higher than that [$\sim(J/\eta_0)^2$] due to random fields fluctuations. A critical temperature may be obtained from the comparison of the length scales beyond which the effect of the temperature and the random fields on the domain wall roughening are significant. The dynamics of the system at low temperature like $T = 0.50$ will match that at $T = 0$. Later we show that the time evolution of the three quantities introduced earlier is governed by the minimization of the total energy for the domain formation. Thus as long as T is small, ground state (GS) can be approached gradually as $t \to \infty$. So after a long time the final state is statistically the same as the ground state (GS), i.e., the overlap between the GS and the corresponding finite-T state is close to unity for T small. As a whole, coarsening proceeds through a compromise between the strength of the exchange interaction and the random fields with the thermal fluctuations serving only to renormalize the strengths of these couplings [31].

We begin our analysis with the idea put forward by Imry and Ma [1]. They argued that if one reverses the spins within a domain of linear size R, the energy cost $E_{\rm ex}$ is proportional to the domain wall area, i.e., $E_{\rm ex} \propto JR^{d-1}$, where d is the spatial dimension. This energy increase has to be compared with the energy gain from the interaction with the random fields. The central limit theorem tells that the mean-squared random field energy $E_{\rm RF}^2$ inside a region of volume R^d is $\sim\eta_0^2 R^d$. The total energy involved in the creation of a domain of linear size R is therefore

$$E(R) \approx JR^{d-1} - \eta_0 R^{d/2}. \tag{2}$$

The first term of Eq. (2) represents the contribution due to the boundary of the domain with linear size $R(t)$. The second term represents the contribution due to the fluctuations of the random fields in the bulk of the domain of linear size $R(t)$. On the basis of the above argument, Imry and Ma concluded that the lower critical dimensionality (LCD) of the RFIM is two. This argument is based on domains having flat interfaces. The question arises if this argument would hold even in presence of rough interfaces with fluctuating curvature. To address this concern, Binder [30] reformulated the problem in terms of the interfaces and had shown that the interface roughness is negligible if $d > 2$, confirming that the LCD of RFIM is two. However, Binder shows that the domain wall energy has a logarithmic correction, which introduces a breakup length scale, $L_b \sim \exp(A[J/\eta_0]^2)$, below which Imry-Ma argument is valid. Well-defined interfaces of domains are meaningful for length scales less than L_b. The system size considered here is below the L_b and therefore it makes sense to consider Eq. (2) as the starting point of our analysis, although Eq. (2) disregards the logarithmic correction. Taking a cue from Eq. (2), the surface energy of all the domains $E_{\rm ex}^{\rm t} \sim J\Pi(\eta_0,t)$ and the mean-squared bulk energy contained in all the domains of the system due to random fields $E_{\rm RF}^{\rm t2} \sim \frac{\eta_0^2 L^d}{\Lambda(\eta_0,t)}$, as the density of domains is inversely related to their characteristic volume.

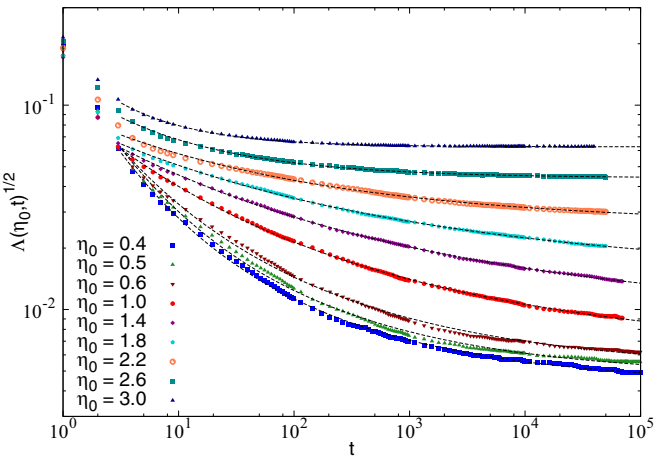

FIG. 1. (Color online) The plot of the function $\Pi(\eta_0,t)$ against time. The dotted lines are the best fits according to the scaling relation (7).

Thus the energy density ($\epsilon = E^t/L^d$) involved in the creation of all the domains in the system is given by

$$\epsilon(\eta_0,t) \approx \frac{J\Pi(\eta_0,t)}{L^d} - \frac{\eta_0}{\Lambda(\eta_0,t)^{1/2}}. \tag{3}$$

As $L \to \infty$, the surface energy contribution vanishes. This is true for any growing volume. For finite system size, the contribution from the surface energy term cannot be neglected. So the variation of either $\Pi(\eta_0,t)$ or $\Lambda(\eta_0,t)^{1/2}$ with time will govern the coarsening of the system. The log-log plot of $\Pi(\eta_0,t)$ and $\Lambda(\eta_0,t)^{1/2}$ against time are shown in Figs. 1 and 2, respectively. It is evident from Eq. (3) that the coarsening of the system energetically favors the minimization of the total length of the interfaces and also the decrease in the number of the domains. This is observed in Figs. 1 and 2, respectively. In view of the above discussions, we can redefine the coarsening as simply the minimization of $\epsilon(\eta_0,t)$ and the kinetic coarsening will be characterized by the scaling behavior of $\Pi(\eta_0,t)$ and $\Lambda(\eta_0,t)^{1/2}$. As time flows, small domains coalesce to form

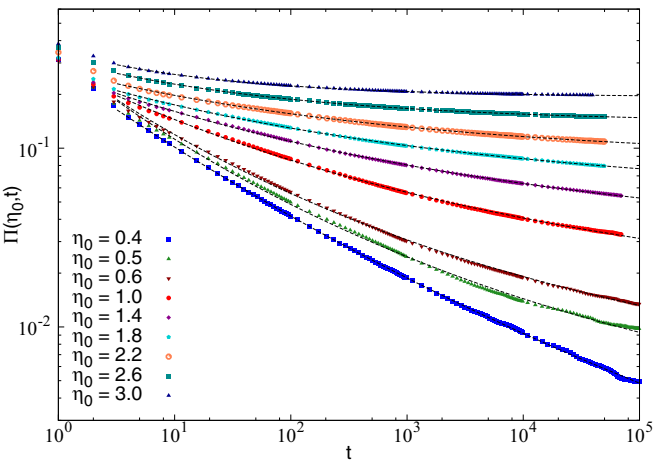

FIG. 2. (Color online) The plot of the function $\Lambda(\eta_0,t)^{1/2}$ against time. The dotted lines are the best fits according to the scaling relation (7).

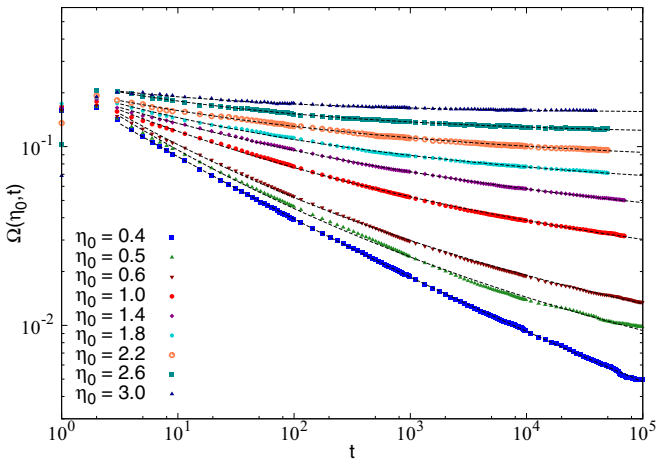

FIG. 3. (Color online) Plot of $\Omega(\eta_0,t)$ against time along with the best fits according to (7).

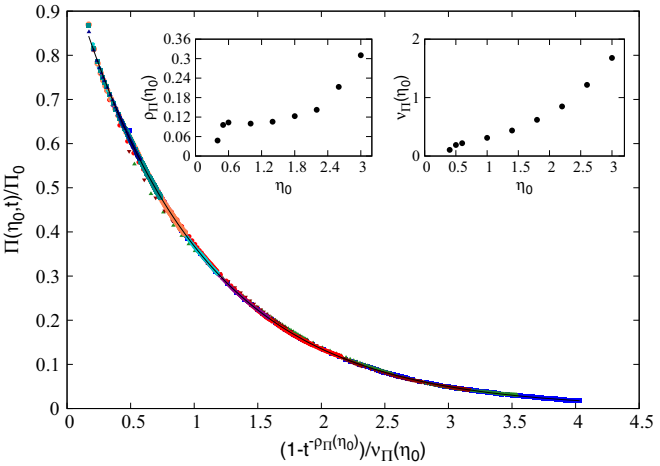

FIG. 4. (Color online) Plot of the data collapse of the function $\Pi(\eta_0,t)$. The inset shows the variation of $\rho_\Pi(\eta_0)$ and $\nu_\Pi(\eta_0)$ against η_0.

relatively larger domains and from Eq. (2), it is clear that a domain with a typical average size R should grow in such a way that the total length of all the interfaces of all the domains present in the system shrinks in order to minimize the energy of the system. The domain with largest mass should grow in the same fashion during its dynamical evolution. Therefore, $\Omega(\eta_0,t)$ is expected to exhibit similar behavior as that of $\Pi(\eta_0,t)$. Log-log plot of $\Omega(\eta_0,t)$ against time is shown in Fig. 3. In this context, Seppälä and Alava [32] showed that below a critical random field strength, the largest domain spans the system and the two-dimensional (2D) RFIM shows a percolation transition. This was supported by some later studies [7,33]. We also check that below a critical random field strength η_c, the largest cluster is a spanning one with a fractal dimension 1.89 ± 0.02. Above η_c, the largest cluster is finite. The value of η_c of course depends on temperature. In a recent article [34] we also reported that below a critical random field strength, the 2D RFIM exhibits long range order (LRO). We now provide the theory of kinetic coarsening. In general, we denote by $\Psi(\eta_0,t)$ the three quantities. A careful observation of the graphs suggests that the initial and asymptotic behavior of the generalized function $\Psi(\eta_0,t)$ is given by

$$\begin{aligned} &\Psi(\eta_0,t) \to \Psi_0 \text{ (const)}, \quad \text{as } t \to 1 \\ &\text{and } \Psi(\eta_0,t) \to \Psi_0 e^{-1/\nu(\eta_0)}, \quad \nu(\eta_0) > 0, \quad \text{as } t \to \infty, \end{aligned} \tag{4}$$

where $\nu(\eta_0)$ is a disorder-dependent scaling exponent. The decay rate of the function $\Psi(\eta_0,t)$ at any time step for a fixed η_0 should depend on the following factors: first, on the value of the function itself at this time step. Second, from the nature of the variation of the functions, it is evident that the rate of decay of the function $\Psi(\eta_0,t)$ also depends on the particular time step. As the time flows, the rate of decay of $\Psi(\eta_0,t)$ slows down and this dependence is taken as a power-law decay. In addition to these factors, another η_0-dependent parameter should be there for controlling the decay rate of $\Psi(\eta_0,t)$. This parameter considers the wandering of the interfaces in presence of the random fields. Thus the decay rate of $\Psi(\eta_0,t)$ is given by

$$\frac{d\Psi}{dt} \sim -a(\eta_0)\Psi t^{-\mu(\eta_0)}, \quad \mu(\eta_0) > 0, \tag{5}$$

$a(\eta_0)$ is a disorder-dependent parameter. Integrating,

$$\Psi(\eta_0,t) = \Psi_0 \exp\left[\frac{t^{-(\mu(\eta_0)-1)}}{\nu(\eta_0)} + k(\eta_0)\right], \tag{6}$$

where $k(\eta_0)$ is a constant of integration and $\nu(\eta_0) = \frac{\mu(\eta_0)-1}{a(\eta_0)}$. Now from (4) as $t \to 1$, $\Psi(\eta_0,t) \to \Psi_0$, which gives $k(\eta_0) = -1/\nu(\eta_0)$ and as $t \to \infty$, $\Psi(\eta_0,t) \to \Psi_0 e^{-1/\nu(\eta_0)}$, which gives $\mu(\eta_0) > 1$. Thus the functional form of $\Psi(\eta_0,t)$ is given by

$$\Psi(\eta_0,t) = \Psi_0 \exp\left[-\frac{1-t^{-\rho(\eta_0)}}{\nu(\eta_0)}\right], \tag{7}$$

where $\rho(\eta_0) = \mu(\eta_0) - 1 > 0$. The scaling behavior (7) of the functions characterizing the kinetic coarsening shows an universal nature with two disorder-dependent exponents $\rho(\eta_0)$ and $\nu(\eta_0)$. The validity of the the scaling relation (7) can be confirmed from the plots of the data collapse of the functions $\Pi(\eta_0,t)$, $\Lambda(\eta_0,t)^{1/2}$, and $\Omega(\eta_0,t)$ with the corresponding exponents (ρ_Π,ν_Π),$(\rho_\Lambda,\nu_\Lambda)$, and (ρ_Ω,ν_Ω) [35]. The plots are shown in Figs. 4–6, respectively. From scaling relation (7) the initial time behavior of $\Psi(\eta_0,t)$ is given by

$$\Psi(\eta_0,t) = \Psi_0 t^{-\rho(\eta_0)/\nu(\eta_0)} \quad \text{for} \quad t \ll \exp(1/\rho). \tag{8}$$

Thus the function $\Psi(\eta_0,t)$ shows a power-law decay with the exponent $\frac{\rho(\eta_0)}{\nu(\eta_0)}$ until the characteristic time scale $t_\times \sim e^{1/\rho(\eta_0)}$. This initial linear behavior in log scale is observed from Figs. 1–3, respectively. It is observed from the insets of Figs. 4–6, where the variation of $\rho(\eta_0)$ and $\nu(\eta_0)$ against η_0 is shown, that as $\eta_0 \to 0$, $\rho(\eta_0) \to 0$ and $\nu(\eta_0) \to 0$ with $\frac{\rho}{\nu}$ finite, which implies that the power-law decay continues for longer time as $\eta_0 \to 0$. This behavior is quite expected because for weak random field strength, the decay of $\Psi(\eta_0,t)$ is dominated by the exchange interaction. $\Psi(\eta_0,t)$ asymptotically approaches the value $\Psi_0 e^{-1/\nu(\eta_0)}$. Physically it means that as $t \to \infty$, the domains cease to grow. The

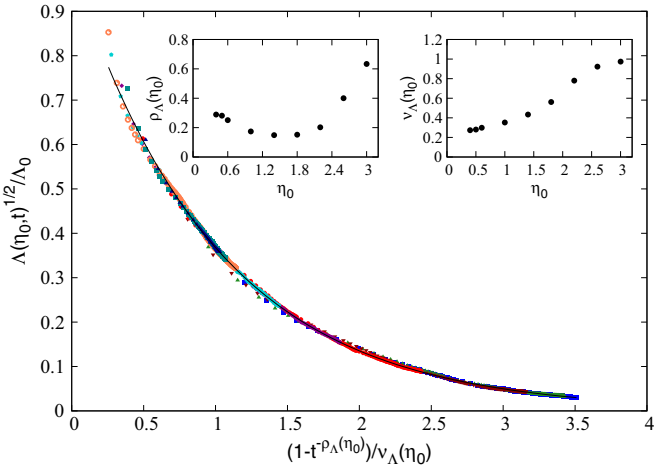

FIG. 5. (Color online) Plot of the data collapse of the function $\Lambda(\eta_0,t)^{1/2}$. The inset shows the variation of $\rho_\Lambda(\eta_0)$ and $\nu_\Lambda(\eta_0)$ against η_0.

dynamic behavior of the average linear domain size can also be predicted from the scaling relation (7). The typical average linear size $R(\eta_0,t)$ is given by

$$R(\eta_0,t)^d \sim \frac{L^d}{\Lambda(\eta_0,t)}$$
$$R(\eta_0,t) \sim \Lambda(\eta_0,t)^{-1/2} \quad \text{for} \quad d=2. \tag{9}$$

From simple calculations, the initial time, late time, and the asymptotic nature of $R(\eta_0,t)$ are obtained as

$$\begin{aligned} R(\eta_0,t) &\sim t^{\rho_\Lambda/\nu_\Lambda} \quad \text{for} \quad t \ll e^{1/\rho_\Lambda} \\ &\sim (\ln t)^{1/\nu_\Lambda} \quad \text{for} \quad e^{1/\rho_\Lambda} \ll t \ll \infty \\ &\sim e^{1/\nu_\Lambda} \quad \text{for} \quad t \to \infty. \end{aligned} \tag{10}$$

We interpret the ratio $\frac{\nu_\Lambda(\eta_0)}{\rho_\Lambda(\eta_0)}$ as the nonuniversal dynamic exponent $z(\eta_0)$ corresponding to the early time power-law

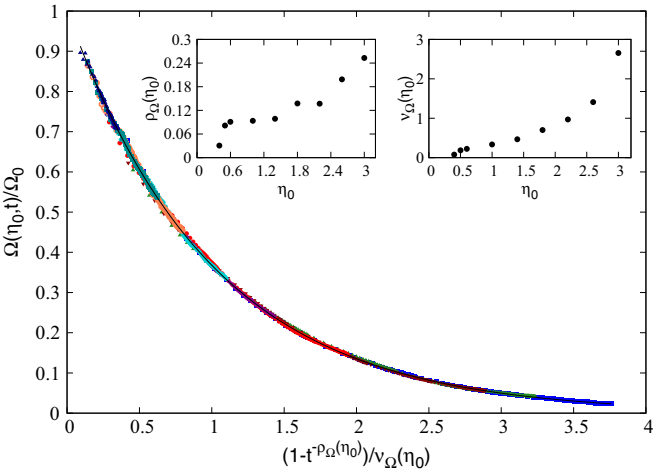

FIG. 6. (Color online) Plot of the data collapse of the function $\Omega(\eta_0,t)$. The inset shows the variation of $\rho(\eta_0)$ and $\nu(\eta_0)$ against η_0.

growth of $R(\eta_0,t)$. Also, the barrier exponent for the late time regime is interpreted as $\nu_\Lambda(\eta_0)$. Thus the scaling relation (7) successfully reproduces the recent claims [9,16,17,25] that the growing length scale $R(\eta_0,t)$ shows a dynamic crossover from a preasymptotic algebraic growth to asymptotic slow logarithmic growth. Another essential feature corresponding to the growth of $R(\eta_0,t)$ is contained in the proposed scaling law. At $t \to \infty$, the value of $R(\eta_0,t)$ approaches $e^{1/\nu_\Lambda(\eta_0)}$. This avoids the asymptotic divergence of $R(\eta_0,t)$. Thus, the behavior of the average linear domain size $R(\eta_0,t)$ in the entire time regime can physically be explained with the help of the scaling relation (7). In view of the above discussion, $\nu(\eta_0)$ is interpreted as follows. The scaling relation (7) shows that as $\nu(\eta_0) \to 0$, $\Psi(\eta_0,t) \to 0$ for $t \to \infty$. It means that the pinning interaction starts dominating as $\nu(\eta_0)$ increases. Thus, $\nu(\eta_0)$ is responsible for the stiffness of the domain wall. This is also obvious from the late time dynamics of $R(\eta_0,t)$ which is governed by the exponent $\nu(\eta_0)$ only [see Eq. (10)]. The domain wall would become stiffer with the increase of $\nu(\eta_0)$ As $t \to 1$, $\Psi(\eta_0,t)$ reaches a fixed value Ψ_0, independent of η_0. It is to be noted that the scaling relation (7) suggests that as $t \to \infty$, $\Psi(\eta_0,t)/\Psi_0 = \exp(-1/\nu)$. So at $t \to \infty$, the number of domains relative to their initial value converges to a well-defined value and the ratio is a measure of the entropy of the system [1]. Thus if the ratio is known, the value of $\nu(\eta_0)$ can be determined and from the insets of Figs. 4–6, the value of η_0 may be found corresponding to a particular $\nu(\eta_0)$. Therefore the infinite time limit of the scaling relation converges to a well-defined thermodynamic quantity that would fix the value of η_0.

We end this article with a few comments. Although we present results for a particular temperature and for a particular system size, we check that the same scaling relation holds good for other temperatures and other system sizes as well. However, the system size has to be below the breakup length scale and the temperature should not be so high that the thermal fluctuations become relevant. We would also like to point out that the Hamiltonian given by Eq. (1) depends on J, η_0, and T or, more precisely, on the ratio J/T and η_0/T. J/T being fixed in the present work, the quantities of our interest depend on η_0/T only. This means irrespective of any particular value of η_0 and T, the ratio of η_0 and T would govern the coarsening of the system. This adds generality to the scaling relation (7). Although we arrived at the scaling relation for the 2D RFIM, this relation also corroborates the recent claim of a possible crossover from a early time power-law growth to a late-time logarithmic growth in Ising model with random coupling and random dilution [26,27]. Certainly many more simulations on different systems are required to confirm the generic nature of the scaling relation (work in this direction is in progress).

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