Finite kicked environments and the fluctuation-dissipation relation

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In this work we derive a generalized map for a system coupled to a kicked environment composed of a finite number N of uncoupled harmonic oscillators. Dissipation is introduced via the interaction between system and environment which is switched on and off simultaneously (kicks) at regular time intervals. It is shown that kicked environments naturally generate a non-Markovian rotated dynamics, describe more complicated system-environment couplings which involve position *and* momentum, and satisfy an unusual fluctuation-dissipation relation. As an example, the motion of a kicked Brownian particle is discussed.

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I. INTRODUCTION

The classical Langevin equation (LE) is a phenomenological equation applicable to a large variety of physical systems coupled to a surrounding heat bath [1–3]. Without discussing in detail the obvious relevance of such an equation, it elucidates the role of environments and describes the system dynamics due to complicated surroundings. Roughly speaking it includes a dissipation induced by the bath, memory effects, and noise via the fluctuating term which satisfies the fluctuation-dissipation relation (FDR). It also shows that surrounding effects can mainly be described by the bath spectral density. In the case of weak system-bath couplings, the generalized LE can be derived from a microscopic model composed of a system interacting bilinearly (in position) with a surrounding bath composed of N harmonic oscillators (HOs) [4]. Equations of motion for the HOs can be solved analytically and the resulting equation of motion for the system only is exactly the generalized Langevin equation. The word generalized refers to the explicit appearance of the memory integral in the LE. The quantum LE has also been derived [4–7], to mention a few works, even though with more complicated FDRs.

If there exists such an important equation valid for a large variety of continuous dynamical systems, it is natural to ask if there is something similar for dynamical systems described by maps. The dynamics of such a generalized map should also describe a large variety of discrete physical systems. The main goal of the present work is to derive such a generalized map starting from the microscopic model composed of a system coupled to an environment constituted of HOs. The essential distinctions to the usual continuous model is that our environment is composed of a *finite* number N of uncoupled *kicked* HOs, and the coupling between system and environment is switched on and off instantaneously (kicks) at regular times. While the effects of finite but nonkicked environments have already been studied in classical problems [8–13], kicked couplings and environments have been used to describe quantum decoherence due to the nonlinear dynamics of the environment [14,15]. Our approach is different from the dynamical systems of Langevin type presented by [16], which analyzed the dynamics of a kicked damped particle, where the kicks coming from the environment are deterministic and generated by specific nonlinear chaotic maps. The Langevin-like map has also been studied in a totally different approach for systems without baths kicks [17].

As we will see, dissipation in the system is induced by the kicked couplings and a non-Markovian dynamics naturally appears due to kicked environments. The derived map is quite general since it is applicable to any system and contains features of realistic physical properties which are very actual, namely kicked couplings, kicked environments which induce bilinear couplings in the momentum, finite baths [18,19], and memory effects. A possible, very recent experimental realization of kicks could be implemented for atoms in the Bose-Einstein condensate [20,21]. The laser is switched on and off and induces loss of atoms inside one well of the optical standing wave, representing the dissipation. External kicking forces can also be used to control quantum two-level systems [22,23].

Results are presented in the following way. Section II presents the derivation of the generalized map. In Sec. III the dissipation constant from the map is discussed and in Sec. IV the properties generated by the kicked environment are analyzed, namely the intrinsic non-Markovian dynamics, the fluctuation-dissipation relation, the rotated dynamics, and couplings between system and environment which depend on the momentum. In Sec. V we discuss the dynamics of a free particle coupled to the kicked environment and, finally, in Sec. VI we summarize our results.

II. GENERALIZED MAP

We start with the time-dependent Hamiltonian

$$H(t) = H_S(t) + H_I(t) + H_B(t), \tag{1}$$

where H_S, H_I, and H_B are the Hamilton functions of the system, interaction, and bath, respectively. They are written as follows:

$$H_S(t) = \frac{P^2}{2M} + V(X)\delta_\tau, \tag{2}$$

$$H_I(t) = -\sum_{i=1}^{N} \Gamma_i x_i X \delta_\tau, \tag{3}$$

$$H_B(t) = \sum_{i=1}^{N} \left[\frac{p_i^2}{2m_i} + \frac{m_i \omega_i^2 x_i^2}{2} \delta_\tau \right], \tag{4}$$

where $\delta_\tau = \sum_{n=0}^{\infty}\delta(\frac{t}{\tau}-n)$, (X,P) are the conjugated position and momentum of the system, while (x_i,p_i) are the conjugated position and momenta of the N bath HOs $(i=1,2,\ldots,N)$. For the purpose of the present work the interaction between system and bath is considered bilinear with interaction intensity given by Γ_i. The Hamiltonian (1) describes a one system particle with mass M, and N bath particles with masses m_i, which receive respectively a "kick," at times $t=n\tau$, from the system potential $V(X)$ and the harmonic potential $m_i\omega_i^2x_i^2/2$. The interaction between system and particle is turned on at the kicks.

The equations of motion from the microscopic model (1) can be integrated between subsequent times t_n and t_{n+1}. Using the notation $P(t_n)\to P_n, X(t_n)\to X_n$, and for the bath variables $x_i(t_n)\to x_n^{(i)}$ and $p_i(t_n)\to p_n^{(i)}$, the scalings $x^{(i)}\to x^{(i)}\sqrt{m_i\omega_i^2/E_i}$ and $p^{(i)}\to p^{(i)}/\sqrt{m_iE_i}$ [where E_i is defined in Eq. (21)], we obtain in dimensionless variables

$$P_{n+1} = P_n + F_S(X)|_{X=X_n} + \sum_{i=1}^{N}\gamma_i x_n^{(i)}, \tag{5}$$

$$X_{n+1} = X_n + P_{n+1}, \tag{6}$$

$$p_{n+1}^{(i)} = p_n^{(i)} - k_i x_n^{(i)} + \frac{\gamma_i}{k_i}X_n, \tag{7}$$

$$x_{n+1}^{(i)} = x_n^{(i)} + k_i p_{n+1}^{(i)}, \tag{8}$$

where $F_S(X) = -\partial v(X)/\partial X$ with $\gamma_i = \Gamma_i\tau^2/\sqrt{m_iM}$, where $(i=1,2,\ldots,N)$ being $k_i\equiv\omega_i\tau$ for each oscillator. Here the discrete time steps are given in units of the period τ. All quantities are defined just after the kicks. We have used the same symbols for dimensionless variables, but now with $v(X)$ being the dimensionless system potential.

Equations (7) and (8) are linear in all physical quantities and can be solved analytically. After straightforward calculations we obtain the following solutions for the environment coordinates:

$$x_n^{(i)} = g_{n-1}^{(i)}x_0^{(i)} + k_i f_{n-1}^{(i)}p_0^{(i)} + \gamma_i\sum_{n'=0}^{n-1}X_{n'}f_{n-n'-1}^{(i)}, \tag{9}$$

$$p_n^{(i)} = -k_i f_{n-1}^{(i)}x_0^{(i)} + \left(g_{n-1}^{(i)} + k_i^2 f_{n-1}^{(i)}\right)p_0^{(i)} + \frac{\gamma_i}{k_i}\sum_{n'=0}^{n-1}X_{n'}\left(g_{n-n'-1}^{(i)} + k_i^2 f_{n-n'-1}^{(i)}\right), \tag{10}$$

where $g_{-1}^{(i)}=1$, $g_0^{(i)}=1-k_i^2$, and $f_0^{(i)}=1$. In other words, the coefficients can be generated by the recursive relation:

$$g_{n+1}^{(i)} = g_n^{(i)} - k_i^2 f_{n+1}^{(i)}, \tag{11}$$

$$f_{n+1}^{(i)} = f_n^{(i)} + g_n^{(i)}, \tag{12}$$

provided that the initial conditions satisfy $0\leqslant k_i<2$. For $k_i\geqslant 2$ the above recursive relations are such that any initial condition will diverge after few iterations. Substituting the solution Eq. (9) in Eq. (5) we obtain

$$P_{n+1} = P_n + F_S(X)|_{X=X_n} + \sum_{i=1}^{N}\gamma_i\left[g_{n-1}^{(i)}x_0^{(i)} + k_i f_{n-1}^{(i)}p_0^{(i)} + \gamma_i\sum_{n'=0}^{n-1}X_{n'}f_{n-n'-1}^{(i)}\right]. \tag{13}$$

In order to bring this expression in a more appropriate form we rewrite the last sum term as

$$\sum_{n'=0}^{n-1}X_{n'}f_{n-n'-1}^{(i)} = \sum_{n'=0}^{n-1}\frac{X_{n'}}{k_i^2}\left[g_{n-n'-2}^{(i)} - g_{n-n'-1}^{(i)}\right] = \frac{1}{k_i^2}\left[g_{-1}^{(i)}X_n - g_{n-1}^{(i)}X_0 - \sum_{n'=1}^{n}g_{n-n'-1}^{(i)}P_{n'}\right], \tag{14}$$

where we have used the maps for coefficients Eqs. (11) and (12) and the fact that $X_{n'} = X_{n'+1} - P_{n'+1}$. In this way we obtain the final map for the system only as follows:

$$P_{n+1} = P_n + F_S^{\text{eff}}(X)|_{X=X_n} + F_n - \sum_{n'=1}^{n}K_{n,n'}P_{n'} - \sum_{i=1}^{N}\frac{\gamma_i^2}{k_i^2}g_{n-1}^{(i)}X_0, \tag{15}$$

$$X_{n+1} = X_n + P_{n+1}, \tag{16}$$

where we used $F_s^{\text{eff}} = -\partial v_{\text{eff}}(X)/\partial X$ with $v_{\text{eff}}(X) = v(X) - \sum_{i=1}^{N}\frac{\gamma_i^2}{k_i^2}X_n^2/2$, F_n is the force defined as

$$F_n \equiv \sum_{i=1}^{N}\gamma_i\left[g_{n-1}^{(i)}x_0^{(i)} + k_i f_{n-1}^{(i)}p_0^{(i)}\right], \tag{17}$$

and

$$K_{n,n'} \equiv \sum_{i=1}^{N}\left(\frac{\gamma_i^2}{k_i^2}\right)g_{n-n'-1}^{(i)} \tag{18}$$

is the memory kernel, or memory-friction kernel.

The system of Eqs. (15) and (16) is the generalized map, our main result, which has some similarities to the generalized LE from the continuous model [4]. The generalized map gives the dynamics for the system and shows explicitly what are the effects coming from the kicked environment. These effects, which will be discussed in the following, can be summarized as the dissipation dynamics, memory effects contained in the sum $\sum_{n'=1}^{n}K_{n,n'}P_{n'}$ (this is the origin of the term generalized, used here), the fluctuating force F_n, the fluctuation-dissipation relation, the inherent kicked property of the environment, and the finite sum over the bath oscillators (in the derivation of the continuous Langevin description from the microscopic model, the number of bath oscillators is taken to be infinite).

III. CONTRACTION RATE (λ)

Even though the system plus environment is a conservative problem, due to energy exchanges between them the system dynamics may behave dissipatively. The contraction rate in the phase space of the system can be obtained from the Jacobian of the maps (15) and (16) and is given by $\lambda = 1 - \sum_{i=1}^{N} \gamma_i^2/k_i^2$. Dissipation occurs when $0 \leqslant \lambda < 1$, going from the overdamping limit $\lambda = 0$ to close to the conservative limit $\lambda = 1$. It is clear to see that dissipation is strongly dependent on the number N of bath oscillators, on the kick intensities k_i and on the coupling strength to each oscillator γ_i. However, it is not dependent on memory effects. Choosing $\gamma_i = \gamma$ and $k_i = k$, we obtain $\lambda = 1 - N\gamma^2/k^2$ and it is reasonable to scale γ^2 with N so that an effective coupling exists, which is $\gamma_{\rm eff} = \gamma/\sqrt{N}$. This agrees with results obtained using linear response theory [12].

IV. PROPERTIES OF THE KICKED ENVIRONMENT

In this section we discuss separately the main relevant dynamical properties generated by the finite kicked environment. All these properties are essential in the description of the system of interest via the generalized map (15) and (16).

A. Rotated environment

In order to understand the dynamics of the kicked environment, we look here at the dynamics of only one kicked HO of the bath. For this we use $\Gamma_i = 0 \Rightarrow \gamma_i = 0$ and Eqs. (7) and (8) can be solved analytically. The solutions are

$$x_n^{(i)} = \left[\cos(n\theta_i) - \frac{k_i}{2\sqrt{1-k_i^2/4}}\sin(n\theta_i)\right]x_0^{(i)} + \left[\frac{1}{\sqrt{1-k_i^2/4}}\sin(n\theta_i)\right]p_0^{(i)}, \tag{19}$$

$$p_n^{(i)} = -\left[\frac{k_i}{\sqrt{1-k_i^2/4}}\sin(n\theta_i)\right]x_0^{(i)} + \left[\cos(n\theta_i) + \frac{k_i}{\sqrt{1-k_i^2/4}}\sin(n\theta_i)\right]p_0^{(i)}, \tag{20}$$

with $\cos\theta_i = 1 - k_i^2/2$ and $\sin\theta_i = k_i\sqrt{1-k_i^2/4}$. The effect of the kick is such that one obtains a rotated ellipse in the phase space $(x_n^{(i)}, p_n^{(i)})$. For each oscillator of the bath we can write

$$H_{\rm eff}(x,p) = \frac{x^2}{2} + \frac{p^2}{2} - \frac{k}{2}xp = E, \tag{21}$$

where we omit the index i. This actually represents points that are over the orbit in phase space of one oscillator of the bath, and the above quantity is the total constant energy E of the ellipse. In Fig. 1 we depicted three trajectories for the equation $\frac{p^2}{2E} + \frac{x^2}{2E} - \frac{xp}{2E/k} = 1$ with the same initial condition $(x_0,p_0) = (0.0,\sqrt{2E})$, but different values of the dimensionless parameter $k = 0.2, 0.8, 1.2$. For $k \to 0$ the phase-space dynamics becomes a circle. As k increases the circle transforms into an ellipse with the principal axis along the line $p = x$. It is also interesting to observe that for some values of k_i the angle θ_i can assume a rational value. In such cases

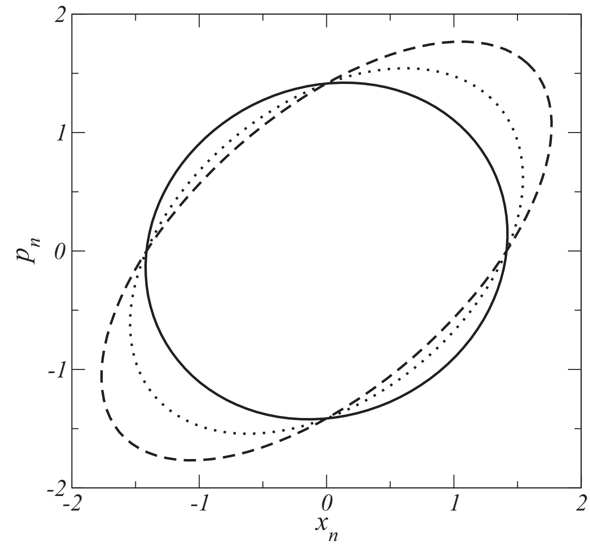

FIG. 1. Phase space for three trajectories of the bath map, Eqs. (7) and (8), with $\gamma_i = 0$ and $k = 0.2$ (continuous line), $k = 0.8$ (dashed), and $k = 1.2$ (dot-dashed) for 10^5 iterates.

the environment dynamics is periodic and the fluctuating force can have some particular properties which may be of interest.

In the case of one kicked HO it is possible to obtain distributions for the position and momentum of each oscillator, defined respectively through the general expression as follows:

$$\Phi(x) = A\int_{-\infty}^{\infty}\delta(H_{\rm eff}(x,p) - E)dp = \frac{1/\pi}{\sqrt{\frac{2E}{1-k^2/4} - x^2}}, \tag{22}$$

$$\Phi(p) = B\int_{-\infty}^{\infty}\delta(H_{\rm eff}(x,p) - E)dx = \frac{1/\pi}{\sqrt{\frac{2E}{1-k^2/4} - p^2}}, \tag{23}$$

where A and B are normalization constants. The dimensionless constant E can be obtained through initial conditions and it is equal to $H_{\rm eff}(x_0,p_0)$. The above distributions are plotted (dashed line) respectively in Figs. 2(a) and 2(b) for $k = 0.8$. Numerical results (histogram) were realized and fully agree with the distributions. When compared to the nonkicked HO, the above distributions have the additional term $-k^2/4$ in the denominator [24]. Consequently, in the kicked HOs the amplitude of the oscillations increases. The physical reason for this is that the particle is free to move between the kicks, so it can go furthest before the kick restricts the motion inside the harmonic potential.

B. Intrinsic non-Markovian dynamics

For practical reasons, the treatment of a real bath causing dissipation is usually based on the introduction of the spectral density $J(\omega) = \pi/2\sum_{i=1}^{N}\Gamma_i^2/(m_i\omega_i)\delta(\omega - \omega_i)$ [25]. In our model, the HOs frequencies ω_i are contained in the definition

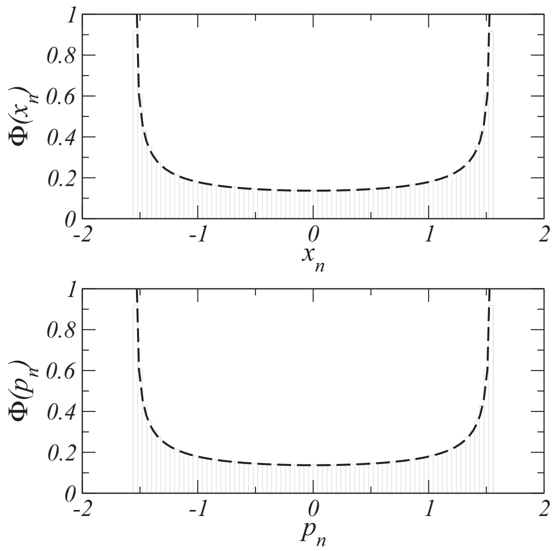

FIG. 2. (a) Distributions for position $\Phi(x)$ and (b) for momentum $\Phi(p)$ for one initial condition and 10^5 iterates. The bars show the histograms obtained by iteration of the map (7), (8) (for $\gamma_i = 0$) and the dashed curves are given by Eqs. (22) and (23).

of the scaling constant $k_i = \tau\omega_i$. Thus we can define a spectral density in the following form:

$$J(k) = \frac{\pi}{2}\sum_{i=1}^{N}\left(\frac{\gamma_i^2}{k_i}\right)\delta(k-k_i), \tag{24}$$

so that the memory kernel (18) becomes

$$K_{n,n'} = \frac{2}{\pi}\int_0^\infty dk\frac{J(k)}{k}\cos\left[(n-n')\theta(k)\right] - \frac{1}{\pi}\int_0^\infty dk\frac{J(k)}{\sqrt{1-k^2/4}}\sin\left[(n-n')\theta(k)\right], \tag{25}$$

with $\tan[\theta(k)] = (k\sqrt{1-k^2/4})/(1-k^2/2)$. When $k>2$ the angle θ becomes imaginary and the above integrals diverge. In addition, it is worth remembering that for $k>2$ the dynamics of the map (11) and (12) for the environment also diverges. Such a behavior for the friction kernel and the map is not of interest for now.

For the moment we assume to have an infinite number of HOs ($N\to\infty$) so that $J(k)$ is a continuous function of k. We can choose $J(k)=\eta k$, which would correspond to the ohmic case [25] (Markovian limit) $J(\omega)=\eta\omega$ from the continuous case, where η is the viscosity coefficient. For this particular choice of $J(k)$ we obtain

$$K_{n,n'} = \frac{2\eta}{\pi}\int_0^\infty dk\,\cos\left[(n-n')\theta(k)\right] - \frac{\eta}{\pi}\int_0^\infty dk\frac{k}{\sqrt{1-k^2/4}}\sin\left[(n-n')\theta(k)\right]. \tag{26}$$

Since $\theta(k)$ depends on k in a complicated way, we clearly recognize that both integrals did not give the $\delta_{n,n'}$ which are expected in the Markovian limit. In order to observe the Markovian limit we transform the integrals to $d\omega$,

$$K_{n,n'} = \frac{2\eta\tau}{\pi}\int_0^\infty d\omega\,\cos\left[(n-n')\theta(\omega)\right] - \frac{\eta\tau^2}{\pi}\int_0^\infty d\omega\frac{\omega}{\sqrt{1-(\tau\omega)^2/4}}\sin\left[(n-n')\theta(\omega)\right]. \tag{27}$$

The integration over ω can now be taken from 0 to ∞, but in order to avoid imaginary values of $\theta(\omega)$, we have to assume that τ goes faster to zero than ω increases. Expanding $\theta(\omega)\approx\tau\omega+\tau^2\omega^2/24$ and neglecting quadratic terms in τ we obtain

$$K_{n,n'} \sim \frac{2\eta\tau}{\pi}\int_0^\infty d\omega\,\cos\left[(n-n')\tau\omega)\right] = \frac{2\eta}{\pi}\delta_{n,n'}, \tag{28}$$

which gives the correct Markovian limit. Thus in order to observe the Markovian limit we have to look at smaller time scales $\tau\to 0$ (continuous time), $N\to\infty$ and allow larger bath frequencies. This is very much in the spirit of the LE, where the time is continuous and the bath frequencies are larger, i.e., the bath dynamics occurs on a faster time scale than the system.

In general, however, it is interesting to observe from the expression for the memory kernel (25) that solely the time discreteness of the kicked environment is capable of generating a non-Markovian dynamics of the system. Physically, it expresses the fact that the bath requires a finite time to respond to any fluctuation in the system motion. In the Markovian limit this time should disappear, as we observed above when $\tau\to 0$ and $N\to\infty$. In addition, further analysis for specific $J(k)$ in Eq. (25) could be realized which may generate special dynamics on the system. This analysis, however, is not the purpose of the present work. It is also interesting to observe that since the kick of one bath HO is $k_i=\tau\omega_i$, any distribution for the k_i corresponds to a distribution of the frequencies ω_i. However, we could assume one fixed ω with distinct τ_i. In such a case, we have a distribution of the kicking times which plays the role of the spectral density. It is a distribution of the collision times of the bath HOs with the system. This is related to what has been done in the work [26], which analyzed dissipation and fluctuation for a randomly kicked particle. They used a Poisson distribution for the waiting times (between kicks) and a distribution for which the average time between the kicks diverges.

C. Defining a temperature for the kicked bath and the fluctuation-dissipation relation

As in the continuous LE, we may interpret the force F_n as a fluctuating force representing the microscopic collisions of the bath particles with the system and it is reasonable to expect that $\langle F_n\rangle = 0$. The question remains about the possible relation (or not) of the autocorrelation functions of F_n with the memory kernel when kicked environments are present. The subject of this section is to answer this question.

Consider the situation in which the bath is initially in equilibrium and uncoupled with the system. The distribution for the bath is assumed to be

$$\rho(x_0,p_0) = Z^{-1}e^{-\beta\sum_{i=1}^{N}H_{\text{eff}}(x_0^{(i)},p_0^{(i)})}, \tag{29}$$

where $\beta \propto 1/T$, T is the temperature, and Z is the partition function given by

$$Z = \int_{-\infty}^{\infty} e^{-\beta \sum_{i=1}^{N} H_{\rm eff}(x_0^{(i)},p_0^{(i)})} dx_0^{(i)} dp_0^{(i)} \tag{30}$$

$$= \left(\frac{2\pi}{\beta}\right)^N \prod_{i=1}^{N} \frac{1}{\sqrt{1-k_i^2/4}}. \tag{31}$$

The Hamiltonian $H_{\rm eff}(x_0^{(i)},p_0^{(i)})$ corresponds to the constant of motion for the harmonic map described in Sec. IV A. Thus the environment composed of kicked HOs (or the dynamics on the rotated ellipse) allows us to define a thermal bath with temperature T for the kicked environment. It is important to mention here that this definition is only possible if the rotated Hamiltonian $H_{\rm eff}(x_0^{(i)},p_0^{(i)})$ is used.

With the above distributions the fluctuating force Eq. (17) obeys $\langle F_n\rangle = 0$. For the autocorrelation we obtain

$$\langle F_n F_{n'}\rangle = \sum_{i=1}^{N} \frac{\gamma_i^2}{k_i^2 \beta\left(1-k_i^2/4\right)} \cos\left[(n-n')\theta_i\right], \tag{32}$$

where we have used the solution for x_n, Eq. (19), and coefficients g_n and f_n. This must be compared with the friction kernel from Eq. (18), where

$$g_{n-n'-1}^{(i)} = \cos\left[(n-n')\theta_i\right] - \frac{k_i}{2\sqrt{1-k_i^2/4}} \sin\left[(n-n')\theta_i\right]. \tag{33}$$

As an immediate consequence we have, for the autocorrelation function,

$$\langle F_n F_{n'}\rangle = \frac{K_{n,n'} + \Psi_{n,n'}}{\beta}, \tag{34}$$

with

$$\Psi_{n,n'} = \sum_{i=1}^{N} \frac{\gamma_i^2}{(1-k_i^2/4)} \left\{ \frac{1}{4}\cos\left[(n-n')\theta_i\right] + \frac{\sqrt{1-k_i^2/4}}{2k_i} \sin\left[(n-n')\theta_i\right] \right\}.$$

Looking at Eq. (34) we observe the additional term $\Psi_{n,n'}$ which is responsible for the more complicated FDR obtained in this kicked environment. However, in the limit $k_i \to 0$ (which can be obtained by taking $\tau \to 0$ and keeping ω_i fixed), Eq. (34) reduces to the usual FDR $\langle F_n F_{n'}\rangle = K_{n,n'}/\beta$. Since this is the same limit from which we obtained the Markovian limit, we can say that the usual FDR is obtained in the Markovian limit.

D. Rotating back the kicked environment

As we observed in Secs. IV A and IV C, the phase-space dynamics of the kicked HO is a rotated ellipse and the usual FDR is not satisfied. On the other hand, it is known that for a bath composed of the usual nonkicked HOs, the phase-space dynamics is along a circle and the FDR is valid. Thus the following interesting question remains: which effect generates the complicated FDR, the rotated ellipse of the kicked HOs, and/or the kicked properties? To answer this we search for a coordinate system for the kicked HO for which the ellipse is not rotated. This can be checked when the crossed term in the constant of motion from Eq. (21) vanishes. Such coordinates are expressed by

$$\begin{pmatrix}\tilde{x}_n\\ \tilde{p}_n\end{pmatrix} = R(\phi)\begin{pmatrix}x_n\\ p_n\end{pmatrix}, \tag{35}$$

where $R(\phi)$ is the rotation matrix of a system of coordinates and corresponds to a rotation of coordinates (x_n,p_n) by an angle ϕ. Since $R^{-1}(\phi) = R(-\phi)$ and $R(\phi)R(-\phi) = R(-\phi)R(\phi) = I$, which is the identity matrix, then

$$\begin{pmatrix}\tilde{x}_{n+1}\\ \tilde{p}_{n+1}\end{pmatrix} = R(\phi) J R^{-1}(\phi)\begin{pmatrix}\tilde{x}_n\\ \tilde{p}_n\end{pmatrix}, \tag{36}$$

with J being the Jacobian matrix of the original kicked system. As observed in Sec. IV A the principal axis of the ellipse is along the line $p = x$. Thus the angle to rotate it back to the line $p = 0$ is $\phi = \pi/4$. As a consequence, we obtain a new map for the rotated coordinates system [we omit the index (i)]

$$\tilde{p}_{n+1} = \left(1-\frac{k^2}{2}\right)\tilde{p}_n - \left(1-\frac{k}{2}\right)k\tilde{x}_n + \frac{\gamma}{\sqrt{2}}\frac{(1-k)}{k}X_n, \tag{37}$$

$$\tilde{x}_{n+1} = \left(1+\frac{k}{2}\right)k\tilde{p}_n + \left(1-\frac{k^2}{2}\right)\tilde{x}_n + \frac{\gamma}{\sqrt{2}}\frac{(1+k)}{k}X_n, \tag{38}$$

where the solutions for $\gamma = 0$ are

$$\tilde{x}_n = \tilde{x}_0 \cos(n\theta) + \sqrt{\frac{2+k}{2-k}}\,\tilde{p}_0 \sin(n\theta), \tag{39}$$

$$\tilde{p}_n = -\sqrt{\frac{2-k}{2+k}}\,\tilde{x}_0 \sin(n\theta) + \tilde{p}_0 \cos(n\theta). \tag{40}$$

The new constant of motion becomes

$$\mathcal{H}(\tilde{x},\tilde{p}) = \left(\frac{1}{2}-\frac{k}{4}\right)\tilde{x}^2 + \left(\frac{1}{2}+\frac{k}{4}\right)\tilde{p}^2, \tag{41}$$

which is the nonrotated ellipse.

We also note that we have new coefficients for $(\tilde{x}_0,\tilde{p}_0)$, which we will call $\tilde{g}_{n-1}$ and $\tilde{f}_{n-1}$; then, for $\gamma \neq 0$,

$$\tilde{x}_n^{(i)} = \tilde{x}_0^{(i)}\tilde{g}_{n-1}^{(i)} + \sqrt{\frac{2+k_i}{2-k_i}}\,\tilde{p}_0^{(i)}\tilde{f}_{n-1}^{(i)} + \frac{\gamma_i}{k_i\sqrt{2}}\sum_{n'=0}^{n-1} X_{n'}\left(\tilde{g}_{n-n'-1}^{(i)} + \sqrt{\frac{2+k_i}{2-k_i}}\,\tilde{f}_{n-n'-1}^{(i)}\right), \tag{42}$$

$$\tilde{p}_n^{(i)} = -\sqrt{\frac{2-k_i}{2+k_i}}\,\tilde{x}_0^{(i)}\tilde{f}_{n-1}^{(i)} + \tilde{p}_0^{(i)}\tilde{g}_{n-1}^{(i)} + \frac{\gamma_i}{k_i\sqrt{2}}\sum_{n'=0}^{n-1} X_{n'}\left(\tilde{g}_{n-n'-1}^{(i)} - \sqrt{\frac{2-k_i}{2+k_i}}\,\tilde{f}_{n-n'-1}^{(i)}\right). \tag{43}$$

The rotation of coordinate x_n implies the replacement of the position in Eq. (5) by $(\tilde{x}_n - \tilde{p}_n)/\sqrt{2}$. This leads to

$$\begin{aligned} P_{n+1} &= P_n + F_S(X)|_{X=X_n} + \sum_{i=1}^{N} \frac{\gamma_i}{\sqrt{2}}\big(\tilde{x}_n^{(i)} - \tilde{p}_n^{(i)}\big) \\ &= P_n + F_S(X)|_{X=X_n} + \mathcal{F}_n \\ &\quad + \sum_{i=1}^{N} \frac{\gamma_i^2}{\sqrt{1-k_i^2/4}} \sum_{n'=0}^{n-1} X_{n'} \tilde{f}_{n-n'-1}^{(i)}, \end{aligned} \tag{44}$$

with the fluctuating force $\mathcal{F}_n$,

$$\begin{aligned} \mathcal{F}_n \equiv \sum_{i=1}^{N} \frac{\gamma_i}{\sqrt{2}} \Bigg[\tilde{x}_0^{(i)} \Bigg(\tilde{g}_{n-1}^{(i)} + \sqrt{\frac{2-k_i}{2+k_i}} \tilde{f}_{n-1}^{(i)} \Bigg) \\ + \tilde{p}_0^{(i)} \Bigg(\sqrt{\frac{2+k_i}{2-k_i}} \tilde{f}_{n-1}^{(i)} - \tilde{g}_{n-1}^{(i)} \Bigg) \Bigg]. \end{aligned} \tag{45}$$

Now $\tilde{g}_n^{(i)} = \cos[(n+1)\theta_i]$ and $\tilde{f}_n^{(i)} = \sin[(n+1)\theta_i]$, with θ_i being the same as before. It is interesting to note that the map (44) contains system-bath couplings which include not only the position of the HOs but also their momentum. Therefore, it is possible to say that kicked environments are able to describe more complicated system-bath couplings which include terms proportional to $X\sum_{i=1} p^{(i)}$.

The partition function, now written as $Z = \int_{-\infty}^{+\infty} e^{-\beta\mathcal{H}(\tilde{x}_0,\tilde{p}_0)} d\tilde{x}_0 d\tilde{p}_0$, has the same value as before. One can show that $\langle \mathcal{F}_n \rangle = 0$ and the autocorrelation function has the same form as given in Eq. (32), but with the new $\tilde{g}_n^{(i)}$. Thus, by using unrotated coordinates, we still obtain the unusual FDR. Therefore, it is not the rotated environment that induces the unusual FDR but its kicked properties.

V. FREE PARTICLE IN THE FINITE KICKED ENVIRONMENT: THE KICKED BROWNIAN MOTION

As an example we study in this section the dynamics of a free particle $[v(X) = 0]$ coupled to the finite kicked environment. We discuss two physical situations: no memory effects, where analytical results can be obtained and numerically a map with just one memory term.

Markovian case. We assume that all memory terms from the generalized map are zero, so the map is

$$P_{n+1} = \lambda P_n + F_n, \tag{46}$$

$$X_{n+1} = X_n + P_{n+1}, \tag{47}$$

with $\lambda \equiv (1 - K_{n,n})$ being the dissipation constant, where the bath average gives $\langle F_n \rangle = 0$ with $\langle F_n^2 \rangle = \sum_{i=1}^{N} \gamma_i^2 / [k_i^2(1 - k_i^2/4)]$ from Eq. (32). The above simplified map can be obtained by (a) choosing particular relations between γ_i and k_i in Eq. (18), so that all $K_{n,n'}$ for $n \neq n'$ vanish, or (b) as a particular case of the generalized map proposed in the conclusion, which is not restricted to a bath composed of uncoupled HOs and must not satisfy Eq. (18).

For the momentum we have the following solution written as a sum of independent variables:

$$P_n = \sum_{l=0}^{n-1} \lambda^l F_{n-l-1}. \tag{48}$$

The characteristic function, defined as the average value of the Fourier transform of the probability density function is equal to

$$g_n(k) = \langle e^{ikP_n} \rangle = \prod_{l=0}^{n-1} \langle e^{ik\lambda^l F_{n-l-1}} \rangle = e^{-k^2 b_n/2}, \tag{49}$$

with $b_n = \langle F_n^2 \rangle \frac{(1-\lambda^{2n})}{(1-\lambda^2)}$. The average is taken over the bath realizations. Then the probability density function, which can be evaluated through inverse Fourier transform of characteristic function is given by

$$\rho(P_n) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g_n(k) e^{-ikP_n} dk \tag{50}$$

$$= \frac{1}{\sqrt{2\pi b_n}} e^{-P_n^2/2b_n}, \tag{51}$$

which gives

$$\langle P_n \rangle \equiv \int_{-\infty}^{+\infty} \rho(P_n) P_n dP_n = 0, \tag{52}$$

$$\langle P_n^2 \rangle = b_n = \frac{1-\lambda^{2n}}{1-\lambda^2} = \frac{1-(1-\gamma^2)^{2n}}{1-(1-\gamma^2)^2}. \tag{53}$$

In the last expressions, consider γ as the effective coupling parameter. Proceeding in the same way, we can obtain the solution for the position, written in terms of a sum of independent variables,

$$X_n = \sum_{l=0}^{n-1} \frac{1}{K_{n,n}} (1 - \lambda^{l+1}) F_{n-l-1}. \tag{54}$$

For the characteristic function we find

$$G_n(k) = \prod_{l=0}^{n-1} \left\langle \exp \frac{ik}{K_{n,n}} (1 + \lambda^{l+1}) F_{n-l-1} \right\rangle = e^{-k^2 d_n/2},$$

where

$$d_n = \frac{\langle F_n^2 \rangle}{K_{n,n}^2} \left[\frac{2\lambda - 2\lambda^{n+1} - 2\lambda^{2+n} + \lambda^{2+2n} - n + (n+1)\lambda^2}{\lambda^2 - 1} \right]. \tag{55}$$

As a consequence, we obtain $\rho(X_n) = \frac{e^{-X_n^2/2d_n}}{\sqrt{2\pi d_n}}, \langle X_n \rangle = 0$, and $\langle X_n^2 \rangle = d_n$.

This example nicely shows that the diffusion increases linearly in time like $d_n \sim Dn$, with the diffusion constant $D = 1/K_{n,n}^2 = 1/(\lambda - 1)^2$. These results are very similar to a Brownian particle coupled to a Markovian environment. The main effect coming from the kicked bath is a particular value for the dissipation constant λ.

Non-Markovian case. Here we choose the map with just one memory term:

$$P_{n+1} = \lambda P_n + K_{n,n-1} P_{n-1} + F_n, \tag{56}$$

$$X_{n+1} = X_n + P_{n+1}, \tag{57}$$

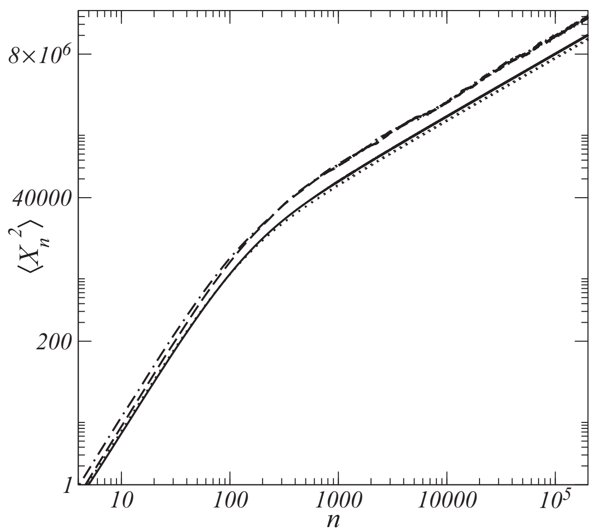

FIG. 3. Plotted is $\langle X_n^2 \rangle$ for $\gamma = 0.1/\sqrt{N}$ as a function of n for the Markovian case [continuous ($N = 100$) and dotted ($N = 1000$) lines] and non-Markovian case [dashed ($N = 100$) and dot-dashed ($N = 1000$) lines].

where $\langle F_n \rangle = 0$, $\langle F_n^2 \rangle = \sum_{i=1}^{N} \gamma_i^2/[k_i^2(1 - k_i^2/4)]$, and $\langle F_n F_{n-1} \rangle = \sum_{i=1}^{N} \gamma_i^2 \cos[\theta_i]/[k_i^2(1 - k_i^2/4)]$ obtained from Eq. (32). Figure 3 shows $\langle X_n^2 \rangle$ as a function of n in the log-log plot, and compares the analytical Markovian case with the non-Markovian one. While normal diffusion ($\sim n^{1.0}$) is observed for longer times $n \gtrsim 2000$, accelerated superdiffusive motion with $\sim n^{2.7}$ is observed for smaller times. In all cases the quantity $\langle P_n^2 \rangle$ converges to a finite value.

VI. CONCLUSIONS

This work shows that kicked environments generate a very rich and unexpected dynamics on the system coupled to it. Dissipation in the system is introduced by turning on and off, at regular times, the coupling to the environment. To show the effects of kicked environment we derive a generalized map starting from a microscopic model composed by a system coupled bilinearly to a finite number N of harmonic oscillators. The main features induced by kicked environments and couplings can be summarized as the intrinsic non-Markovian dynamics (the Markovian dynamics was obtained in the continuous time limit), rotated phase-space dynamics of the HOs of the environment, definition of a temperature for the kicked environments, unusual fluctuation-dissipation relation, and more complicated system-environment couplings which take into account the momentum of the HOs. All these features are valid independent of the system of interest.

Since discrete dynamical systems are usually easy to study numerically, the present generalized map represents a possibility to model complicated phenomena and couplings found in the real system. In fact, we derived the generalized map from the microscopic model, but our results allow us to propose the generalized map in a more general sense in the following form:

$$P_{n+1} = \lambda P_n + F_S(X)|_{X=X_n} + F_n - \sum_{n'=1}^{n-1} K_{n,n'} P_{n'},$$

$$X_{n+1} = X_n + P_{n+1}.$$

Here $F_S = -\partial v(X)/\partial X$ is the system force, the dissipation constant is $\lambda = (1 - K_{n,n})$, F_n is the fluctuating force, and $K_{n,n'}$ is the dissipation kernel which can be related to the spectral density $J(\omega)$ of the environment and contains all information coming from the bath. All these quantities can be chosen as needed since we do not have to restrict the bath to uncoupled HOs, as we did in this work. *If* we assume to have uncoupled HOs in the environment, then $\langle F_n \rangle = 0$, $K_{n,n'}$ is given by Eq. (18), the unusual FDR from Eq. (34) must be satisfied, and a non-Markovian dynamics is expected. This allows us to affirm that the usual FDR, $\langle F_n F_{n'} \rangle = K_{n,n'}/\beta$, can only be used in the above discrete map model when the spectral density of the environmental HOs is so complicated that Eq. (35) is zero, or the environment is composed by more complicated oscillators (maybe *coupled* HOs, *nonlinear* oscillators, or *chaotic* oscillators, to give some possible examples).

One possible implementation of the generalized map with kicks was mentioned in the Introduction in the context of cold atoms [20,21]. We also would like to conjecture further applications of the kicked environment (rotated), namely, to describe rotating heat baths in a Kerr black hole [27] or the Faraday rotation in nonlinear optics. This is a phenomenon of the interaction between light and a magnetic field in a medium, and it causes a rotation of the polarization plane of the light. From this point of view, our kicked environment could represent a bath composed of a light interacting with distinct media.

The next interesting topics which can be analyzed further are choosing specific spectral densities $J(\omega)$, the numerical analysis to understand the importance of kicked environments in distinct systems, and deriving the quantum version of the generalized map.

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