Universal shocks in the Wishart random-matrix ensemble. II. Nontrivial initial conditions

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We study the diffusion of complex Wishart matrices and derive a partial differential equation governing the behavior of the associated averaged characteristic polynomial. In the limit of large-size matrices, the inverse Cole-Hopf transform of this polynomial obeys a nonlinear partial differential equation whose solutions exhibit shocks at the evolving edges of the eigenvalue spectrum. In a particular scenario one of those shocks hits the origin that plays the role of an impassable wall. To investigate the universal behavior in the vicinity of this wall, i.e., in the vicinity of a critical point, we derive an integral representation for the averaged characteristic polynomial and study its asymptotic behavior. The result is a Bessoid function.

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I. INTRODUCTION

The Wishart random-matrix ensemble [1], a multidimensional generalization of the χ^2 distribution, has proven, over the many years since its invention, to be one of the most prominent examples of the vast applicability of random-matrix theory. It has become an important tool in multivariate statistics [2], helping to understand a broad range of phenomena occurring in such fields as population structure study [3], financial data analysis [4], or image processing [5]. When it was realized that it can describe the information capacity of a multiple input multiple output system [6–8], the otherwise called Laguerre ensemble changed the face of multichannel information theory. Moreover, being closely related to so-called chiral random matrices, the Wishart matrix shares, in a narrow universal window in the vicinity of the zero eigenvalue, spectral properties with the Dirac operator in Euclidean quantum chromodynamics, thus portraying the spontaneous breakdown of chiral symmetry through the famous Banks-Casher formula [9]. Finally, matrices from the Laguerre ensemble appear in quantum information theory [10], the research of conducting mesoscopic systems [11], or chaotic scattering in cavities [12].

The study of static properties of random matrices proves to be highly rewarding. Yet, as realized already by Dyson [13], introducing some additional dynamics can be equally if not more fruitful. In the case of the Wishart ensemble, an evolving matrix was first defined through a Brownian motion of real and complex matrix entries in [14,15] and [16,17], respectively. More recently [18], such a stochastic process was generalized to arbitrary values of the Dyson parameter β, in particular for $\beta \in (0,2]$. In the meantime, the theory of nonintersecting Brownian motions or the so-called vicious walkers was developed. The subject that originated from the works of de Gennes on fibrous structures [19] and Fisher on wetting and melting [20] was linked to random-matrix theory [21–24] and led to many developments, including a physical realization of the statistical properties of Wishart matrices through fluctuations of nonintersecting interfaces in thermal equilibrium [25]. For additional physical applications of the Laguerre ensemble we refer the reader to the prequel of this paper [26].

Both in random-matrix and vicious walker theories, a central role is played by (multi)orthogonal polynomials, their Cauchy transforms, and the related, characteristic, and inverse characteristic polynomials. This is because these polynomials are the building blocks of correlation functions and they govern the universal asymptotic behavior of probability distributions [27–30]. It is an ongoing challenge to uncover the properties of these objects, in particular, those related to the Laguerre ensemble.

In a previous work [26] we studied the stochastic evolution of a Wishart matrix for trivial initial conditions corresponding to vanishing eigenvalues. In that setting, the associated characteristic polynomial coincides with a time-dependent, monic, orthogonal Laguerre polynomial, which we have shown to satisfy a certain, exact (i.e., it is valid for any matrix size N), complex, partial differential equation. This in turn allowed us to recover the universal Airy and Bessel asymptotic behaviors of the characteristic polynomial at the edges of the spectrum as associated with hydrodynamical-like shocks arising from the solution of a related nonlinear partial differential equation governing the evolution of the resolvent in the large-N limit.

Here we show that the same stochastic process, but with nontrivial initial conditions, i.e., initialized with a Wishart matrix possessing a single N-degenerate eigenvalue $a^2 \neq 0$, allows us to identify a microscopic eigenvalue scaling associated with a novel asymptotic behavior of the characteristic polynomial. The phenomenon occurs at the origin, precisely when it is hit by the diffusing spectrum or, in the hydrodynamic language of [26], when the shock wave reaches the origin that plays the role of an impassible wall. To achieve this, we prove that the characteristic polynomial satisfies the above-mentioned partial differential equation for any initial condition. In this scenario, the model can be viewed as a Wishart matrix perturbed by a source and there are no polynomials, orthogonal in the classical sense, associated with this setting. Note that this is the reason why the derivation requires the use of more sophisticated methods than those employed in [26]. Moreover, it was through the studies of the

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Gaussian unitary random-matrix ensemble with an external source [31,32] that the asymptotic Pearcey behavior at the critical point was discovered [33–35]. In our setting, it would arise through a diffusion of a Hermitian matrix initiated with at least two distinct eigenvalues, at the point of merging of the spectra. In the case of the Laguerre ensemble, the additional symmetry imposes a different functional form, associated with modified Bessel functions of the first kind.

The multiple orthogonal polynomials associated with the modified Bessel functions of the first kind were first studied in [36,37]. They were used to build a kernel for a chiral Gaussian unitary ensemble perturbed by a source in [38]. The critical behavior studied here, however, was not identified. This was done in the context of nonintersecting squared Bessel paths in [39], where the integral representation of the limiting kernel was derived with Riemann-Hilbert techniques. Finally, this kernel reduces to the so-called symmetric Pearcey kernel identified through the studies of random growth with a wall [40,41]. Our work differs from those above by the use of completely different methods. We follow strictly the diffusing Wishart matrix and focus on the averaged characteristic polynomial rather than on the kernel itself. This allows us to obtain a unified picture of the behavior near the critical point.

This paper has the following structure. We start by defining the Brownian walk of the elements of a Wishart matrix and state the partial differential equations fulfilled by the associated characteristic polynomial and its logarithmic derivative, an inverse Cole-Hopf transform (the proof is left for Appendix A). The latter coincides with the resolvent (or Green's function) in the large-N limit. By exploiting the method of complex characteristic, we subsequently determine the resolvent for our new set of initial conditions. This allows us to formally identify the new critical point and the large-N scaling of the eigenvalue density in its vicinity. We then recover the explicit arbitrary N solution of the partial differential equation for the characteristic polynomial, expand it in the vicinity of the origin at the time of the collision, and show that it is asymptotically a version of the so-called Bessoid function. We conclude with a summary.

II. FORMAL SETTING

We consider an $N \times N$ random matrix of the following form:

$$L(\tau) = K^{\dagger}(\tau)K(\tau), \tag{1}$$

where the entries of K, an $M \times N$ ($M > N$) matrix, evolve in time τ according to

$$dK_{ij}(\tau) = dx_{ij} + i dy_{ij} = b_{ij}^{(1)}(\tau) + i b_{ij}^{(2)}(\tau), \tag{2}$$

where $b_{ij}^{(1)}(\tau), b_{ij}^{(2)}(\tau)$ are two independent sets of free Brownian walks:

$$b_{ij}^{(c)}(\tau) = \zeta_{ij}^{(c)}(\tau) d\tau, \tag{3}$$

$$\left\langle \zeta_{ij}^{(c)}(\tau) \right\rangle = 0, \tag{4}$$

and

$$\left\langle \zeta_{ij}^{(c)}(\tau)\zeta_{kl}^{(c')}(\tau') \right\rangle = \tfrac{1}{2}\delta^{cc'}\delta^{ik}\delta^{jl}\delta(\tau - \tau'). \tag{5}$$

We define $\nu \equiv M - N$ and the rectangularity as $r \equiv N/M$.

A (Gaussian) probability is related to the free Brownian motions, which allows us to define the averaged characteristic polynomial associated with the matrix L:

$$Q_N^{\nu}(z,\tau) \equiv \langle \det[z - L] \rangle. \tag{6}$$

It is shown in Appendix A that $Q_N^{\nu}(z,\tau)$ satisfies the partial differential equation

$$\partial_\tau Q_N^{\nu}(z,\tau) = -z\partial_{zz} Q_N^{\nu}(z,\tau) - (\nu+1)\partial_z Q_N^{\nu}(z,\tau) \tag{7}$$

for any initial condition. The same equation was obtained in [26] for the particular initial condition $L(\tau = 0) = 0$. It was shown there that its solutions are in this case the time-dependent, monic, Laguerre polynomials.

We proceed by performing the inverse Cole-Hopf transform on $Q_N^{\nu}(z,\tau)$. Namely, we define $f_N = \frac{1}{N}\partial_z \ln[Q_N(z,\tau)]$. Equation (7) then yields the following equation for f_N:

$$\partial_\tau f_N + 2Nzf_N\partial_z f_N + Nf_N^2 = -(2+\nu)\partial_z f_n - z\partial_{zz} f_N. \tag{8}$$

After rescaling the time according to $\tau \to \frac{r\tau}{N}$ [26,42], this equation becomes

$$\partial_\tau f_N + r\left(2zf_N\partial_z f_N + f_N^2\right) + (1-r)\partial_z f_N = -\frac{r}{N}(2\partial_z f_N + z\partial_{zz} f_N). \tag{9}$$

In the large-N limit $f_N(z,\tau) = G(z,\tau) \equiv \frac{1}{N}\langle \mathrm{Tr}\frac{1}{z - L(\tau)}\rangle$, we recover

$$\partial_\tau G(z,\tau) = (r-1)\partial_z G(z,\tau) - 2rzG(z,\tau)\partial_z G(z,\tau) - rG^2(z,\tau), \tag{10}$$

in agreement with [43]. The partial differential equations (7) and (10) form the backbone of this paper. Solving the latter, in the following section, will allow us to recover the large-N limit spectrum of eigenvalues and identify the scaling of the level density in the vicinity of the edges, in particular that near the origin. The former, on the other hand, as shown in Sec. IV, admits an asymptotic solution that describes the universal behavior near the critical point. This new solution is the main result of this paper.

III. BULK OF THE SPECTRUM AT LARGE N

As announced above, we choose for the initial condition $L(\tau = 0) = \mathbb{1}^{N\times N}a^2$. This, according to Eq. (6), translates into $Q_N^{\nu}(z,\tau = 0) = (z - a^2)^N$ and, in the large-N limit, $G(z,\tau = 0) = \frac{1}{z - a^2}$. We focus in this section on $G(z,\tau)$ and solve Eq. (10) using the method of complex characteristics. This method transforms Eq. (10) into the following three ordinary differential equations [26]:

$$\frac{dz}{dp} = 1 - r + 2rzG, \tag{11}$$

$$\frac{d\tau}{dp} = 1, \tag{12}$$

$$\frac{dG}{dp} = -rG^2. \tag{13}$$

Here z_0 and p are auxiliary variables such that $z(p = 0) = z_0 + a^2$, $\tau(p = 0) = 0$, and $G(p = 0) = \frac{1}{z_0}$. Solving the last

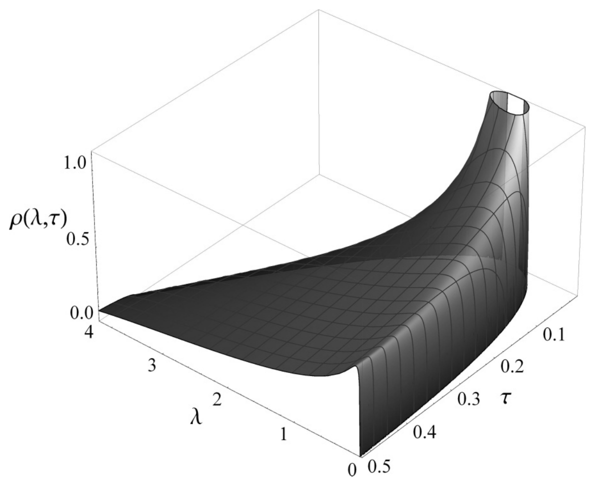

FIG. 1. Computed eigenvalue probability density function for $a = 1$ and $r = 1$.

two equations yields $p = \tau$ and

$$G = \frac{1}{r\tau + z_0}. \tag{14}$$

We are therefore left with

$$\frac{dz}{d\tau} = 1 - r + \frac{2rz}{r\tau + z_0}, \tag{15}$$

which is solved by

$$z = (z_0 + r\tau)\left(1 + \frac{\tau}{z_0} + a^2\frac{\tau r + z_0}{z_0^2}\right). \tag{16}$$

The characteristic curves are parametrized by z_0. By eliminating z_0 in Eq. (14) one gets the following implicit cubic equation for $G(z,\tau)$:

$$z = \frac{1}{G(z,\tau)} + \frac{\tau}{1 - r\tau G(z,\tau)} + a^2\frac{1}{[1 - r\tau G(z,\tau)]^2}. \tag{17}$$

The proper solution of this equation yields the eigenvalue density via the usual Sochocki-Plemelj formula. An illustration of this density and its time dependence is given by Fig. 1. One can also reconstruct the spectrum directly from the characteristic curves, as shown in Appendix B.

From now on we will work in the $r = 1$ limit since only then can the eigenvalues reach the origin. This is realized when we let N and M go to infinity, keeping ν constant and finite.

In the above derivation we assumed that the mapping between z and z_0 is one to one, that is, it can be inverted. This is the case except at points $z_{0_c}(\tau)$, such that $dz/dz_0 = 0$, where a singularity occurs. We obtain the following equation for z_{0_c}:

$$z_{0c}^3 - z_{0c}\tau(2a^2 + \tau) - 2a^2\tau^2 = 0. \tag{18}$$

The equation defines the locations of the shock waves, which coincide with the edges of the spectrum, otherwise found as points for which the solution of (17) gains a nonzero imaginary part. This is because the shocks mark the branch points of the Green's function. From this equation we deduce that the left edge reaches 0 at $\tau_c = a^2$.

We can expand the Green's function in the vicinity of this critical point. Setting $\tau = a^2$, in the leading order, for z close to zero, we obtain from (17)

$$G(z,\tau) - \frac{2}{3\tau} \propto z^{-1/3}. \tag{19}$$

This implies that the spectral density in a narrow strip s around the origin has to scale like $\rho(\lambda) \sim \lambda^{-1/3}$, therefore total number of eigenvalues in this strip $[n(s)]$ is equal to $N\int_s \lambda^{-1/3}d\lambda \sim Ns^{2/3}$. This binds the average eigenvalue spacing (corresponding to one eigenvalue per bin) to scale as $s \sim N^{-3/2}$. If we further allow ourselves to move around the critical point within the time domain, a careful expansion of $G(z,a^2 + t)$ will show that t has to be of the order of $N^{-1/2}$. In the beginning of the paper we defined the evolution of the matrix elements K_{ij} so that $\langle|K_{ij}|^2\rangle \sim \tau$. The time is rescaled overall by $N^{-3/2}$ and therefore this diffusive character of the dynamics is preserved in the microscopic regime defined in the vicinity of the critical point.

We are now equipped with enough information to study, in the following section, the large-N M asymptotics of the characteristic polynomial. Before we do that, let us briefly consider a scenario with a more complicated initial condition. In particular, suppose $L(\tau = 0) = \mathrm{diag}(a_1,\ldots,a_N)$ with a_i being positive numbers, $1 < T \leqslant N$ of which are distinct. This leads to a higher degree implicit polynomial equation for $G(z,\tau)$ in (17). The spectrum, at the beginning, occupies T disjoint intervals on the positive side of the real axis. As they merge, the microscopic behavior of the average characteristic polynomial in the vicinity of the associated point is given by the Pearcey function. Hitting the origin by the leftmost interval yields the same behavior as the one we will now study for a simpler initial condition.

IV. CHARACTERISTIC POLYNOMIAL AT THE CRITICAL POINT

Recall the partial differential equation for the averaged characteristic polynomial (7):

$$\partial_\tau Q_N^\nu(z,\tau) = -\frac{1}{M}z\partial_{zz}Q_N^\nu(z,\tau) - \frac{\nu+1}{M}\partial_z Q_N^\nu(z,\tau). \tag{20}$$

One can check with an explicit calculation that

$$Q_N^\nu(z,\tau) = \mathcal{C}\tau^{-1}z^{-\nu/2}\int_0^\infty y^{\nu+1}\exp\left(M\frac{z-y^2}{\tau}\right)I_\nu \times\left(\frac{2iMy}{\tau}\sqrt{z}\right)Q_N^\nu(-y^2,0)dy, \tag{21}$$

an integral involving the initial condition, is a solution of (20) for any N and ν. Note that I_ν is the modified Bessel function. [For $\nu = 0$, after a change of variables, $Q_N^0(w^2,\tau)$ can also be found in [44], as a solution of a two-dimensional heat equation with central symmetry.] The constant $\mathcal{C}$ is determined by matching the solution with the initial condition $Q_N^\nu(z,0)$. Note that [45] $\lim_{|x|\to\infty} I_\nu(x) \simeq \frac{1}{\sqrt{2\pi x}}e^x$, valid for $|\arg(x)| < \frac{\pi}{2}$ and here $x = \frac{2iMy\sqrt{z}}{\tau}$ so that $\arg(z) \neq 0$. In the limit of $\tau \to 0$, the saddle point approximation method enables us to deduce that $\mathcal{C} = i^{-\nu}2M$. We therefore obtain an integral representation for the averaged characteristic polynomial associated with a freely diffusing Wishart-type matrix of arbitrary size and for arbitrary initial conditions consistent with the symmetry of the ensemble. Let us mention additionally that it was recently derived in [46], with combinatorial methods, for a static Wishart matrix perturbed by a source.

We now turn to the specific case of $Q_N^\nu(z,0) = (z-a^2)^N$. In the limit of M and N going to infinity, with ν constant, the exponent, arising in the integral from the expansion of the modified Bessel function and the exponentiation of the initial condition, is dominated by values of y in the vicinity of the saddle points given by the solutions of the equation

$$y - i\sqrt{z} - \frac{\tau y}{a^2+y^2} = 0. \tag{22}$$

The three saddle points merge at $y=0$ for $z=0$ and $\tau = a^2$. Moreover, as predicted in the previous section, the critical behavior occurs when $|z| \sim N^{-3/2}$ and $|\tau - a^2| \sim N^{-1/2}$. One can therefore expand the natural logarithm $\ln(a^2+y^2) \approx \ln(a^2) + \frac{y^2}{a^2} - \frac{y^4}{2a^4}$. Furthermore, we set $\tau = a^2 + N^{-1/2}a^2 t$ and $z = N^{-3/2}a^2 s$, with $\arg(s) \neq 0$. To recover the proper asymptotics we rescale the integration variable by defining $y = N^{-1/4}au$. The limiting behavior becomes

$$\begin{aligned} &Q_N^\nu(N^{-3/2}a^2 s, a^2 + N^{-1/2}a^2 t) \\ &\quad\approx (-a^2)^N N^{(\nu+1)/2} s^{-\nu/2} \int_0^\infty u^{\nu+1} \\ &\quad\quad\times \exp\left(-\frac{1}{2}u^4 + u^2 t\right) I_\nu(2iu\sqrt{s})\, du, \end{aligned} \tag{23}$$

the announced result. Let us mention that for $\nu = -\frac{1}{2}$, (23) takes the form of

$$(i\pi)^{-1/2} s^{-1/4} \int_0^\infty \exp\left(-\frac{1}{2}u^4 + u^2 t\right)\cos(2u\sqrt{s})\, du \tag{24}$$

and is called the symmetric Pearcey integral through its connection with the symmetric Pearcey kernel arising for phenomena of random surface growth with a wall [40].

Moreover, for positive integer ν, as the Wishart ensemble is connected to chiral random matrices, it has an analog in the integral describing the statistical properties of the Dirac operator around its zero eigenvalue, at the moment of chiral symmetry breaking in Euclidean quantum chromodynamics [47]. The averaged characteristic polynomial of a diffusing complex chiral matrix is namely defined by

$$\tilde{Q}^\nu_{M+N}(w,\tau) \equiv \left\langle \det \begin{pmatrix} w & -K^\dagger \\ -K & w \end{pmatrix} \right\rangle \tag{25}$$

and related to its Wishart counterpart through $\tilde{Q}^\nu_{M+N}(w,\tau) = w^\nu\, Q_N^\nu(z = w^2, \tau)$. Its critical point analysis is analogical.

Finally, (23) was known earlier in optics. In particular

$$B(x,y) \equiv \int_0^\infty u \exp(iu^4 + iu^2 y) I_0(iux)\, du \tag{26}$$

is recognized as the Bessoid canonical function of order zero and appears in the description of the rotationally symmetric cusp (cuspoid) diffraction catastrophe [48–50]. Note that the behavior of the two differ as the $\sqrt{s}$ is complex while x is real and because the exponent in the latter has a complex phase. We are inspired, however, by the analogy and call (23) simply the Bessoid function.

V. CONCLUSION

In this paper we have continued our study of matrices belonging to the Wishart ensemble and performing a white-noise-driven Brownian walk. Our derivation of the partial differential equation fulfilled by the associated averaged characteristic polynomial allows us to inspect this process for arbitrary initial conditions and size of the matrix N. Here we differ from the prequel of this paper, where the method used permitted only a study of a trivial initial condition, for which the average characteristic polynomial coincides with a Laguerre polynomial.

The inverse Cole-Hopf transform of the characteristic polynomial obeys a nonlinear partial differential equation. In the large matrix size limit its solutions contain shocks that are positioned at the moving edges of the spectrum. For a matrix diffusion initiated from a nonzero N degenerate eigenvalue, when the shock reaches the origin, a distinct universal behavior of the eigenvalues occurs in a window shrinking like $N^{-3/2}$ while N grows to infinity. This phenomenon is encapsulated by the asymptotics of the averaged characteristic polynomial. We have derived its integral representation and studied it in the vicinity of the critical point. The resulting limiting behavior is described by an integral that we call Bessoid function.

In future research, we plan to address the problem of time and space correlations of the products and rations of characteristic polynomials. Another fascinating direction is including an arbitrary potential for the elements of the Wishart matrix to diffuse in. We hypothesize that such a generalization of this stochastic process would show universality of the obtained results.

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APPENDIX A: DERIVATION OF THE PARTIAL DIFFERENTIAL EQUATION

Here we derive the partial differential equation governing the evolution of the averaged characteristic polynomial associated with a diffusing Wishart matrix (7). The real and imaginary parts of each of the elements of the matrix K evolve according to the same diffusion equation:

$$\begin{aligned} \frac{d}{d\tau} P_{ji}^{(1)} &= \frac{1}{4}\frac{d^2}{dx_{ji}^2} P_{ji}^{(1)}, \\ \frac{d}{d\tau} P_{ji}^{(2)} &= \frac{1}{4}\frac{d^2}{dy_{ji}^2} P_{ji}^{(2)}. \end{aligned} \tag{A1}$$

The initial conditions are arbitrary. If, however, we intend to stay in the realm of Wishart-type random matrices, they cannot violate the symmetry of the ensemble. Since the elements evolve independently, the joint probability density is $P(x,y,\tau) = \prod_{j,i,c} P_{ji}^{(c)}$ and it obeys the equation

$$\partial_\tau P(x,y,\tau) = \frac{1}{4}\sum_{j,i}\left(\partial_{x_{ji}x_{ji}} + \partial_{y_{ji}y_{ji}}\right)P(x,y,\tau). \quad \text{(A2)}$$

Let η represent a column of complex Grassman variables η_i where $i \in \{1,2,\ldots,N\}$. The averaged characteristic polynomial, associated with (1), can be expressed in terms of the following integral:

$$Q_N^\nu(z,\tau) = \int D(\eta,\bar{\eta},x,y)\exp[\bar{\eta}(z - K^\dagger K)\eta]P(x,y,\tau), \quad \text{(A3)}$$

where the integration measure is $D(\eta,\bar{\eta},x,y) \equiv \prod_{i,j,k} d\eta_k d\bar{\eta}_i dx_{ji}dy_{ji}$. This form allows us to proceed to the main part of the proof. Acting with the time derivative on $Q_N^\nu(z,\tau)$, noticing that only the probability distribution $P(x,y,\tau)$ depends on time, and exploiting (A2), one gets

$$\partial_\tau Q_N^\nu(z,\tau) = \frac{1}{4}\int D(\eta,\bar{\eta},x,y)\exp[\bar{\eta}(z - K^\dagger K)\eta] \times \sum_{j,i}\left(\partial_{x_{ji}x_{ji}} + \partial_{y_{ji}y_{ji}}\right)P(x,y,\tau). \quad \text{(A4)}$$

At this point, we integrate by parts with respect to the variables x_{ij} and y_{ij} and proceed by executing the resulting differentiation [recall that $(K^\dagger K)_{ij} = \sum_k (x_{ki} - iy_{ki})(x_{kj} + iy_{kj})$]. After a brute force calculation one obtains

$$\partial_\tau Q_N^\nu(z,\tau) = -\int D(\eta,\bar{\eta},x,y)\bar{\eta}\eta(M + \bar{\eta}K^\dagger K\eta) \times \exp[\bar{\eta}(z - K^\dagger K)\eta]P(x,y,\tau). \quad \text{(A5)}$$

The first term in the sum in the integrand, namely, $M\bar{\eta}\eta$, can be represented as a differentiation with respect to z acting on the exponential factor and taken out of the integral over Grassmann variables. The second term $\bar{\eta}\eta(\bar{\eta}K^\dagger K\eta)$, on the other hand, can be identified as a differentiation of the exponent with respect to Grassmann variables. This jointly amounts to

$$\partial_\tau Q_N^\nu(z,\tau) = -M\partial_z Q_N^\nu(z,\tau) + \int D(\eta,\bar{\eta},x,y)\bar{\eta}\eta\exp(\bar{\eta}\eta z) \times \sum_i \bar{\eta}_i\partial_{\bar{\eta}_i}\exp(-\bar{\eta}K^\dagger K\eta)P(x,y,\tau). \quad \text{(A6)}$$

Again, we integrate by parts, this time in the Grassmann variables. This yields

$$\partial_\tau Q_N^\nu(z,\tau) = -M\partial_z Q_N^\nu(z,\tau) + \int D(\eta,\bar{\eta},x,y)\exp(-\bar{\eta}K^\dagger K\eta) \times \sum_i \partial_{\bar{\eta}_i}[\bar{\eta}\eta\exp(\bar{\eta}\eta z)\bar{\eta}_i]P(x,y,\tau). \quad \text{(A7)}$$

Performing the differentiation results in

$$\partial_\tau Q_N^\nu(z,\tau) = -z\partial_{zz}Q_N^\nu(z,\tau) + (N - M - 1)\partial_z Q_N^\nu(z,\tau), \quad \text{(A8)}$$

which concludes the proof. Calculations exploiting the methods used here can be found, for example, in [51].

APPENDIX B: ANALYSIS OF THE CHARACTERISTICS

Here we show how the large-N properties of the eigenvalue spectrum are encoded in the characteristics. The complex characteristic curves are defined in the (z,τ) hyperplane by Eq. (16), namely,

$$z = (z_0 + r\tau)\left(1 + \frac{\tau}{z_0} + a^2\frac{\tau r + z_0}{z_0^2}\right). \quad \text{(B1)}$$

These are labeled by the values of the complex variable z_0. They are not straight lines as in the case of the usual Burgers equation. Let us define $z = x + iy$ and $z_0 = x_0 + iy_0$. Notice that for $\tau = 0$, $z = z_0$, so that if a characteristic starts from a purely real point z_0, then z remains real at all times. For simplicity we set $a = 1$ and, as we are interested in the scenario where the spectrum hits the origin, $r = 1$. By taking the real and the imaginary parts of Eq. (B1) one gets

$$x = \frac{2\tau^2 x_0^2}{\left(x_0^2 + y_0^2\right)^2} + \frac{\tau[\tau(x_0 - 1) + 2x_0]}{x_0^2 + y_0^2} + 2\tau + x_0 + 1 \quad \text{(B2)}$$

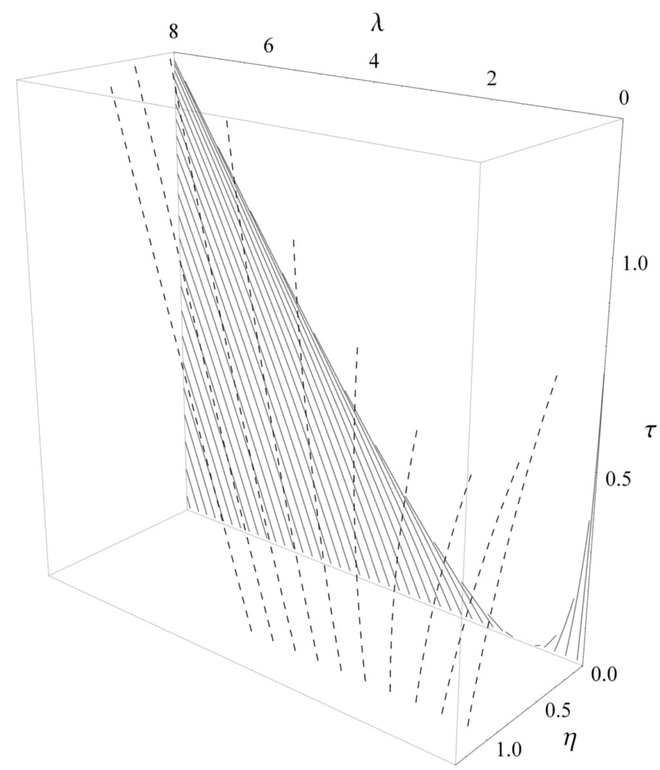

FIG. 2. Sample of characteristic curves remaining real through the evolution depicted with solid lines in the plane $y = 0$. They cross each other on the edges of the large-N limit spectrum. The dashed lines are examples of characteristic curves that start at complex points (here with $y_0 = 1$). At a specific value of τ, they cross the plane $y = 0$ in the area of the nonzero eigenvalue probability density.

and

$$y = y_0\left(1 - \frac{2\tau^2 x_0}{\left(x_0^2+y_0^2\right)^2} - \frac{(\tau+2)\tau}{x_0^2+y_0^2}\right) \equiv y_0 Y(x_0,y_0). \quad \text{(B3)}$$

We also have [cf. Eq. (14)]

$$G = \frac{1}{\tau+z_0} = \frac{\tau+x_0-iy_0}{(\tau+x_0)^2+y_0^2}. \quad \text{(B4)}$$

Moreover, the spectral density is given by $\rho(x,\tau) = -\frac{1}{\pi}\mathrm{Im}G|_{y=0^+}$ and therefore

$$\rho(x,\tau) = \left.\frac{y_0}{\pi\left[(\tau+x_0)^2+y_0^2\right]}\right|_{y=0^+}, \quad \text{(B5)}$$

where x appears on the right-hand side through (B2) and (B3). The limit $y = 0^+$ (recall that $y = y_0 Y$) can be accessed in four ways, by $y_0 \to 0^+$ with $Y(x_0,y_0) > 0$, $y_0 \to 0^-$ with $Y(x_0,y_0) < 0$, and $Y(x_0,y_0) \to 0^+$ with $y_0 > 0$ or $Y(x_0,y_0) \to 0^-$ with $y_0 < 0$. The first two give zero spectral density at points occupied by the characteristics curves defined by

$$x = (x_0+\tau)\left(1 + \frac{\tau}{x_0} + \frac{\tau+x_0}{x_0^2}\right). \quad \text{(B6)}$$

These are the curves that remain on the plane of real z throughout the evolution (solid lines depicted in Fig. 2). The last two conditions, together with Eq. (B2), define the characteristic curves that cross the $y = 0$ plane at a specific time τ (depicted as dashed lines in Fig. 2) and reconstruct the nonzero part of the spectral density, realizing the main goal of this calculation.

The edge of the spectrum in the real z plane is defined by $y_0 = 0$ and $Y(x_0,y_0) = 0$ fulfilled simultaneously, namely,

$$x_0^3 - \tau(\tau+2)x_0 - 2\tau^2 = 0. \quad \text{(B7)}$$

This coincides with (18).

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